Academic Research Publishing Group

Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 2, No. 9, pp: 98-101, 2016
URL: http://arpgweb.com/?ic=journal&journal=17&info=aims

Proximinality in Topological Vector Spaces

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Abstract: In this paper we introduce the concept of proximinality in topological vector spaces. Some results are proved.
Keywords: Proximinal sets; Topological vector spaces.
JEL Classification: Primary 46B20; Secondary 46C15.

1. Introduction

Let *X* be a Banach space and *E* be any set in *X*. For, $x \in X$ set d(*x*,*E*) = . $\inf\{\|x-y\| : y \in E\}$ The set *E* is called proximinal if for any $x \in X$ there exists $z \in E$ such that $d(x,E) = \|x-z\|$. The point *z* is called the closest point of *E* to *x*. The concept of proximinal sets in Banach spaces goes back to fourties of the last century, [1].

Proximinal sets in operator spaces was studied in Deeb and Khalil [2]. The problem of best approximation in $L^p(I,X)$, were studied in Khalil [3] ,Khalil and Deeb [4] and Khalil and Saidi [5]. The concept of proximinality, so was de..ned in metric and normed spaces. But it never been de..ned and studied in general topological vector spaces. In this paper we study remotality in topological vector spaces, where no metric or norm is available. We hope that this will be a new direction in theory of best approximation.

2. Basic Definitions

Definition 2.1: Let X be a vector space. A subset E of X is called **balanced** if $\forall \alpha \in \mathbb{R}, |\alpha| \leq 1$ we have $\alpha E \subseteq E$. Where $\alpha E = \{\alpha e : e \in E\}$.**E** is called **absolutely convex** if it is convex and balanced, and is called **symmetric** if $E = -E$. Where $-E = \{-e : e \in E\}$. E is called **absorbing** if for each $x \in X$, there exists $r \in \mathbb{R}$ such that $x \in \alpha E, \forall \alpha > r$, where $\alpha E = \{\alpha e : e \in E\}$.

Let (X, τ) be a topological vector space. X is called **locally convex** if there is a local basis at $\{0\}$ consisting of open, balanced convex sets.

Observe that, if (X, τ) is a topological vector space then the translation function $f : X \to X : x \to x + a$, for a fixed $a \in X$, is a continuos (1-1) function. Hence every neighborhood base of $\{0\}$ can be translated to be a neighborhood base for any given x in X.

Through out this paper, we assume that $0 \in (X, \tau)$ has a **balanced symmetric convex absorbing** neighborhood, which we denote by $U(0)$.

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Remark 2.2. (i). $U(a) = U(0) + a$ is a balanced symmetric convex absorbing neighborhood of $\{a\}$.

(ii). $tU(0)$ is balanced symmetric convex absorbing neighborhood of $\{0\}$, $\forall t > 0$.

(iii).A subset E of X is called **bounded** if there exists some $t \geq 0$, such that $E \subseteq tU(0)$

Notation2.3

(i) $tU(0) = \{ty : y \in U(0)\}$. Notice that if $0 < t \leq 1$ then $tU(0) \subseteq U(0)$, and if $1 < t < \infty$, then $U(0) \subseteq tU(0)$.

(ii) $tU(x) = tU(0) + x$. Which means we enlarged $U(x)$ in all directions by t, keeping x fixed. We will call x the center of $U(x)$.

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Definition 2.4. For any $x, y \in (X, \tau)$ we define:

(i) $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$, and we will call it the segment joining x to y.

(ii) $[x, y, -] = [x, y] \cup \{z \in X : y \in [x, z]\}$.

(iii) $[-, x, y] = [x, y] \cup \{z \in X : x \in [z, y]\}$.

(vi) $U(x, [x, y])$ denotes the open neighborhood centered at x, with $y \in \partial U(x, [x, y])$. So the whole segment $[x, y]$ is in $U(x, [x, y])$,except y.

Assumption

Let (X, τ) be a topological vector space that satisfies the following property:

For any $x \in X$, $x \notin \partial\overline{U(0,1)}$, there exists $t > 0$ such that $tx \in \partial\overline{U(0,1)}$. Now, let $U(x, [x, -tx])$ be the open neighborhood centered at x, with $-tx \in \partial U(x, [x, -tx])$.

So, $U(0, [0, tx]) = U(0, 1) \subseteq U(x, [x, -tx])$, and let us call this property of (X, τ) the radial inclusion property.

Further, we assume that if $x \notin U(0)$, such that $tx \in \partial\overline{U(0,1)}$, then $U(0) \cap U(x, [x, tx]) = \phi$.

Observe that, this property is not satisfied in every topological vector space. For example the space $(\mathbb{R}, \tau_{dis})$, since for any x, y, we have $[x, y] \setminus \{x\} \subseteq \partial U(x, [x, y])$.

Lemma 2.5. For every $x, y \in X$, there exists $t \in (0, \infty)$ such that $y \in \partial(tU(x))$, the **boundary** of $tU(x)$

Proof. Let us prove the result for $x = 0$. Let y be any element in X. If $y \notin \partial U(0)$, then $[0, y, -] \cap U(0) \neq \varphi$. More precisely, $[0, y, -] \cap \partial U(0) = \{w\}$. Then $y = tw$, and $y \in \partial tU(0)$.Since the translation map is continuos, then this result is still true for any $x \in X$.

3. Proximinality in Topological Vector Spaces

Here we introduce a definition of closest points and proximinal sets. The new definition can be applied in topological vector spaces that are not necessarily normed spaces.

Definition 3.1

Let $E \subseteq (X, \tau)$ be any closed subset of X . For $x \in X$, an element $e \in E$ is called a **closest** element in E to x if there exists a $t > 0$ such that

$$E \cap \partial(tU(x)) \supseteq \{e\} \text{ and } E \cap tU(x) = \phi$$

The set of closest elements to x in E will be denoted by $P(x, E)$.

E is called **proximinal** if every $x \in X$, x has a closest element in E. In other words $P(x, E) \neq \phi$ for all $x \in X$

Definition 3.2. A set E in X is called **ball - compact** if $E \cap t\overline{U(0)}$ is compact for all $t > 0$.

In normed spaces, every finite dimensional subspace E is ball - compact.

Theorem 3.1. Let E be a closed ball - compact set in (X, τ) . Then E is proximinal.

proof: Let $x \in X/E$. Since X/E is open, then there exists some $s > 0$ such that $E \cap sU(x) = \phi$. Let $t > 0$ such that $E \cap tU(x) \neq \phi$. Let t_n be a sequence of positive rationals which is decreasing and such that

$$E \quad \cap kU(x) = \phi \text{ where } k \leq \lim t_n \quad ....(*)$$

.

Choose x_n from $E \cap t_n U(x)$. Since t_n is decreasing, then

$$E \cap t_1 U(x) \supseteq E \cap t_2 U(x) \supseteq E \cap t_3 U(x) \supseteq ....$$

Thus (x_n) can be considered as a sequence in $E \cap t_1 U(x)$. But E is ball - compact.

Hence (x_n) has a subsequence (x_{n_k}), say, such that $x_{n_k} \longrightarrow e$. Now $e \in E$ since E is closed. Further,

$$e \in \bigcap_{n=1}^{\infty} (E \cap \overline{t_n U(x)})$$

, since $x_n \in \overline{t_n U(x)}$ which is a shrinking sequence of closed sets. Thus

$$e \in E \cap (\bigcap_{n=1}^{\infty} \overline{t_n U(x)})$$

Hence

$$e \in \bigcap_{n=1}^{\infty} \overline{t_n U(x)}$$

However

$$\bigcap_{n=1}^{\infty} \overline{t_n U(x)} = t_o \overline{U(x)}$$

where $t_o = \lim t_n$. Since

$$\overline{t_1 U(x)} \supseteq \overline{t_2 U(x)} \supseteq \overline{t_3 U(x)}................,$$

then

$$e \in E \text{ and } e \in (\overline{t_o U(x)})$$

Further from $(*)$ we get

$$e \notin t_o U(x)$$

Hence $e \in \partial(tU(x))$.Thus e is closest element to x.

Theorem 3.2. The set $\overline{U(0)}$ is proximinal in (X, τ)

Proof. If $x \in \overline{U(0)}$, then $P(x, E) = \{x\}$. So let $x \in X \setminus \overline{U(0)}$. Since $\overline{U(0)}$ is absorbing, then there exits some $s > 0$ such that $sx \in \partial\, \overline{U(0)}$.

Consider the set $U[x, [x, sx]]$. So $sx \in \partial U[x, [x, sx]]$. But $sx \in \partial\, \overline{U(0)}$. Hence by the assumption on (X, τ), we get $U[x, [x, sx]] \cap U(0) = \phi$.

Thus sx is the closest element to x in $U(0)$.

Acknowledgment

The authors would like to thank the referee for his sound comments and suggestions.

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