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On Nonlocal Initial Problems for Fuzzy Differential Equations and Environmental Pollution Problems

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Abstract: We present the properties of fuzzy solutions of the nonlocal initial problems for fuzzy differential equations under generalized Hukuhara differentiability (NIP for FIDEs) by the point of view of Hausdorff metric space, for example, existence, uniqueness, boundedness, ... and stability of solutions. The different types of solutions NIP for FDEs are generated by the usage of two different concepts of fuzzy derivative in the formulation of a differential problems. The examples are given to illustrate these results.

Keywords: Hausdorff metric space; The positive lyapunov - like function; Fuzzy differential equations under generalized hukuhara differentiability (FDEs); The nonlocal initial problems (NIP).

1. Introduction

The fuzzy set was published in 1965 and later by Zadeh at the University of Berkeley, California - USA [1]. In 1970 Mamdani Queen Mary School, London - UK developed and applied fuzzy logic to control a steam engine control instead of classical techniques. Also in the 70s of XX century, in Germany, Zimmermann fuzzy logic to the problem of decision theory. Based on the theory of fuzzy numbers Tagaki early 1980s, Sugeno, Kendel A. and WJ Byatt has in turn introduced the model equations under fuzzy form. In 1987, Kaleva [2] mapped introduce fuzzy, the Hukuhara derivative for fuzzy sets, the fuzzy metric space E^n most practical problems can be modeled as fuzzy differential equations (FDEs) (see [3-5]). In 90s and later, many mathematicians, for example: V. Lakshmikantham, Nieto J.,... (see[6-11]), have given model Cauchy problem for differential equations and fuzzy theory.

The method of fuzzy mapping was initially introduced by Chang and Zadeh [12]. Later, Dubois and Prade [13] presented a form of elementary fuzzy calculus based on the extension principle [14].Bede and Gal [15] suggested two defnitions for the fuzzy derivative of fuzzy functions. The first method was based on H-difference notation and was further investigated by Kaleva [2]. Several approaches were later proposed for FDEs and the existence of their solutions (e.g. [12, 13, 16-18]). There are several approaches to the study of fuzzy differential equations. One popular approach is based on H- differentiability. The approach based on H-derivative has the disadvantage that it leads to solutions which have an increasing length of their support. For some references on fuzzy equations and applications of fuzzy dynamics, we in [19], [20], [21], [22], [23], and other recent works such as the study of some topological properties and structure of the solutions to the initial valued problem for fuzzy differential systems (see [10], [24]). We know quite clearly that, in [2, 6-13, 16-18, 21-28] the authors have investigated the some properties of solutions of the local initial - valued problems for fuzzy differential equations (LIP for FDEs):

$$D_H x(t) = f(t, x(t)), \quad x(t_0) = x_0 \in E^n \tag{1.1}$$

where the symbol D_H denotes the classical Hukuhara derivative.

In [16] the authors have studied FDEs under strongly generalized differentiability of fuzzy-number-valued functions. In this case the derivative exists and the solution of a fuzzy differential equation may have decreasing length of the support, but the uniqueness is lost. Therefore, our point is that the generalization of the concept of H - differentiability can be of great help in study of local initial problems (LIP) for fuzzy differential equations under generalized Hukuhara differentiability (see [21], [23]).

In this paper, we present the structure of solutions for fuzzy nonlocal initial problems, that means the properties of solutions of NIP for FDEs under generalized Hukuhara differentiability:

$$D_H^g x(t) = f(t, x(t)), \quad t \ldots 0 \quad x(0) = x_0 + h(t_1, t_2, t_3, \ldots, t_p, x(\cdot)) \in E^n \quad \forall t_p,, t, \tag{1.2}$$

where the symbol D_H^g denotes the generalized Hukuhara derivative.

It's clear that the NIP (1.2) is very different of LIP (1.1).

The paper is organized as follows: in section 2, we recall some basic concepts and notations which are useful in next sections. In sections 3 we present the properties of set solutions $x(t) \in E^n$ to the nonlocal initial problem for fuzzy differential equations under generalized Hukuhara differentiability (NIP for FDEs). In sections 4 we present the some examples for simulation of these problems. In the last section, we give the conclusion and acknowledgements.

2. Preliminaries

Let $K_{CC}(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Given $A, B \subset K_{CC}(\mathbb{R}^n)$, the Hausdorff distance between A and B is defined as follows:

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|_{\mathbb{R}^n}, \sup_{b\in B}\inf_{a\in A}|a-b|_{\mathbb{R}^n}\},$$

where $\|.\|_{R^n}$ denotes usual the Euclidean norm in $\mathbb{R}^n$. It is known that $(K_{CC}(\mathbb{R}^n), d_H)$ is a complete metric space and if the space $(K_{CC}(\mathbb{R}^n), d_H)$ is equipped with the natural algebraic operations of addition and nonegative scalar multiplication, i.e. for $A, B \subset K_{CC}(\mathbb{R}^n$ and $\lambda \in \mathbb{R}$

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\},$$

then $K_{CC}(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

Denote $E^n = \{\omega : \mathbb{R}^n \to [0,1]$ such that ω satisfies (i) - (iv) below,

(i) ω is normal, i.e. there exists $z_0 \in \mathbb{R}^n$ such that $\omega(z_0) = 1$;

(ii) ω is fuzzy convex, i.e. $\omega(\lambda z_1 + (1-\lambda)z_2) \geq \min\{\omega(z_1), \omega(z_2)\}$, for any $0 \leq \lambda \leq 1$ and $z_1, z_2 \in \mathbb{R}^n$;

(iii) ω is upper semicontinuous;

(iv) $[\omega]^0 = cl\{z \in \mathbb{R}^n : \omega(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^n, ||\cdot||)$. The element $\omega \in E^n$ is called a fuzzy number or fuzzy set.

The set $[\omega]^\alpha = \{z \in \mathbb{R}^n : \omega(z) \geq \alpha, 0 < \alpha \leq 1\}$ is called the α - level set.

For $\omega \in E^n$ one has that $[\omega]^\alpha \in K_{cc}(\mathbb{R}^n$ for every $\alpha \in [0,1]$.

For all $0 \leq \alpha \leq \beta \leq 1$ then we have $[\omega]^\beta \subset [\omega]^\alpha \subset [\omega]^0$.

For two fuzzy sets $\omega_1, \omega_2 \in E^n$, we denote $\omega_1 \leq \omega_2$ if and only if $[\omega_1]^\alpha \subset [\omega_2]^\alpha$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function then according to Zadeh's extension principle we can extend $g : E^n \times E^n \to E^n$ by the formula $g(\omega_1, \omega_2)(z) = \sup_{z=g(z_1,z_2)} \min\{\omega_1(z_1), \omega_2(z_2)\}$. It is well known that if g is continuous then $[g(\omega_1, \omega_2)]^\alpha = g([\omega_1]^\alpha, [\omega_2]^\alpha), \forall \omega_1, \omega_2 \in E^n, \alpha \in [0,1]$. Especially for addition and scalar multiplication in fuzzy number space we have:

Define $H_0 : E^n \times E^n \to [0, \infty)$ by the expression $H_0[\omega_1, \omega_2] = \sup\{d_H([\omega_1]^\alpha, [\omega_2]^\alpha) : 0 \leq \alpha \leq 1\}$ is the distance between $\omega_1, \omega_2 \in E^n$, where $d_H([\omega]^\alpha, [\omega]^\alpha)$ is the Hausdorff distance between two set $[\omega_1]^\alpha, [\omega_2]^\alpha \subset K_{cc}(\mathbb{R}^n)$. It is easy to see H_0 is a metric in E^n. In fact (E^n, H_0) is a complete space. Some properties of metric H_0 are as follows

$H_0[\omega_1 + \omega_3, \omega_2 + \omega_3] = H_0[\omega_1, \omega_2]$;

$H_0[\lambda\omega_1, \lambda\omega_2] = |\lambda| H_0[\omega_1, \omega_2]$;

$H_0[\omega_1, \omega_2] \leq D_0[\omega_1, \omega_3] + H_0[\omega_3, \omega_2]$, for every $\omega_1, \omega_2, \omega_3 \in E^n$ and $\lambda \in \mathbb{R}$. Let us denote $\theta^n \in E^n$ the zero element of E^n as follows

$$\theta^n(z) = \begin{cases} 1 & \text{if } z = \overline{0} \\ 0 & \text{if } z \neq \overline{0} \end{cases}$$

where $\overline{0}$ is the zero element of $\mathbb{R}^n$

Let $u, v \in E^n$. The set $w \in E^n$ satisfying $w = u + v$ is known as the geometric difference of the set u and v and is denoted by the symbol u-v .

Let $x, y : [a,b] \to E^n$ be a fuzzy function, that means $[x(t)]^\alpha = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)]$ and $[y(t)]^\alpha = [\underline{y}(t,\alpha), \overline{y}(t,\alpha)], \forall \alpha \in [0,1]$. We say that

(i) scalar product $\lambda x(t)$ exists if $[\lambda x(t)]^\alpha = \lambda k, k \in [x(t)]^\alpha = \lambda[x(t)]^\alpha, \forall t \in [t_0, T], \alpha \in [0,1]$;

(ii) fuzzy product $z(t) = x(t).y(t)$ exists if

$$\underline{z}(t,\alpha) = \min\left\{\underline{x}(t,\alpha).\underline{y}(t,\alpha), \underline{x}(t,\alpha).\overline{y}(t,\alpha), \overline{x}(t,\alpha).\underline{y}(t,\alpha), \overline{x}(t,\alpha).\overline{y}(t,\alpha)\right\};$$

$$\overline{z}(t,\alpha) = \max\left\{\underline{x}(t,\alpha).\underline{y}(t,\alpha), \underline{x}(t,\alpha).\overline{y}(t,\alpha), \overline{x}(t,\alpha).\underline{y}(t,\alpha), \overline{x}(t,\alpha).\overline{y}(t,\alpha)\right\}.$$

Let $x, y : [a,b] \to E^n$ be a fuzzy function, that means $[x(t)]^\alpha = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)]$ and $[y(t)]^\alpha = [\underline{y}(t,\alpha), \overline{y}(t,\alpha)], \forall \alpha \in [0,1]$. We say that $x(t) \prec y(t)$ if and only if satisfies one of the followings:

a) $\underline{x}(t,\alpha) \geq \underline{y}(t,\alpha), \overline{x}(t,\alpha),, \overline{y}(t,\alpha), \forall t \in [t_0, T], \alpha \in [0,1];$

b) $[x(t)]^\alpha \subseteq [y(t)]^\alpha, \forall t \in [t_0, T], \alpha \in [0,1];$

c) $H_0[x(t), \theta],, \; H_0[y(t), \theta], \forall t \in [t_0, T]$

Let $x, y : [a,b] \to E^n$ be the fuzzy functions. We say that exist a geometric difference Hukuhara between x(t) and y(t), if exist the fuzzy fuction z(t), such that:

$$x(t) \,!\; y(t) = z(t) \Leftrightarrow x(t) = y(t) + z(t)$$

Let $x, y : [a,b] \to E^n$ be the fuzzy functions. We say that exist a general difference Hukuhara between x(t) and y(t), if exist the fuzzy fuction z(t), such that:

$$x(t) \ominus_{gH} y(t) = z(t) \Leftrightarrow \begin{cases} \text{a. } x(t) = y(t) + z(t) & \text{or} \\ \text{b. } y(t) = x(t) + (-1)z(t) \end{cases}$$

Let $x : [a,b] \to E^n$ be a fuzzy function, that means $[x(t)]^\alpha = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)], \forall \alpha \in [0,1]$. We say that a fuzzy function x is first type Hukuhara differentiable (classical Hukuhara differentiable) at $t_0 \in [a,b]$ if there exists an element $D_H x(t_0) \in E^n$ such that the limits exists

$$\lim_{h \to 0^+} h^{-1}\left(x\left(t_0 + h\right) \ominus x\left(t_0\right)\right) \text{ and } \lim_{h \to 0^+} h^{-1}\left(x\left(t_0\right) \ominus x\left(t_0 - h\right)\right) = D_H^g x(t) \qquad (2.1)$$

Here the limits are taken in the metric space (E^n, H_0) and at boundary points we consider only the one-side derivatives.

Let $x : [a,b] \to E^n$ and $t \in (a,b)$. We say that fuzzy function x is generalized Hukuhara differentable at t, if there exists $D_H^g x(t) \in E^n$, such that either

(H^{g1}) for all $h > 0$ sufficiently small, the H- differences, $x\left(t+h\right) \ominus x(t), x(t) \ominus x(t-h),\; D_H^g x(t) \in E^n$ exist and the limits (in the metric H_0):

$$\lim_{h\searrow 0^+} H_0\left[\frac{x(t+h)\ominus x(t)}{h}, D_H^g x(t)\right] = \lim_{h\searrow 0^+} H_0\left[\frac{x(t)\ominus x(t-h)}{h}, D_H^g x(t)\right] = 0,$$

or

(H^{g2}) for all $h > 0$ sufficiently small, the H- differences $x(t)\ominus x(t+h), x(t-h)\ominus x(t)$, $D_H^g x(t)\in E^n$ exist and the limits

$$\lim_{h\searrow 0^+} H_0\left[\frac{x(t)\ominus x(t+h)}{-h}, D_H^g x(t)\right] = \lim_{h\searrow 0^+} H_0\left[\frac{x(t-h)\ominus x(t)}{-h}, D_H^g x(t)\right] = 0$$

or

(H^{g3}) for all $h > 0$ sufficiently small, the H- differences $x(t+h)\ominus x(t), x(t)\ominus x(t-h)$, $D_H^g x(t)\in E^n$ exist and the limits

$$\lim_{h\searrow 0^+} H_0\left[\frac{x(t+h)\ominus x(t)}{h}, D_H^g x(t)\right] = \lim_{h\searrow 0^+} H_0\left[\frac{x(t-h)\ominus x(t)}{-h}, D_H^g x(t)\right] = 0$$

or

(H^{g4}) for all $h > 0$ sufficiently small, the H- differences $x(t)\ominus x(t+h), x(t-h)\ominus x(t)$, $D_H^g x(t)\in E^n$ exist and the limits

$$\lim_{h\searrow 0^+} H_0\left[\frac{x(t)\ominus x(t+h)}{-h}, D_H^g x(t)\right] = \lim_{h\searrow 0^+} H_0\left[\frac{x(t)\ominus x(t-h)}{h}, D_H^g x(t)\right] = 0$$

In this definition, case (H^{g1}) corresponds to the classic H-derivative, so this differentiability concept is a generalization of the Hukuhara derivative $D_H^g x(t)\in E^n$ and $[D_H^g x(t)]^\alpha = [\underline{x}'(t,\alpha), \overline{x}'(t,\alpha)]$. In this paper we consider only the two first generalized H- differentiabilities. In the other cases, the derivative is trivial because it is reduced to a crisp element.

3. Main Results

3.1. The Existence and Uniqueness of Solutions to the NIP

Definition 3.1 [**Nonlocal initial problems for fuzzy differential equations**] Let's conside the nonlocal initial problems for fuzzy differential equations (NIP for FDEs) under generalized Hukuhara differentiability:

$$D_H^g x(t) = f(t, x(t)),\quad t\dots 0 \quad x(0) = x_0 + h(t_1, t_2, t_3, \dots, t_p, x(\cdot)) \in E^n \quad \forall t_p,, t, \tag{3.1}$$

where $f, h : [0, T]\times E^n \to E^n$ are fuzzy continuous multifunctions, fuzzy state set $x(t)\in E^n$, $t\in[0,T], 0 < t_1 < t_2 < \dots < t_p < T, x_0 \in E^n$. The symbol $h(t_1, t_2, t_3, \dots, t_p, x(\cdot))$ is used in the sence that in the place of $(\cdot)$, such that $x(0) = x_0 + h(t_1, t_2, t_3, \dots, t_p, x(\cdot)) \in E^n$ plays as the nonlocal conditions and we can substitute only elements of the set $\{t_1, t_2, \dots, t_p\}$.

Definition 3.2 [**Fuzzy solution**] The fuzzy mapping set $x(t)\in C^1[[0,T], E^n]$ is said to be a solution of NIP for FDEs (3.1) on [0,T] if it satisfies (3.1) with generalized Hukuhara derivative $D_H^g x(t)\in E^n$ by t and it is presented by:

$$x(t) = x_0 + h(t_1, t_2, t_3, \dots, t_p, x(\cdot)) + \int_0^t f(s, x(s))ds \tag{3.2}$$

if x is (H^{g1}) - differentiable, or

$$x(t) = x_0 + h(t_1, t_2, t_3, \dots, t_p, x(\cdot)) \ominus (-1)\left(\int_0^t f(s, x(s))ds\right) \tag{3.3}$$

if x is (H^{g2}) - differentiable.

A solution of the Hukuhara integral equations (3.2) (or (3.3)) is equivalent a solution of the NIP for FDEs (3.1) on [0,T].

Assume that the fuzzy functions $f:\left[0,T\right]\subset\mathbb{R}^{+}\to E^{n},\ h:[0,T]^{p}\times E^{n}\to E^{n}$ satisfy the following hypotheses:

(**hf**) There exists a function $c_1>0$ such that:

$$H_0[f(t,x(t)),\theta^n]\leq c(1+H_0\left[x(t),\theta^n\right]),\ \forall t\in[0,T],x(t)\in E^n;$$

(**hh**) There exists a constant M>0 such that:

$$H_0[h(t_1,t_2,t_3,\dots,t_p,x(\cdot)),\theta^n]\leq M, 0<t_1<t_2<\dots<t_p\leq T, x(\cdot)\in E^n.$$

Theorem 3.1 *Let* $x_0\in E^n$ *and* $H_0[x_0,\theta^n]\leq M_1$ *with* $M_1\in\mathbb{R}^{+}$. *If the* $f(t,x(t))$ *are fuzzy continuous multifunctions and the nonlocal conditions* $h(t_1,t_2,t_3,\dots,t_p,x(\cdot))$ *satisfy the hypotheses* (**hf**) - (**hh**), *then the nonlocal nitial problems for* FDEs (3.1) *has the unique solution in* E^n. *Furthermore:*

Let the sequence $\{x_{k+1}\}:[t_0,t_0+q]\to E^n$ *given by*

$$x_{k+1}(t)=x_0+h(t_1,t_2,t_3,\dots,t_p,x_k(\cdot))+\int_0^t f\left(s,x_k\left(s\right)\right)ds, \tag{3.4}$$

is well-defined for any k = {0,1,2,..}. *Then the problem* (3.1) *has a unique fuzzy solution which is* (H^{g1}) - *differentiable on* [0,T].

Let the sequence $\{x_{k+1}\}:[t_0,t_0+q]\to E^n$ *given by*

$$x_{k+1}(t)=x_0+h(t_1,t_2,t_3,\dots,t_p,x_k(\cdot))\ominus(-1)\left(\int_0^t f\left(s,x_k\left(s\right)\right)ds\right) \tag{3.5}$$

is well-defined for any k = {0,1,2,..}. *Then the NIP* (3.1) *has a unique fuzzy solution which is* (H^{g2}) - *differentiable on* [0,T].

Proof: By inductive method, obtaining for $t\in[t_0,t_0+r]$, we infer that the sequence $x_k(t)$ is uniformly convergences to x(t).

Lemma 3.1 *Let* x(t) *fuzzy continuous multifunctions on the* $\alpha-$*level, such that*

$$\left[x(t)\right]^{\alpha}=\left[\underline{x}^{\alpha}(t),\overline{x}^{\alpha}(t)\right],\forall\alpha\in\left[0,1\right] \tag{3.6}$$

(i) *If* x (t) *is* (H^{g1}) - *differentiable then* $\left[D_H^g x(t)\right]^{\alpha}=\left[\underline{x}'^{\alpha}(t),\overline{x}'^{\alpha}(t)\right]$; (3.7)

(ii) If x (t) *is* (H^{g2}) - *differentiable then* $\left[D_H^g x(t)\right]^{\alpha}=\left[\overline{x}'^{\alpha}(t),\underline{x}'^{\alpha}(t)\right]$. (3.8)

Proof: By definitions of (H^{g1}) and (H^{g2}) of the fuzzy continuous multifunctions on the $\alpha-$level.

Theorem 3.2 *If the* x(t), f(t,x(t)) *are fuzzy continuous multifunctions on the* $\alpha-$*level and the nonlocalconditions* $h(t_1,t_2,t_3,\dots,t_p,x(\cdot))$ *satisfy the hypotheses* (**hf**), (**hh**) *then the NIP for FDEs* (3.1) *has the unique solution in* E^n. *Furthermore:*

Let system ordinary differential equations:

$$\begin{cases}\underline{x}_1^{\alpha'}(t) &= \underline{f}^{\alpha}(t,\underline{x}^{\alpha}(t),\overline{x}^{\alpha}(t))\\ \overline{x}_1^{\alpha'}(t) &= \overline{f}^{\alpha}(t,\underline{x}^{\alpha}(t),\overline{x}^{\alpha}(t))\end{cases} \tag{3.9}$$

and system ordinary differential equations:

$$\begin{cases}\underline{x}_1^{\alpha'}(t) &= \overline{f}^{\alpha}(t,\underline{x}^{\alpha}(t),\overline{x}^{\alpha}(t))\\ \overline{x}_1^{\alpha'}(t) &= \underline{f}^{\alpha}(t,\underline{x}^{\alpha}(t),\overline{x}^{\alpha}(t))\end{cases} \tag{3.10}$$

with the initial conditons $x(0) = x_0 + h(t_1, t_2, t_3, \ldots, t_p, x(\cdot)) \in E^n$ *have the unique solutions, then theproblem* (3.1) *has a unique fuzzy solution* $\left[x(t)\right]^\alpha = \left[\underline{x}^\alpha(t), \overline{x}^\alpha(t)\right]$(*for each* $\alpha \in \left[0,1\right]$), *which is generalized Hukuhara differentiable on* [0,T].

Proof: Replacing the NIP for FDEs (3.1) by two systems of ordinary differential equations (3.9) - (3.10) for the fuzzy continuous multifunctions on the $\alpha-$level, we have a unique fuzzy solution $\left[x(t)\right]^\alpha = \left[\underline{x}^\alpha(t), \overline{x}^\alpha(t)\right]$ (for each $\alpha \in \left[0,1\right]$)which is generalized Hukuhara differentiable on [0,T].

3.2.The Boundedness of Solutions to the NIP

Definition 3.3 [**The boundedness of solutions**] A fuzzy solution $x(t) \in E^n$ of NIP for FDEs (3.1) is called:

(i) B- bounded if there exists positive constant K such that

$H_0[x(t), \theta^n] \le K$ for all $t \in [0, T]$.

(ii) EB- bounded if there exists positive constant K such that

$H_0[x(t), \theta^n] \le K.e^{(-\beta.t)}, \forall t \in [0, T]$

Lemma 3.3 [**The extension of Gronwall - Bellman inequality**] *Assume that the real functions* $r(t) > 0,\ a > 0,\ b > 0$ *on* $\left[0, T\right]$ *satisfy* $r(t) \le a + b\int_0^t r(s)ds$ *then* $r(t) \le a.\exp(bt)$.

Theorem 3.3 *Let* $x_0 \in E^n$ *and* $H_0[x_0, \theta^n] \le M_1$ *with* $M_1 \in \mathbb{R}^+$ *If the fuzzy function* f(t,x(t)) *satisfies the hypotheses* (**hf**) *and the nonlocal conditions* $h(t_1, t_2, t_3, \ldots, t_p, x(\cdot))$ *satisfy the hypotheses* (**hh**), *then the NIP for FDEs* (3.1) *has the unique B- bounded solution in* E^n.

Proof: (a) Problems of existence and uniqueness are clear.

(b) Problem of (B)- bounded are proved by integral expression (3.2) following

$$H_0\left[x\left(t\right), \theta^n\right] = H_0\left[x_0 + h(t_1, t_2, t_3, \ldots, t_p, x_k(\cdot)) + \int_0^t f\left(s, x\left(s\right)\right)ds, \theta^n\right].$$

By assumptions (**hf**), by Lemma 3.1, and by hypotheses (**hf**) – (**hh**) we obtain

$$H_0\left[x\left(t\right), \theta^n\right] \le H_0\left[x_0, \theta^n\right] + M + l\int_0^t \left(1 + H_0\left[x\left(s\right), \theta^n\right]\right)ds$$

$$\le H_0\left[x_0, \theta^n\right] + M + lt + l\int_0^t H_0\left[x\left(s\right), \theta^n\right]ds$$

Putting $r(t) = H_0\left[x\left(t\right), \theta^n\right], a = H_0\left[x_0, \theta^n\right] + M + lT, b = l$ and by Lemma 3.3, we obtain

$$r(t) = H_0\left[x\left(t\right), \theta^n\right] \le \left(H_0\left[x_0, \theta^n\right] + M + lT)\right)\exp\left(lT\right)$$

Choosing $K = \left(H_0\left[x_0, \theta^n\right] + M + lT)\right)\exp\left(lT\right)$, we have $\sup\limits_{t\in[0,T]} H_0\left[x(t), \theta^n\right] \le K, \forall t \in [0, T]$ and the proof of Theorem 3.3 is completed.

Theorem 3.4. *Assume that the positive Lyapunov - like function* $V \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$*which satisfies the following conditions:*

(i) $|V(t, x(t)) - V(t, \overline{x}(t))| \le L(H_0[x(t), \overline{x}(t)])$ *where*L *is bounded Lipschitz constant, for all* $x(t), \overline{x}(t) \in E^n, b\left(t, H_0[x(t), \theta^n] \le V(t, x(t)) \le a\left(t, H_0[x(t), \theta^n]\right), (t, x) \in \mathbb{R}_+ \times S^c(r)\right.$ *where* b(.), a(t, .) *are increasing functions;*

(ii) $D^+V(t, x(t)) \equiv \lim\limits_{\tau\to 0^+} \sup \frac{1}{\tau}\left\{V\left(t + \tau, x(t) + \tau f(t, x(t))\right) - V(t, x(t))\right\} \le g(t, V(t, x(t)))$, *where* $g \in C[\mathbb{R}_+^2, \mathbb{R}], g(t,0) = 0, \forall t \in \mathbb{R}_+$.

a/ If $g(t, V(t, x(t))) \le 0, \forall t \ge t_0$ *then fuzzy set solution* $x(t) \in E^n$ *of NIP for FDEs* (3.1) *is (B).*

b/ If $g(t, V(t, x(t))) < 0, \forall t \ge t_0$ *or if* $g(t, V(t, x(t))) < -\beta V, \forall t \ge t_0$ *then fuzzy solution* $x(t) \in E^n$ *of NIP for FDEs* (3.1) *is (EB).*

Proof: Setting the function $m(t) = V(t, x(t))$, we have

$$D^+m(t) = D^+V(t, x(t)) \equiv \lim_{\tau \to 0^+} \sup \frac{1}{\tau}\left\{V\big(t+\tau, x(t) + \tau f(t, x(t))\big) - V(t, x(t))\right\} \le g(t, V(t, x(t))),$$

so $D^+m(t) \le g(t, m(t))$, implies that $m(t_0) \le W_0$.

Since $m(t) \le r(t_0, W_0, t)$ where $r(t_0, W_0, t)$ is maximal solution of scarlarequation $\frac{dW}{dt} = g(t, W)$, then $V(t, x(t)) \le V(t_0, x_0)$.

Let $0 < \varepsilon < r, t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(t_0, \varepsilon)$ such that $a(t_0, \delta) < b(\varepsilon)$. We claim that with this δ then (B)- bounded solution. If not, there exists solution $x(t) = x(t_0, x_0, t)$ of the NIP for FDEs (3.1) and $t_1 > t_0$ such that $H_0[u(t_1), \theta^n] = \varepsilon$ and $H_0[x(t), \theta^n] \le \varepsilon < r, t_0 < t < t_1$.

Wherever $H_0[x_0, \theta^n] < \delta$ because $V(t, x(t)) \le V(t_0, x_0), t_0 < t < t_1$, then

$$b(\varepsilon) = b(H_0[x(t_1), \theta^n]) \le V(t_1, x(t_1)) \le V(t_0, x_0) \le a(t_0, H_0[x_0, \theta^n]) \le a(t_0, \delta) < b(\varepsilon)$$

this contradiction proves that the fuzzy set solution $x(t) \in E^n$ is B- bounded.

In the case, if $g(t, V(t, x(t))) < 0$ (or $D^+V(t, x(t)) < -\beta V(t, x(t))$) then we have $V(t, x(t)) \le V(t_0, x_0), \forall t \ge t_0$ and the fuzzy set solution is (B).

We need prove that $\lim_{t \to +\infty} H_0[x(t), \theta^n] = 0$. We consider $D^+V(t, x(t)) < -\beta V(t, x(t))$ then $V(t, x(t)) \le V(t_0, x_0).e^{[-\beta(t-t_0)]}, \forall t \ge t_0$. If (BE) is not true, given ε_0, we choose

$$T = T(t_0, \varepsilon_0) = \frac{1}{\beta} \ln \frac{a(t_0, \delta)}{b(\varepsilon_0)} + 1 \text{ then}$$

$$b(H_0[x(t), \theta^n]) \le V(t, x(t)) \le a(t_0, \delta).e^{[-\beta(t-t_0)]} \le b(\varepsilon), \forall t \ge t_0 + T$$

this contradiction proves that the fuzzy set solution $x(t) \in E^n$ is EB- bounded.

3.3. The Comparisons of Solutions to the NIPs

We consider nonlocal initial problems – NIP for two FDEs type (3.1):

$$D_H^g x(t) = f_1(t, x(t)), \quad t \ldots 0 \quad x(0) = x_0 + h(t_1, t_2, t_3, \ldots, t_p, x(\cdot)) \in E^n \quad \forall t_p,, t, \qquad (3.11)$$

$$D_H^g x(t) = f_2(t, x(t)), \quad t \ldots 0 \quad x(0) = x_0 + h(t_1, t_2, t_3, \ldots, t_p, x(\cdot)) \in E^n \quad \forall t_p,, t, \qquad (3.12)$$

where $f_j : [0, T] \times E^n \to E^n$ is fuzzy continuous multifunctions, fuzzy state set $x(t), y(t) \in E^n$, $h : [0, T]^p \times E^n \to E^n$ is fuzzy continuous multifunctions and $t \in [0, T]\ 0 < t_1 < t_2 < \ldots < t_p < t \le T, x_0 \in E^n$

The symbol $h(t_1, t_2, t_3, \ldots, t_p, x(\cdot))$ is used in the sence that in the place of $(\cdot)$ we can substitute only elements of the set $\{t_1, t_2, \ldots, t_p\}, j = 1, 2$.

Theorem 3.5 *Assume that,* $H_0[x(0), y(0)] \le \delta_0$ *and for* $(t, x) \in \mathbb{R}_+ \times S^c(r)$, *the fuzyy mappings* f_j *satisfy the following conditions:*

$H_0\left[\int_{t_0}^{t} f_1(s, x(s))ds, \int_{t_0}^{t} f_2(s, y(s))ds\right] \le L\eta\int_{t_0}^{t} H_0[x(s), y(s)]ds$ *where* L *is bounded constant, for all* $x(t), y(t) \in E^n$ *and* $t \in \mathbb{R}_+$, *then we have the following estimation*:

$$H_0[x(t), y(t)] \le (\delta_0 + \eta).\exp[L\eta(t - t_0)] \qquad (3.13)$$

Proof: Proof of the Theorem 3.5 by ussing the Gronwall- Bellman's Lemma for estimation of supper distance H_0.

3.4. The Global Existence and Uniqueness of Solutions to the NIP

Next, we shall establish the global existence and uniqueness of solutions to the NIP for FDEs (3.1).

Theorem 3.6 *Assume that the assumptions of Theorem3.4 hold. In addition, assume ellse the fuzzyfunction* $f \in C(\mathbb{R}^+ \times E^n, E^n)$ *satisfy that* $H[f(t, x), f(t, y)] \le g(t, H_\sigma[x, y])$ *for* $x(t), y(t) \in E^n$ *and* $w(t) \equiv 0$ *is only solution of*

$$\frac{dw}{dt} = g(t, w), w(0) = 0 \qquad (3.14)$$

for $t \ge 0$. *Then the* NIP for FDEs (3.1) *has a unique solution on* $[0, \infty)$ *for each* $H^{gj}, j = 1, 2$ *case.*

Proof: We prove that for the case of (H^{g2})- differentiability, the proof of the other case is similar. Since $x(t), y(t) \in E^n$ aresolutions to the NIP for FDEs (3.1), we have: for h > 0, small enough, there exist the Hukuharadifference $x(t-h) \ominus x(t), y(t-h) \ominus y(t)$. Now for $t \in \mathbb{R}^+$, setting $m(t) = H_0[x(t), y(t)]$ we have:

$$m(t-h) - m(t) = H_0[x(t-h), y(t-h)] - H_0[x(t), y(t)]$$
$$\le H_0[x(t-h), x(t) + (-1)hf(t, x(t))] + H_0[x(t) + (-1)hf(t, x(t)), y(t) + (-1)hf(t, y(t))]$$
$$\le H_0[(t-h), x(t) + (-1)hf(t, x(t))] + H_0[y(t) + (-1)hf(t, y(t)), y(t-h)]$$
$$+ hH_0[f(t, x(t)), f(t, y(t))]$$

from which we get

$$\frac{m(t-h) - m(t)}{h} \le H_0\left[\frac{x(t-h) \ominus x(t)}{-h}, f(t, x(t))\right] + H_0\left[f(t, y(t)), \frac{y(t-h) \ominus y(t)}{-h}\right]$$
$$+ H_0[f(t, x(t)), f(t, y(t))]$$

Taking liminf as $h \to 0^+$ yields

$$D^- m(t) = \liminf_{h \to 0^+} \frac{1}{h}\left[m(t-h) - m(t)\right] \le g(t, H_0[x, y]) = g(t, |m_t|_\sigma)$$

which together with the fact that $H_0[\varphi_0, \psi_0] \le x_0$ and by using Theorem 3.3 and Theorem 3.4 we obtain $H_0[x(t), y(t)] \le r(t, t_0, x_0), \quad t \ge t_0$. The proof is complete.

Corollary 3.1. *Under assumptions of Theorem3.4, if we suppose in addition that there exists* $L > 0$ *such that* $H_0[f(t, x), f(t, y)] \le L.H_0[x, y]$, *then for* $t \ge 0$ *the NIP for FDEs* (3.1) *has a unique solution on* $[0, \infty)$. for each $H^{gj}, j = 1, 2$.

3.5. The Stability Properties of Trivial Solutions to the NIP

Assume that NIP for FDEs (3.1) has the trivial set solution $v(t)$ thatmeans $f(t, v(t)) = \theta^n$. Put $S(r) = \{u(t) \in E^n : H_0[u(t), \theta^n] < r\}$ - neighbourhood of the zero set point.

Definition 3.4 [**Stability of solutions by Lyapunov's mean**] The trivial set solution θ^n of NIP for FDEs (3.1) is said to be:

(LS) stable by Lyapunov's mean if for each $\varepsilon > 0$ and t > 0, there exists a $\delta = \delta(0,\varepsilon)$ such that $H_0[x_0,\theta^n] < \delta$ implies $H_0[x(t),\theta^n] < \varepsilon$ for $t \geq 0$.

(ALS) asymptotically stable by Lyapunov's mean if it is stable and $\lim_{t\to+\infty} H_0[x(t),\theta^n] = 0$.

(ELS) exponentially stable by Lyapunov's mean if $H_0[x(t),\theta^n] \leq \beta(H_0[x_0,\theta^n],0)\exp[-\alpha(t)], t > 0$.

where $\beta(H_0[.,.],0) : [0,1]\times\mathbb{R}_+ \to \mathbb{R}_+$

Theorem 3.7 *Assume that the positive Lyapunov - like function* $V \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$ *which satisfies the following conditions:*

i) $|V(t,x(t)) - V(t,\bar{x}(t))| \leq L.H_0[x(t),\bar{x}(t)]$ *where* L *is bounded Lipschitz constant, for all* $x(t),\bar{x}(t) \in E^n$ *and* $t \in \mathbb{R}_+$;

ii) $b\big(H_0[x(t),\theta^n]\big) \leq V(t,x(t)) \leq a\big(t, H_0[x(t),\theta^n]\big)$ *for* $(t,x) \in \mathbb{R}_+ \times S(r)$ *where* $b(.), a(t,.)$ *are increassing functions;*

iii) $D^+V(t,x(t)) \equiv \limsup_{\tau\to 0^+} \frac{1}{\tau}\left\{V\big(t+\tau, x(t)+\tau f(t,x(t))\big) - V(t,x(t))\right\} \leq g(t,V(t,x(t)))$

where $g \in C\big[\mathbb{R}_+^2, \mathbb{R}\big]$, $g(t,0) = 0$ *for all* $x(t) \in E^n$ *and* $t \in \mathbb{R}_+$

a/ *If* $g(t,V(t,x)) \leq 0, \forall t \geq 0$ *then a trivial set solution of NIP for FDEs* (3.1) *is* (LS).

b/ *If* $g(t,V(t,x(t))) < 0, \forall t \geq 0$ *(or if* $g(t,V(t,x(t))) < -\beta V, \forall t \geq 0$ *then atrivial set solution of NIP for FDEs* (\ref{eq3.1}) *is* (ALS).

Proof: Setting the function $m(t) = V(t, x(t))$, we have

$$D^+m(t) = D^+V(t,x(t)) \equiv \limsup_{\tau\to 0^+} \frac{1}{\tau}\left\{V\big(t+\tau, x(t)+\tau f(t,x(t))\big) - V(t,x(t))\right\} \leq g(t,V(t,x(t))),$$

so $D^+m(t) \leq g(t,m(t))$ implies that $m(0) \leq W_0$. Since $m(t) \leq r(0,W_0,t)$ where $r(0,W_0,t)$ is maximal solution of scarlarequation $\frac{dW}{dt} = g(t,W)$, then $V(t,x(t)) \leq V(0,x_0)$.

Let $0 < \varepsilon < r, 0 \in \mathbb{R}_+$ be given. Choose a $\delta = \delta(0,\varepsilon)$ such that $a(0,\delta) < b(\varepsilon)$. We claim that with this δ then (LS) holds. If not, there is exists $x(t) = x(0,x_0,t)$ of NIP (3.1) and $t_1 > 0$ such that $H_0[x(t_1),\theta^n] = \varepsilon$ and $H_0[x(t),\theta^n] \leq \varepsilon < r, 0 < t < t_1$.Wherenever $H_0[x_0,\theta^n] < \delta$, because $V(t,x(t)) \leq V(0,x_0), 0 < t < t_1$ then

$$b(\varepsilon) = b(H_0[x(t_1),\theta^n]) \leq V(t_1,x(t_1)) \leq V(0,x_0) \leq a(0,H_0[x_0,\theta^n]) \leq a(0,\delta) < b(\varepsilon),$$

this contradiction proves that (LS) holds.

In the case, if $g(t,V(t,x(t))) < 0$ (or $D^+V(t,x(t)) < -\beta V(t,x(t))$) then we have $V(t,x(t)) \leq V(0,x_0), \forall t \geq 0$ and the trivial set solution is (LS). We need prove that

$$\lim_{t\to+\infty} H_0[x(t),\theta^n] = 0$$

We consider $D^+V(t,x(t)) < -\beta V(t,x(t))$ then $V(t,x(t)) \leq V(0,x_0).e^{[-\beta(t)]}, \forall t \geq 0.$

If (ALS) is not holded, given ε_0, we choose $T = T(0,\varepsilon_0) = \frac{1}{\beta}\ln\frac{a(0,\delta)}{b(\varepsilon_0)} + 1$ then

$b(H_0[x(t),\theta^n]) \le V(t,x(t)) \le a(0,\delta).e^{[-\beta(t)]} \le b(\varepsilon),\ \forall t \ge T$,

this contradiction proves that (ALS) holds.

Theorem 3.8: *Assume that the positive Lyapunov - like function* $V \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$ *satisfies thefollowings:*

(i) $|V(t,\bar{x}(t)) - V(t,x(t))| \le L.H_0[\bar{x}(t),x(t)])$ *where* L *is bounded Lipschizt constant, for all* $\bar{x}(t), x(t) \in E^n$ *and* $t \in \mathbb{R}_+$;

(ii) $\exists \lambda_1(t), \lambda_2(t), \lambda_3(t), p, q > 0$ *where* $\lambda_1(t)$ *increassing function such that* $\lambda_1(t)H_0[x(t),\theta^n]^p \le V(t,x(t)) \le \lambda_2(t)H_0[x(t),\theta^n]^q$;

(iii) $D^+V(t,x(t)) \le -\lambda_3 H_0[x(t),\theta^n]^s + K.e^{(-\delta t)},\ \forall t > 0,\ x(t) \in E^n\ ,\ \{\theta^n\}$;

(iv) $\delta > \inf_{t\in\mathbb{R}_+}\{\frac{\lambda_3(t)}{[\lambda_2(t)]^{l/q}}\} > 0$;

(v) $V(t,x(t)) - [V(t,x(t))]^{l/q},, \gamma e^{-\delta t}$ *where* $s, K, l, \gamma, \delta > 0$,

then a trivial set solution of NIP for FDEs (3.1) *is* (ELS).

Definition 3.5 [**Stability of solutions**] The trivial fuzzy solution of the NIP for FDEs (3.1) is said to be:

(S1)] equi-stable of for each $\varepsilon > 0$ and $t_0 \ge 0$ there exists a $\delta = \delta(t_0,\varepsilon)$ such that $H_0[x_0,\theta^n] < \delta$ implies $H_0[x(t),\theta^n] < \varepsilon$ for $t \ge t_0$;

(S2) uniformly stable, if the δ in (S1) is independent of t_0;

(S3) quasi-equi-asymptotically stable, if for each $\varepsilon > 0, t_0 > 0$ there exist a $T = T(t_0,\varepsilon)$ and $\delta_0 = \delta_0(t_0)$ such that $H_0[x_0,\theta^n] < \delta_0$ implies $H_0[x(t),\theta^n] < \varepsilon,\ \forall t > t_0 + T$;

(S4) quasi-uniformly asymptotically stable, if $\delta_0 = \delta_0(t_0)$ and T in (S3) are independent of t_0;

(S5) equi-asymptotically stable, if (S1) and (S3) hold simultaneously;

(S6) uniformly asymptotically stable, if (S2) and (S4) hold simultaneously;

(S7) exponentially asymptotically stable, if $H_0[x(t),\theta^n] \le \beta(H_0[x_0,\theta^n],0)\exp[-\gamma.t)],\ t > 0$ where $\beta(H_0[.,.],0) : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$.

Remark 3.1: According to the Definition 3.4 and Definition 3.5, we have:

(S1) $\Leftrightarrow$ (LS).
(S6) $\Leftrightarrow$ (ALS).
(S7) $\Leftrightarrow$ (ELS).
(S6) or (ALS) $\Rightarrow$ (S3).
(S6) or (ALS) $\Rightarrow$ (S4).
We have to prove (S2) and (S6).

Theorem 3.9: *Assume that the positive Lyapunov - like function* $V \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$ *satisfies the followings:*

(i) $|V(t,\bar{x}(t)) - V(t,x(t))| \le L.H_0[\bar{x}(t),x(t)]$ *where* L *is bounded Lipschitz constant, for all* $t \in \mathbb{R}_+$

(ii) $b\big(H_0[x(t),\theta^n]\big) \le V(t,x(t)) \le a\big(t,H_0[x(t),\theta^n]\big)$, *for* $(t,x(t)) \in \mathbb{R}_+ \times S(r)$ *where* $b(.), a(t,.)$ *are increasing functions;*

(iii) $D^+V(t,x(t)) \equiv \limsup_{\tau\to 0^+} \frac{1}{\tau}\{V\big(t+\tau, x(t)+\tau f(t,x(t)) + \tau[\lambda(t)f(t,x(t)) + \lambda'(t)x(t)]\big) - V(t,x(t))\}$
$\le g(t,V(t,u(t)))$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}],\ g(t,0) = 0,\ \forall x(t) \in E^n$ *and* $t \in \mathbb{R}_+$ *Further more*

a/ *If* $g(t,V(t,x(t))) \le 0, \forall t \ge t_0$ *then* (S2) *holds.*

b/ *If* $g(t,V(t,x(t))) < 0, \forall t \ge t_0$ (*or if* $g(t,V(t,x(t))) < -c(H_0[x(t),\theta^n]),\ \forall t \ge t_0$) *then* (S6) *holds.*

Proof: The condition (iii) with a/ (or b/) guarantees that $V(t, x(t))) \le V(0, x(t_0))), \forall t \ge t_0$.

a/ Let $\forall\varepsilon > 0, \forall t \ge t_0$. Choose $\delta = \delta(\varepsilon)$,such that $a(t_0, \delta) < b(\varepsilon)$ and $H_0[x(t_0), \theta^n] \le \delta$, implies that $H_0[x(t), \theta^n] < \varepsilon$ that means (S2) holds.

If it is not true, then $\forall\delta > 0, \exists\varepsilon_0(\delta) > \delta$ and $a(t_0, \delta) < b(\varepsilon)$ such that $x(t) = x(t_0, x_0, t)$ is a set solution of (3.1), which satisfies $H_0[x(t), \theta^n] \ge \varepsilon_0$. On the other hand, we have

$$b(\delta) \le b(\varepsilon_0) = b\big(H_0[x(t), \theta^n]\big) \le b\big(H_0[x(t), \theta^n]\big) \le V(t, x(t)) \le$$
$$\le V(t_0, x_0) \le a\big(t_0, (H_0[x(t_0), \theta^n] \le a\big(t_0, H_0[x(t_0), \theta^n]\big) = a(0, \delta) \le b(\delta)$$

This contradiction proves (S2).

b/ If $g(t, V(t, x(t))) < -c(H_0[x(t), \theta^n]), \forall t \ge t_0$ we have (S2), that means $\forall\varepsilon > 0, \exists\delta(\varepsilon), \forall t \ge t_0 : H_0[x(t_0), \theta^n] \le \delta$ implies that $H_0[x(t), \theta^n] \le \varepsilon, \forall t \ge t_0$.

Suppose that (S6) doesn't hold, that means $\forall\delta > 0, \exists\varepsilon(\delta) = \varepsilon_0 > \delta, \exists T = 1 + \dfrac{a(t_0, \delta)}{c.\varepsilon_0}$

such that $H_0[x(t), \theta^n] \ge \varepsilon_0, \forall t \in [t_0, T]$. Since $g(t, V(t, x(t))) < -c(H_0[x(t), \theta^n]), \forall t \ge t_0$ then

$$V(t, x(t)) \le V(t_0, x(t_0)) - c\int_{t_0}^{t} H_0[x(t), \theta^n]dt, \forall t \in [t_0, T] \text{ Implies}$$

$0 \le V(T, x(T)) \le a(t_0, \delta) - c.\varepsilon_0.T < 0$ This contradiction proves (S6).

4. Illustrations

4.1 Example

We have an example for the change of oxygen concentration in water by classical equations:

$$x'(t) = -Kx(t),\ x(0) = x_0,\quad t \in [0, 100] \tag{4.1}$$

where, x (t)- oxygen levels of concentration by mg / liter; K- Solubility coefficient under certain cut, usually taken 0.038mg/day; x (0) - oxygen concentration - local initial condition.

In [24] the authors repeat (4.1) by the model of LIP for FDEs:

$$D_H x_\alpha(t) = -Kx_\alpha(t),\ x(0) = [90 + \alpha, 110 - \alpha],\ t \in [0, 100] \tag{4.2}$$

where $x_\alpha(t) = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$.

Indeed, measurements of the levels x (t) is fuzzy, that means $x_\alpha(t) = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$ because it depends on many factors: humidity, wind, traffic flow and accuracy of measuring equipment: commonly used measurement time is 100 days.

Let us consider example (4.2) by the following nonlocal initial problem for the fuzzy differential equation (NIP for FDEs):

$$D_H^g x(t) = -Kx(t) \tag{4.3}$$

$$x(0) = x_\alpha(0) + \frac{1}{2}x_\alpha(20) + \frac{1}{2^2}x_\alpha(40) + \ldots + \frac{1}{2^p}x_\alpha(t_p) \tag{4.4}$$

$$\forall t_p < t, x_\alpha(0) = [-1 + \alpha, 1 - \alpha], \alpha \in [0, 1], t \in [0, T] \tag{4.5}$$

x (T) - a final concentration oxygen levels achieved 0.625mg / liter.

Case 1. Suppose that x(t) in initial problem for level fuzzy differential equation (4.3) with nonlocal conditions (4.4) - (4.5) is first type Hukuhara differentiable (H^{g1}). Because x is (H^{g1}) - differentiable, then

$$x(t) = x_\alpha(0) + \frac{1}{2}x_\alpha(20) + \frac{1}{2^2}x_\alpha(40) + \ldots + \frac{1}{2^p}x_\alpha(t_p) + \int_0^t f(s, x(s))ds, \forall t > t_p. \tag{4.6}$$

On the other hands, by Lemma 3.2, we get $[x_1'(t)]^\alpha = -K[(\underline{x}_1^\alpha(t)), (\overline{x}_1^\alpha(t))]$ that means:

$$\begin{cases} \underline{x}_1'^{\alpha}(t) & = -K\underline{x}_1^{\alpha}(t) \\ \overline{x}_1'^{\alpha}(t) & = -K\overline{x}_1^{\alpha}(t) \end{cases}$$

We have the fuzzy solution of nonlocal initial problem - NIP (4.3) - (4.5) under first type of Hukuhara differentiable (H^{g1}): $[x(t)]^{\alpha} = [\underline{x}^{\alpha}(0)e^{-Kt}, \overline{x}^{\alpha}(0)e^{-Kt}]$. Finally, we have a solution that is (H^{g1})-differentiable

$$x(t) = [x_{\alpha}(0) + \frac{1}{2}x_{\alpha}(20) + \frac{1}{2^2}x_{\alpha}(40) + ... + \frac{1}{2^p}x_{\alpha}(t_p)]e^{-Kt}, \forall t > t_p \tag{4.7}$$

$$x(0) = x_{\alpha}(0) + \frac{1}{2}x_{\alpha}(20) + \frac{1}{2^2}x_{\alpha}(40) + ... + \frac{1}{2^p}x_{\alpha}(t_p), \forall t_p < t,$$

$$x_{\alpha}(0) = [-1+\alpha, 1-\alpha], \alpha \in [0,1], t \in [0,T],$$ x(T) = 0.625mg/liter.

Case 2. Suppose that x(t) in nonlocal initial - value problem for level fuzzy differential equation (4.3) - (4.3) is second type Hukuhara differentiable (H^{g2}). Because x is (H^{g2}) - differentiable, then

$$x(t) = x_0 + h(t_1, t_2, t_3, ..., t_p, x(\cdot)) ! \ (-1)\left(\int_0^t f(s, x(s))ds\right),$$ that means

$$x(t) = x_{\alpha}(0) + \frac{1}{2}x_{\alpha}(20) + \frac{1}{2^2}x_{\alpha}(40) + ... + \frac{1}{2^p}x_{\alpha}(t_p) ! \ (-1)\left(\int_0^t f(s, x(s))ds\right), \forall t > t_p \tag{4.8}$$

$$x(0) = x_{\alpha}(0) + \frac{1}{2}x_{\alpha}(20) + \frac{1}{2^2}x_{\alpha}(40) + ... + \frac{1}{2^p}x_{\alpha}(t_p), \forall t > t_p,$$

$$x_{\alpha}(0) = [-1+\alpha, 1-\alpha], \alpha \in [0,1], t \in [0,T],$$ x(T) = 0.325mg/liter.

Therefore we have the fuzzy solution of nonlocal initial problem - NIP (4.3) - (4.5) under second type of Hukuhara differentiable (H^{g2})

$$[x_2(t)]^{\alpha} = [-K(-1+\alpha)\underline{x}(0)\cos t, -K(1-\alpha)\overline{x}(0)\sin t], x(0) = [\underline{x}(0), \overline{x}(0)], \tag{4.9}$$

$$x(0) = x_{\alpha}(0) + \frac{1}{2}x_{\alpha}(20) + \frac{1}{2^2}x_{\alpha}(40) + ... + \frac{1}{2^p}x_{\alpha}(t_p), \forall t > t_p, x_{\alpha}(0) = [-1+\alpha, 1-\alpha], \alpha \in [0,1], t \in [0,T]$$

Remark 4.1: The fuzzy solution of nonlocal initial problem - NIP (4.3) - (4.5) exists in only case 1, that means solution in form (4.7) because the change of oxygen concentration in water is increased with time.

We have this numerical simulation solution, when K = 0.038, $\alpha = 0.5$

$$x(0) = [-0.5; +0.5] = x_0(0)$$

$$x_{0.5}(20) = x_0(0)e^{-20K} = [-0.5; +0.5]e^{-20K}$$

$$x(0) = [-0.5; +0.5] + \frac{1}{2}[-0.5; +0.5]e^{-20K} = x_1(0)$$

$$x_{0.5}(40) = x(0)e^{-40K} = x_1(0)e^{-40K}$$

$$x(0) = [-0.5; +0.5] + \frac{1}{2}x_{0.5}(20) + \frac{1}{2^2}x_{0.5}(40) = x_2(0)$$

$$x_{0.5}(60) = x_2(0)e^{-60K} = x_2(0)e^{-60K}$$

$$x(0) = x_{0.5}(0) + \frac{1}{2}x_{0.5}(20) + \frac{1}{2^2}x_{0.5}(40) + \frac{1}{2^3}x_{0.5}(60) = x_3(0) \ ......$$

$$x(80) = x_{0.5}(0) + \frac{1}{2}x_{0.5}(20) + \frac{1}{2^2}x_{0.5}(40) + \frac{1}{2^3}x_{0.5}(60) + \frac{1}{2^5}x_{0.5}(80)$$

x(T)=0.625mg/liter

We have this numerical simulation solution, when K= 0.038, $\alpha = 0.5$ shown in Fig.1, and when K = 0.038, $\alpha = 0.75$ that shown in Fig.2

Fig-1. Solution of Example (4.3) - (4.5) in the case 1 when $\alpha = 0.5$

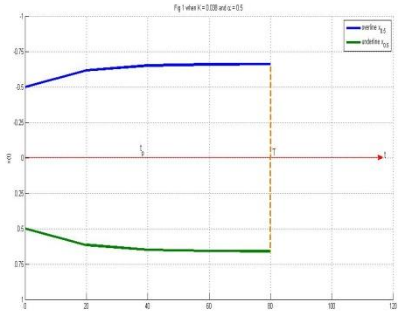
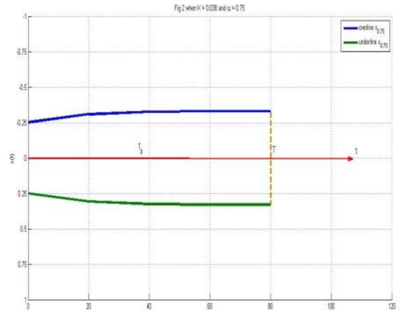


Fig-2.Solution of Example (4.3) - (4.5) in the case 1 when $\alpha = 0.75$.



We have this numerical simulation solution, when K = 0.038, $\alpha = 0.5$ that shown in Fig.3, and when when K = 0.038, $\alpha = 0.75$ shown in Fig.4(in 3D).

Fig-3.Solution of Example (4.3) - (4.5) in the case 1 when $\alpha = 0.5$

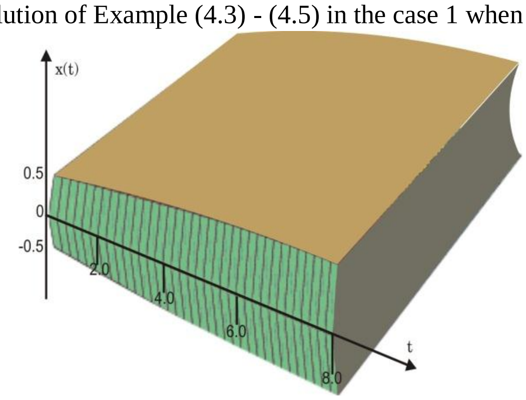
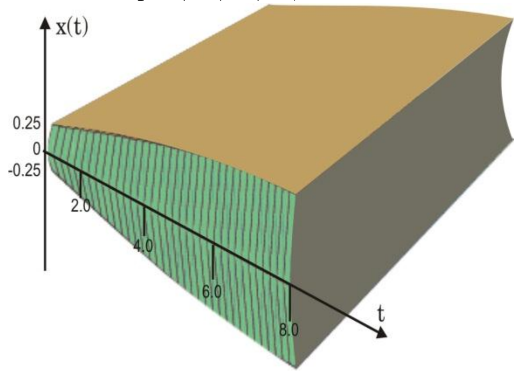


Fig-4. Solution of Example (4.3) - (4.5) in the case 1 when $\alpha = 0.75$.



4.2 Example

We have an example for the change of chemical toxic concentrations in the air, that depends continuously on each measurement point. Let us consider example (4.1) by the following nonlocal initial problem for the fuzzy differential equation (NIP for FDEs):

$$D_H^g x(t) = -\gamma x(t) \tag{4.10}$$

$$x(0) = x_\alpha(0) - \frac{1}{2}x_\alpha(10) - \frac{1}{2^2}x_\alpha(20) - \ldots - \frac{1}{2^p}x_\alpha(t_p), t_p < t \tag{4.11}$$

$$x_\alpha(0) = [-1+\alpha, 1-\alpha], \alpha \in [0,1], t \in [0,T], \text{ with } \gamma = 0.0025 \text{ mg/day, } x(T) = 0.005 \text{ mg/ } m^3 \tag{4.12}$$

Because $x(t) \in E^1$ is α - level fuzzy such that according Lemma 3.2: $[D_H^g x(t)]^\alpha = [\underline{x}'^\alpha(t), \overline{x}'^\alpha(t)]$, and by two types of Hukuhara derivative (H^{g1}) and (H^{g2}) ,then the level fuzzy differential equation (4.10) is similar the followings:

Case 1. Suppose that x(t) in nonlocal initial problem - NIP (4.10) - (4.12) is first type of Hukuhara differentiable (H^{g1}),by Lemma 3.2, we get $[x_1'(t)]^\alpha = -\gamma[(\underline{x}_1^\alpha(t))', (\overline{x}_1^\alpha(t))']$, that means:

$$\begin{cases} \underline{x}_1'^\alpha(t) & = -\gamma \underline{x}_1^\alpha(t) \\ \overline{x}_1'^\alpha(t) & = -\gamma \overline{x}_1^\alpha(t) \end{cases}$$

We have the fuzzy solution of nonlocal initial problem - NIP (4.10) - (4.12) under first type of Hukuharadifferentiable (H^{g1}): $[x_1(t)]^\alpha = [(-1+\alpha)e^{-\gamma t}, (1-\alpha)e^{-\gamma t}]$

$$x(0) = x_\alpha(0) - \frac{1}{2}x_\alpha(10) - \frac{1}{2^2}x_\alpha(20) - \ldots - \frac{1}{2^p}x_\alpha(t_p), t_p < t$$

$$x_\alpha(0) = [-1+\alpha, 1-\alpha], \alpha \in [0,1], t \in [0,T], \gamma = 0.012 mg/\ m^3, x(T) = 0.005 mg/\ m^3$$

Case 2. Suppose that x(t) in nonlocal initial problem - NIP (4.10) - (4.12) is second type of Hukuhara differentiable (H^{g2}), and by Lemma 3.2, we get $\begin{cases} \underline{x}_2'^\alpha(t) & = -\gamma \overline{x}_2^\alpha(t) \\ \overline{x}_2'^\alpha(t) & = -\gamma \underline{x}_2^\alpha(t) \end{cases}$

Therefore we have the fuzzy solution of nonlocal initial problem - NIP (4.10) - (4.13) under second type of Hukuharadifferentiable (H^{g2}):

$$[x(t)]^\alpha = [-\gamma(-1+\alpha)\underline{x}(0)\cos t, -\gamma(1-\alpha)\overline{x}(0)\sin t], x(0) = [\underline{x}(0), \overline{x}(0)], \tag{4.13}$$

$$x(0) = x_\alpha(0) - \frac{1}{2}x_\alpha(10) - \frac{1}{2^2}x_\alpha(20) - \ldots - \frac{1}{2^p}x_\alpha(t_p), t_p < t, \tag{4.14}$$

$$x_\alpha(0) = [-1+\alpha, 1-\alpha], \alpha \in [0,1], t \in [0,T], \gamma = 0.0025 mg/\ day, x(T) = 0.005 mg/\ m^3.$$

Remark 4.2: The fuzzy solution of nonlocal initial problem - NIP (4.10) - (4.12) exists in only case 2, that means solution in form (4.13) - (4.14) because the change of oxygen concentration in the air is descreased with time T = 04 days.

We have this numerical simulation solution, when $\gamma = 0.0025 mg/\ day, \alpha = 0.25$ that shown in Fig.5and when when T= 4 days, $\gamma = 0.0025 mg/\ day, \alpha = 0.25$ shown in Fig.6

Fig-5.Solution of Example (4.10) - (4.12) in the case 2 when $\alpha = 0.25$

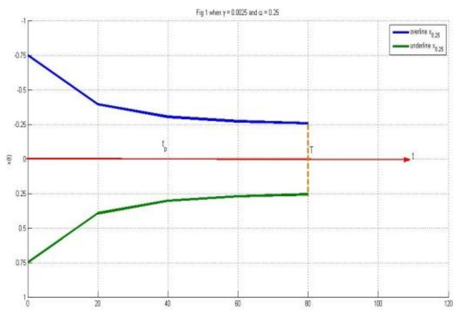
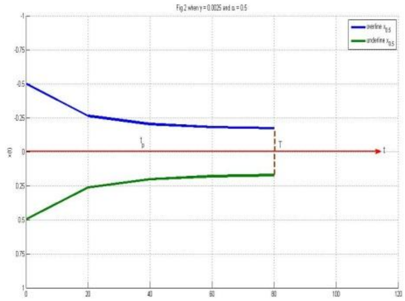


Fig-6.Solution of Example (4.10) - (4.12) in the case 2 when $\alpha = 0.50$



We have this numerical simulation solution, when $\gamma = 0.0025$ mg/day, $\alpha = 0.25$ that shown in Fig.7and when when T= 4 days, $\gamma = 0.0025$ mg/day, $\alpha = 0.50$ shown in Fig.8(in 3D).

Fig-7. Solution of Example (4.10) - (4.12) in the case 2 when $\alpha = 0.25$

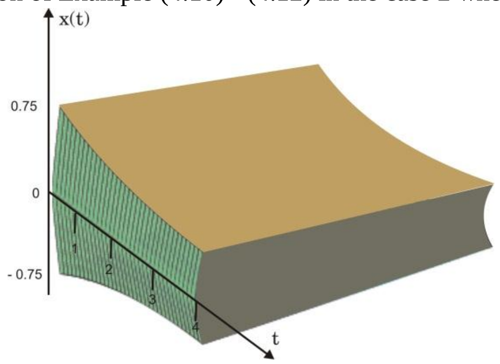
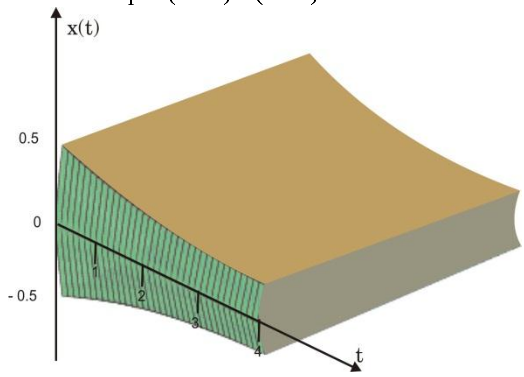


Fig-8. Solution of Example (4.10) - (4.12) in the case 2 when $\alpha = 0.50$.



5. Conclusion

In this work, the existence, uniqueness, boundedness and stability by mapping of fuzzy solutions $x(t) \in E^n$ of the nonlocal initial problems (NIP) for fuzzy differential equations were investigated by the supper distance between fuzzy sets. To illustrate this NIP we consider two examples of water-soluble oxyzen volume and concentration of harmful substances in the air. Similarly, these two examples, we can examine the problem of moisture in the air, the concentration of drug in the human being, etc...

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