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Dynamics of Markets and the Flaws of Prevalent Principles: A Mathematical Note

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Abstract: Both the optimization and equilibrium principles turn out to be more akin to common sense than to science. They have been postulated as describing markets, but lack the required empirical underpinning. Optimization is not a magic cure. In order to particularly circumvent some of the technical obstacles for a control problem, it turns out to be practically effective to reduce the system dynamics to a system of ordinary differential equations of considerably higher dimension, such an approach might replace a theoretical difficulty by a greatly increased computational problem.
Keywords: Integrability condition; Hamiltonian system; Adjoint differential equation.

1. Introduction

In his text book Intermediate Microeconomics. Varian [1] writes that much of the neo-classical theory in economics, finance and management is based on two principles: the optimization principle and the equilibrium principle. In the first people try to choose the best patterns of consumption they can afford. In the second, prices adjust until the amount people demand of something is equal to the amount that is supplied. Both of these principles may sound true. Both of these have been postulated as describing markets but lack the required empirical underpinning. This is because we do not know any universal laws of markets that could be used to explain even qualitatively correctly the phenomenon of economic growth, bubbles, recessions, depression, the lopsided distribution of wealth, the collapse of Marxism, and so on. Adam Smith long ago observed society qualitatively, as stated by Beinhocker [2] and invented the notion of an Invisible Hand that hypothetically should match supply to demand in free markets.

Adam Smith's stabilizing Invisible Hand forms the theoretical basis of the neoclassical equilibrium market model but, because of the lack of socioeconomic laws of nature and because of the non uniqueness in explaining statistical data, we have more difficulties in explain equilibrium than in natural sciences. That is why attempts are being made as shown in Das [3], Das and Okpechi [4] in recent days to replace the standard arguments about equilibrium with some empirically based non equilibrium dynamic models. The principle of optimization especially as it is used in management also lacks theDynamics of markets required empirical underpinning.

2. Equilibrium and Associated Problems

As an example of how easy it is to violate the expectation of stable equilibrium within the confines of optimizing behavior, consider three agents with three assets. The model is defined by assuming individual utilities of the form

$$U_i\ (x) = \min\,(x_1\ \ x_{2,}) \qquad (1.\ .1)$$

And an initial endowment for agent number 1

$$x_0 \ = \ (1, 0, 0) \qquad (1.\ 2)$$

The utilities and endowments of the other two agents are cyclic permutations on the above. Agent k has one item of asset k to sell and none of the other two assets. Recall that in ne0-classical theory the excess demand equation $dp/dt = D\,(p.\ t) - S\,(p, t) = \varsigma\,(p, t)$, where p_k is the price of an asset, D is the demand at that price and S is the corresponding supply and the vector field ς is the excess demand. With demand assumed to be slaved to price in the form $x = D\,(p)$, the phase space is just the n - dimensional space of prices p. That phase space is flat means that global parallelization of flows is possible for integrable systems.

More generally, we could assume that $d\,p/d\,t = f\ (\varsigma\,(p, t)$ where f is any vector field with the same qualitative properties as the excess demand. Whatever the choice, we must be satisfied with studying topological classes of excess demand functions. Because the excess demand functions cannot be uniquely specified by the

theory, given a model, equilibrium is determined by vanishing excess demand, i.e., by $\varsigma = 0$.[1] Stability of equilibrium, when equilibrium exist at all, is determined by the behavior of solutions displaced slightly from an equilibrium point. Note that dynamics require that we specify $x = D\,(p)$, not $p = f\,(x)$ and likewise for the supply schedule. Given a model of excess demand we can start by analyzing the number and character of equilibria and their stability.[2] Beyond that, one can ask whether is motion is integrable[3]. Typically, the notion for $n > 3$ is nonintegrable and may be chaotic or even complex, depending upon the topological class of model considered.[4]
We always assume that $x = D\,(p)$.if we relax the assumption and assume that demand is generated by a production function s

$$x = s\,(x\,,\,v\,,\,t\,) \qquad (1.3)$$

Where v denotes a set of unknown control functions. Assume a discounted utility functional

$$A = \int e^{-bt}\,u\;\;(x,\,v\,,\,t\,)\;\;dtt \qquad (1.4)$$

Where $u\,(x,\;v\;,t\;)$ is the discounted 'utility rate'. We maximize the utility functional A with respect to the set of instruments v. This is a problem in the calculus of variation.

$$\delta\,A = \int dt\;(\;\delta\,(\;e^{-bt}\,(\,u\,+$$

$$\overline{p}'\;\;\delta\,(s(\,x,\,v\,,\,t) - x\,))) = 0 \qquad (1.5)$$

Where p_i' are the Lagrange multipliers?

We use the discounted utility rate $u\,(x,\,v,\,t) = e^{-bt}\;u\,(x,\,v\,,\,t\,)$ with $p = e^{-bt}\,p'$ to find
$h\,(x,\,p\;,\,t\,) = \max\,(\;\varpi\;\;(\,x,\,v,\,t\,) +$

$$\overline{p}\;s\,(\,x,\,v\,,\,t\,)) \qquad (1.6)$$

$$\overline{p}_i\;\;= -\,d\;h\,/\,dx_i \qquad (1.7)$$

$$\dot{x}' = \;d\;h\,/\;d\;\;p_i = s\;\;(x,\,p,\,t) \qquad (1.8)$$

Which is a Hamiltonian system and h is generally time dependent and since h is dependent on time it is not conserved but integrability occurs if there are n global commuting conservation laws. The integrability condition due to n commuting global conservation laws can be written as

$$p = \nabla U\;\;(x) \qquad (1.9)$$

Where for bounded motion, the utility $U\,(\,x\,)$ is multivalued. U is just the reduced action

$$A = \int \overline{p}\;dx \qquad (1.10)$$

In this scenario, a utility function cannot be chosen by the agent but is determined instead by the dynamics. When satisfied the integrability condition (1.9) eliminates chaotic motion (and complexity) from considerations because there is a global differentiable canonical transformation to a coordinate system where the motion is free particle motion described by n commuting constant speed translations on a flat manifold imbedded in the 2 n dimensional phase space. Conservation laws correspond to continuous symmetries of the Hamiltonian dynamical system. In the economic literature p is called the 'shadow price' but the condition (1.10) is just the neo-classical condition for price.

The generic case is that the motion in phase space is nonintegrable, in which case it is typically chaotic. In this case the neo classical condition (1.9) does not exist and both the action

$$A = \int \varpi\;dt \qquad (1.11)$$

and the reduced action (1.10) are path dependent functionals, in agreement with Mirowski [9]. In this case $p = f\,(x)$ does not exist. The main point is that chaotic dynamics,[5] which is more common than simple dynamics, makes it impossible to construct a utility function.

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[1] The underlying reason for this constrain, called Walras Law, is just that capital and capital accumulation are not allowed in ne0-classical theory; neo-classical models assume a pure barter economy, so that the cost of the goods demanded can only equal the cost of the cost of the goods offered for sale. This condition simply means that the motion in the n-dimensional price space is confined to the surface of an n -1dimensional sphere. Therefore the motion is at most n -1 dimesional.

[2] The assumption of uniqueness of a single global equilibrium is equivalent to assuming the universality of the action of the Invisible Hand Independently of initial conditions. Here equilibrium would have to be an attractive fixed point with infinite basin of attraction in price space, see Jovanovic and Schinckus [5]

[3] See McCauley [6, 7]

[4] What the motion looks like for n > 3 is a question that cannot be answered *a priori* without specifying a definite class of models, see Neftci [8]

[5] For an excellent elucidation on chaotic dynamics, see Brock and Hommes [10]

Exercise 1

If we assume that prices are determined by supply and Demand then the simplest model as we can see is

$$\frac{dp}{dt} = \varsigma\ (p.\ t\ )$$

Where ς is excess demand. With the assumption that asset prices in liquid markets are random we have

$$d\ p = r\ (\ p.\ t\ )\ dt\ +\ d\ (\ p.\ t\ )\ \ d\ B\ (t)$$

Where B (t) is a Weiner process.[6] Write in a paragraph in the context of our equilibrium analysis, what does it mean? What will happen if financial prices appear to be random even on the shortest trading time scale?

3. Optimization and Decision Making

The current generation of decision makers has been led into thinking that the problem of effective decision making is an optimization problem. To illustrate, as in Casti [12] one of the many things that can go wrong in optimal decision, assume the system dynamics are given by the scalar linear differential equation

$dx\,/\,dt\ = f\ x\ +\ u$ where u is the decision function and f is a constant. Let it be required to choose u so as to minimize

$$½ \int u^2\ \text{(t)}\ dt \qquad (1.12)$$

Then it is a trivial exercise in the calculus of variations to see that the optimal system trajectory satisfies

$$A_1\ \ \sin\ \ f\,t\ +\ A_2\ \cos\ f\,t \qquad (1.13)$$

Where A_1 and A_2 are constants depending upon the initial and boundary conditions. Note, in particular that if f = 0, then x* (t) = constant , while for any f =0 the system oscillates. Thus even a small change of the parameter f away from zero changes the entire qualitative character of the system trajectory. Furthermore, for any value of f the system is not asymptotically stable when the so-called optimal decision is used.

In the example noted above, it is easy to see the technical factors that account for the instability of the optimal control but this is not the point. The real point is that if the objective is to choose u so as to class system stability, the criterion above does not reflect all of these factors. In fact, it reflects only on consideration: using as little control as possible. The situation is symptomatic in management, business and governmental environments, namely the optimization criterion imposed to simplify the decision problem account for only a limited number of desired system features and, furthermore, the resulting optimal description is generally a discontinuous function of changes in the problem data.

In essence, the problem is here is the problem of 'attainability', that is, the question is: can we get from where we are to where we want to be with the resources available within a specified time horizon.

Consider a dynamical process described by the set of differential equations

$$dx\,/\,dt = f\,(x,\ u),\ x\ (t_0) = x_0 \qquad (1.\ 14)$$

Where $x\ (\ t)\ \in R^n$, $u\ (\ t)\ \varepsilon\ R^m$ and f is a smooth function of its arguments. We assume that it is desired to transfer the system (1 .14).from x_0 to some state x * at time T by application of input u belonging to some attainable Ω. Thus we have the attainability problem. Clearly, the solution to the attainability problem will depend upon the interrelationship of the problem data, i. e, the structure of f and Ω, together with the time T and the initial and terminal state x_0 and x *, respectively[7].

Some problems of importance, however, are not smooth enough to possess gradients. If the non smoothness is with respect to the state vector, a situation common in problems found in an economic setting, gradients do not exist, the value of the adjoint equation itself is lost, and the dynamic nature of perturbation behavior is destroyed. A new approach is required for problems of this type.[8]

Exercise 2

Consider the problem:

$$\text{x* (t) =}\ \ u\,(t)\ \text{-}\ \ d\,(t)\,,\ \ \text{x (0)}\ = 0$$

$$\text{J}\ = \int_0^{\text{t}_1} \text{I}\ \{\,|\,x\ \text{(t)}\ |\,\}\ +1\,/\,2\ u\ (\text{t}^{\,2})\,d\,t$$

$u\ (\ t)\ >\ 0$

[6] See DeBondt and Richard [11]

[7] It takes little imagination to envision far worse surprises occurring when we try to regulate high-dimensional nonlinear processes unfolding in an uncertain environment – the typical sort of problem encountered in economics and management.

[8] Nonexistence of partial derivatives with respect to x more serious The adjoint equation breaks down and cannot easily be repaired by considering of two-sided derivatives or other simple measures. The fundamental dynamic property of the perturbation equations, at the root of the classical theory, breaks down and must be replaced by a new machinery

this problem can be interpreted as production scheduling problem in which x (t) represents inventory at hand (with negative inventory representing back orders) ; d (t) is the demand rate at time t (it is assumed known) and u (t) the control, is the production rate. There is a unit cost for storing inventory and a unit cost, for loss of goodwill in carrying back orders. The cost of production is quadratic.
Find a simple trial solution to the problem or simply see if you can get u (t) = d (t)

4. Conclusion

There are reasons against the notion that equilibrium exists, as is assumed explicitly by the intersecting supply-demand curves. It is shown here how easy it is to violate stability of equilibrium. The neo classical supply-demand curves cannot be expected to exist in the real markets. In order to circumvent some of the technical obstacles for a control problem, it turns out to be practically effective to reduce the system dynamics to a system of ordinary differential equations of considerably higher dimension. Such an approach, however, replaces the theoretical difficulties by a greatly increased computational problem.

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