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Optimal Discretionary Monetary Policy in A Potential Zero Lower Bound Framework

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Abstract: In a recessionary and deflationary framework, the discretionary monetary policy cannot be optimal when the interest rate is already near zero and cannot decrease anymore. Indeed, when the Zero Lower Bound is binding, a negative demand shock implies a decrease in the current economic activity level and deflationary tensions, which cannot be avoided by monetary policy as the nominal interest rate can no longer decrease. The economic literature has then often recommended to target an inflation rate sufficiently above zero in order to avoid the dangers of this Zero Lower Bound (ZLB) constraint. On the contrary, provided the ZLB is not binding, monetary policy can efficiently contribute to the stabilization of economic activity and inflation in case of demand shocks. The variation in interest rates is then all the more accentuated as interest rate smoothing is a more negligible goal for the central bank. The contribution of our paper is to provide a clear analytical New-Keynesian framework sustaining these results. Besides, our analytical modelling also shows that even if the ZLB is currently not binding, the central bank should take into account the dangers of a potential future binding ZLB. Indeed, the interest rate should be decreased the fastest as a negative demand shock and the possibility to reach the ZLB is anticipated for a nearest future period. Our paper demonstrates the necessity of such a 'pre-emptive' active monetary policy even in a discretionary framework, which has the advantage to be time-consistent and to be in conformity with the empirical practices of independent central banks. We don't have to make the strong hypothesis of a commitment monetary policy intended to affect private agents' expectations in order to demonstrate the optimality of such a pre-emptive monetary policy.
Keywords: Monetary policy discretionary policy; New-Keynesian models; Zero lower Bound; Demand shocks.
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1. Introduction

The oil crisis in 1973 demonstrated the dangers of high inflation rates and self-fulfilling inflation expectations. Indeed, in situations of hyperinflation, money and real cash reserves loose part or all of their value. Inflation then causes distortions in the inter-temporal allocation of consumption, due to the reduction in the real rate of return on saving; cash of creditors and saving are then discouraged. A high inflation rate is also more liable to create problems of distortions in relative-prices; besides, the tax system is distortionary, and it is not inflation neutral. Feldstein (1996) thus insists on the inflation-induced distortions in the lifetime allocation of consumption, in the allocation of households' spending between housing (inflation reduces mortgage tax deductions for interest payments) and other forms of consumption, and regarding the difficulties to make financial and portfolio investment decisions (smaller real value of depreciation allowances and higher effective taxation of capital profits). Woodford (2003) also insists on the fact that price stability should be the main goal of a central bank, in order to avoid real distortions: in aggregate employment and output, in the sectorial composition of the economy and in resources allocation.

On another hand, the Japanese context in the 1990s proved the dangers of a deflationary situation. Indeed, as mentioned by many authors (see below), for very low inflation rates, the monetary authority has a very weak margin of manoeuver to reduce the nominal interest rate (which cannot fall below zero) in order to enhance economic activity: it is the problem of the Zero Lower Bound (ZLB) on the interest rate. Indeed, when expected inflation is zero, then it is very difficult for monetary policy to engineer a negative short-run real interest rate. Therefore, once short term rates fall to zero, for weak inflation rates, conventional monetary policy instruments no longer work to stimulate economic activity. In this framework, it seems that the lower the inflationary target, the less stable economic activity will be.

Besides, a positive inflation rate is also useful if firms want to reduce real wages in a framework of nominal wages downward stickiness Akerlof *et al.* (1996) provide a very detailed survey of empirical evidence confirming this relative rigidity in the United-States]. With such downward wage rigidity, employers prefer the strategy of wage freeze, where real wages decline by the amount of inflation. Therefore, we can underline that the empirical observation of a downward price rigidity is closely linked to the context of economies experiencing a positive inflation rate. Schmitt-Grohé and Uribe (2010) also underline other arguments usually mentioned to justify the optimality of a positive inflation rate: incompleteness of the tax system and necessity to tax profits which otherwise would remain untaxed, quality bias in the measurement of inflation. A deflationary situation also implies the risk to increase the real service of the public debt, and to be highly difficult to sustain for public finances. So, fixing the inflation rate at some appropriate level is fundamental; this level should be low but positive: typically 1 to 3% per year according to Fischer (1996).

So, many central banks have tried to define optimal inflation rates; in particular, the statutes of the European Central Bank (ECB) mention a goal of price-stability, which is to maintain "*a year-on-year increase in the Harmonised Index of Consumer Prices (HICP) for the euro area of below 2%*" (article 105 of the Maastricht Treaty), but close to 2% over the medium term. The Federal Reserve calls 2% a '*longer-run goal'*. However, this figure of a 2% inflation rate is mostly conventional, as no econometric study really sustains the value of this optimal inflation rate. Besides, since 2008, the financial and economic crisis has forced the FED and the ECB to decrease their nominal interest rates to levels near zero, without succeeding to avoid the increase in unemployment and the harmful recessionary consequences of the economic crisis. Therefore, this situation has revived the debate on the optimal inflation rate, which should not be too low in order to avoid the Zero Lower Bound.

Furthermore, in Europe, monetary unification involved new difficulties. Indeed, the 'optimal monetary policy' is probably not the same for the member countries of the Economic and Monetary Union (EMU), with distinct inflation and growth rates (cyclical conditions) and levels of economic development (structural conditions). In this context, a common monetary policy and a common interest rate could contribute to increase imbalances (for example in current account positions) and structural divergences in the EMU. Within the Eurozone, when the real interest rate is too low, it accentuates expansionary tensions in countries with above average inflation rates, whereas when it is too high, it accentuates recessionary tensions in countries with below average inflation rates. Indeed, inflation differentials are persistent, as the adjustment through the real exchange rate channel or in relative price-competitiveness between EMU member countries takes a very long time. It is the 'one size fits none' problem of the ECB, whose monetary policy has asymmetric and pro-cyclical consequences on the member countries (Padoa-Schioppa Group, 2012).

Besides, achieving a low inflation level through central bank independence has been an historic accomplishment. Nevertheless, Blanchard *et al.* (2010) mention that the framework of the economic and financial Crisis after 2007 strongly modified the consensus on appropriate macro-economic policies. Monetary policy was supposed to be efficient in targeting a given stable inflation rate with the help of its instrument: the interest rate (Blanchard *et al.*, 2010). However, the economic crisis proved that targeting a core inflation index may not be sufficient to stabilize the out-put gap, and that imbalances (current account deficit, excessive housing investment, etc.) may arise. Mainly, the crisis revived the debate on the cost of the Zero Lower Bound (ZLB), and thus on the necessity to target a higher inflation rate (around 4%) than previously assumed, in order to avoid that this constraint be binding. A too low inflation target will interfere with the success of monetary stabilization policy, because this policy will too frequently be constrained by the Zero Lower Bound on nominal interest rates. This argument began to be taken more seriously by both central bankers and monetary economists after Japan reached the zero bound in the late 1990.

Williams (2009) underlines that the Crisis forced many central banks to reduce their interest rates to levels near zero. Then, the ZLB and the incapacity to reduce further interest rates could be responsible for the helplessness of monetary policy and for the slowness of the recovery in the United-States. Indeed, the empirical estimates of the author show that an inflation target of 2% or lower could have non-negligible costs in terms of macroeconomic stabilization, if monetary authority follows a classical Taylor rule. In the current framework of relatively low steady-state real interest rates (which have decreased from around 2.5% in the 1980s to near 1% during the Crisis in the United-States), the ZLB would argue for a higher targeted and steady-state inflation rate, especially in a context of greater volatility of disturbances. This question of the dangers of the ZLB has been largely underlined and studied in the recent economic literature.

For example, Fuhrer and Madigan (1997) find that with small negative spending shocks, long term real interest rates can decline in order to limit the decreasing path in output. However, for large shocks persisting a few quarters, the output path can be more hardly constrained in a low inflation scenario (near zero) than in a high inflation scenario (near 4%). In the same way, using a standard macroeconomic New-Keynesian model calibrated to U.S. data accounting for the danger of the ZLB, Billi and Kahn (2008), estimate the optimal inflation rate between 0.7% and 1.4% (under extreme model uncertainty) per year as measured by the Personal Consumption Expenditure price index adjusted for the upward measurement error. More precisely, with the help of a small New-Keynesian model, Billi (2007) underlines that the inflation rate maximizing the global well-being of the representative consumer depends on the credibility of commitments of the central bank. If a government could optimally commit to a plan for all future policy decisions, or to an inertial Taylor rule strongly limiting variations in comparison with the previous interest rate, the optimal long run inflation rate could be very low, below 1%. In the opposite case, if a government could re-optimize discretionally during each period, the optimal inflation rate would be much higher, between 13.4% and 16.7% (in case of extreme model misspecification). Therefore, for a government which cannot credibly commit, an inflation target much higher than 2% would be necessary as insurance against the ZLB. Indeed, given its impossibility to create credible inflationary expectations, a government should have a higher inflationary goal, because it is then the only way to avoid the dangers of a deflationary spiral in a context of high persistence in the inflation rate.

In the same way, Eggertsson and Woodford (2003) mention that in a framework where deflation has become a credible risk, it could be necessary to take this into account, for example by targeting a sufficiently high inflation rate even in normal times. A sufficiently high inflation target could allow to avoid the deflationary and recessionary situation when the ZLB is binding. Orphanides and Wieland (1998) use a small structural rational expectations model with forwards looking behaviour by economic agents and staggered wage contracts, calibrated to the U.S situation in the 1980s and in the 1990s. Then, they find that the consequences of the ZLB would be non-linear: they would remain quite insignificant for an inflation target of 2%, but stabilization performances would be hardly deteriorated with a target between 0% and 1%. Thus, the ZLB seems to generate a non-vertical long-run Phillips curve: output falls increasingly short of potential with lower inflation targets, recessions becoming more frequent and longer lasting.

More precisely, Kato and Nishiyama (2003) show that in the presence of a ZLB on the nominal interest rate, the monetary reaction function is nonlinear, more expansionary (the interest rate is smaller), and more 'aggressive' (the interest rate reacts more to variations in the current inflation rate or output gap) than the usual Taylor rule. Approaching the ZLB, the central bank must cut the interest rate more aggressively and conduct a more expansionary monetary policy, in order to prevent the economy from falling in a deflationary spiral. According to this rule, it seems that empirically, the Bank of Japan was conducting a too soft and naïve monetary policy in the early 1990s, whereas the zero interest rate policy was more aggressive and well fitted after 1995. Using a simple calibrated open-economy model, Orphanides and Wieland (2000) also demonstrate that with a ZLB on the interest rate, the optimal monetary policy is a non-linear function of the inflation rate: it should be more 'aggressive' as the nominal interest rate approaches the ZLB. As inflation declines, policy turns expansionary sooner and more aggressively than would be optimal in the absence of the ZLB. Therefore, this implies an upwards bias regarding the inflation level, in order to reduce the variability of inflation due to the ZLB and to the potentiality of a deflationary framework, and in order to improve economic stabilization.

Ball (2013) even assesses that a 2% inflation target is likely to constrain monetary policy in a large fraction of recessions. On the contrary, a 4% inflation target would ease the constraints on monetary policy arising from the zero bound on interest rates, with the result that economic downturns would be less severe. According to him, this benefit would come at minimal cost, because an inflation rate of 4% does not harm an economy significantly. If the interest rate channel is constrained by the ZLB, Orphanides and Wieland (2000) mention that another channel can be very useful to sustain economic activity. In an open-economy framework, the exchange rate channel can intervene. Indeed, in a deflationary framework, monetary easing, even if the interest rate cannot decrease anymore, can contribute to devaluate the real exchange rate, to sustain exports and thus the level of economic activity. Modifying and easing the credit conditions can also be another monetary tool when the ZLB is binding. However, the authors mention that these effects appear as quantitatively small and uncertain, in comparison with the direct interest rate channel.

So, a first implication of the ZLB is that the central bank should target an inflation rate which is higher than previously assumed, in order to avoid the deflationary risk. Besides, a second teaching of the economic literature putting the stress on the ZLB is that optimal monetary policy should be inertial. For example, Jung *et al.* (2005) argue that the optimal path of monetary policy is policy inertia. Indeed, in case of an economic downturn, a central bank should make credible commitment to a future expansionary monetary policy, in order to avoid the deflationary bias of the ZLB. A zero interest rate policy should be continued for a while after the natural rate of interest returns to a positive level, in order to achieve lower nominal long-term interest rates and higher expected inflation. Gelain (2007) uses a backwards looking model with adaptive expectations, auto-regressive inflation and output-gap functions, in order to estimate the behaviour of the ECB for the period 1995-2003. Then, given the strong inflation persistence in Europe, he underlines that a high degree of interest rate smoothing would be optimal for the ECB. Williams (2009) also assesses that an inertial Taylor rule smoothing variations in interest rates would be the most efficient regarding macro-economic stabilization, if it helps anchoring expectations and if the latter are not too disturbed by a prolonged episode of low inflation and interest rates.

In the same way, Woodford (1999) shows, in the context of a simple model of optimizing private-sector behaviour, that an inertial monetary policy is usually optimal, adjusting interest rates only gradually in response to changes in economic conditions. Indeed, small but persistent changes in short-term interest rates in response to shocks allow a larger effect of monetary policy on long-term rates and thus on aggregate demand, for a given degree of overall interest-rate variability. According to the author, a credible commitment to an optimal (highly inertial) feedback rule on the part of the central bank should not require large movements of short-term interest rates in equilibrium; highly persistent low-amplitude variations are sufficient to achieve a desirable degree of inflation stabilization. Therefore, Woodford (1999) assumes that, whereas discretionary monetary policy is usually suboptimal, it would be efficient to make monetary policy '*history-dependent'*. In a framework where the ZLB can be binding, the option, for the central bank, is to create the right expectations regarding how monetary policy will be used after this constraint is no longer binding. During a liquidity trap, a strong policy commitment that interest rate will be kept low for a longer period can create inflationary expectations, reduce economic contraction and sustain demand, even if interest rates cannot further be decreased. Therefore, Eggertsson and Woodford (2003) are also in favour of a history dependent monetary policy relying on past conditions, even though the latter are in principle irrelevant to the degree to which the stabilization goals of the central bank could be achieved from then on.

Evans and Honkapohja (2005) also underline the possibility of a 'liquidity trap', the economy converging and then being trapped in a deflationary and recessionary situation, in macro-economic frameworks where agents have perfect foresight/ rational expectations. However, they show that in the framework of adaptive expectations, fiscal policy is of limited usefulness and must remain passive (an active policy would imply an outbidding of the public indebtedness without increasing prices). On the contrary, raising the threshold for an active monetary policy, a huge increase in monetary supply if inflation is below a threshold and a more aggressive monetary policy can dislodge the economy from the liquidity trap (low inflation steady state), and ensure a return to the targeted higher inflation rate. In the same way, in order avoid deflationary spirals, Evans *et al.* (2008) recommend a combination of aggressive monetary and fiscal policy triggered whenever inflation threatens to fall below an appropriate threshold. In contrast, policies geared toward ensuring an output lower bound are insufficient for avoiding deflationary spirals. Besides, an aggressive monetary policy should better be adopted quite early, before a still non negligible threshold, and should be in place before negative expectations shocks impact the economy.

So, the aim of our paper is to contribute to this debate on the optimal discretionary monetary policy in a framework where the Zero Lower Bound can potentially be binding, and the paper is then organized as follows. Section 2 describes the demand and supply New-Keynesian functions used to derive our main results. Section 3 solves the model and describes the optimal monetary policy in a discretionary framework where the central bank cannot commit. Section 4 describes the optimal monetary policy when the ZLB is binding, while section 5 describes this monetary policy when the ZLB is not binding. The sixth section concludes the paper.

2. The Model

Our study will adopt the standard framework of a small New-Keynesian model [see for example Clarida *et al.* (1999), Evans and Honkapohja (2003a); (Evans and Honkapohja, 2003b) Galí (2008), or Woodford (2003) for a very extensive presentation], made of a representative household and of a representative firm.

Dynamic Stochastic General Equilibrium (DSGE) models are used by the European Central Bank or by the Federal Reserve in formulating monetary policy. Individuals make decisions about consumption and labor supply that maximize their economic well-being subject to constraints based on their wealth. Firms set prices that maximize profits and demand factors of production, such as labor and capital, in ways that minimize their costs. Nominal wage and price rigidities imply that prices are only randomly adjusted according to a markup over current and expected marginal costs. These models are also dynamic: economic variables depend on expectations about future outcomes and variables.

DSGE models account for the inefficiencies inflation causes when it distorts relative prices and leads consumers and firms to make suboptimal decisions. As mentioned by Woodford (2003), the introduction of nominal wages and prices rigidities and temporary differentials in the evolution of relative prices in various sectors of the economy is necessary to justify the usefulness of the price stability goal for the central bank. It also implies differentials between effective and potential output, non-trivial real effects of monetary policy, and therefore it justifies an output stabilization goal for the central bank. Therefore, nominal rigidities and non-neutrality of monetary policy are key characteristics of New-Keynesian models. However, these models usually lack a full description of government spending and taxation, and thus they ignore the distortions that inflation causes in a tax system that is not fully indexed.

All economic variables are expressed as deviations from their non-stochastic and long run steady state values, which could be observed with a growth rate corresponding to the constant trend or potential output growth rate. In order to study the consequences of openness, our model considers many countries and small open economies, i.e. which have not in isolation any influence on global and average variables. The productivity shocks can differ between countries. However, for simplicity, all countries share the same preferences. Financial markets are assumed to be complete both at the national and international level in the framework of a monetary union (risks are fully shared among households), and countries share the same common interest rate.

2.1. The Households and the Demand Equation

Aggregate demand for the country (i) results from the log-linearization of the Euler equation, which describes the representative household's expenditure decisions. We suppose that the economy (i) is populated by a unit measure of households indexed by (i). In our model, the representative household provides labour and it consumes goods. In a given period (T), the representative household/consumer in country (i) maximizes an inter-temporal utility function:

$$max\sum_{t=T}^{\infty}\beta^{t-T}E_T[U_{i,t}] \qquad (1)$$

Where: $E_t()$ is the rational expectation operator conditional on information available at date (t), and (β) is the time discount factor. Prices of goods, interest rates, taxation rates and wages are then taken as given by the representative household. Besides, this maximization is subject to the life time and inter-temporal budget constraint, for whatever date (T) considered at which the actualization is realized:

$$C_{i,T}+E_T[\sum_{t=T}^{\infty}\frac{C_{i,t+1}}{(1+r_{i,t})\dots(1+r_{i,T})}]+E_T[\sum_{t=T}^{\infty}(\frac{i_{t+1}}{1+i_{t+1}})\frac{M_{i,t+1}}{P_{i,t+1}(1+r_{i,t})\dots(1+r_{i,T})}]$$

$$+(\frac{i_T}{1+i_T})\frac{M_{i,T}}{P_{i,T}}=\frac{W_{i,T}}{P_{i,T}}L_{i,T}(1-t_{i,T})+E_T[\sum_{t=T}^{\infty}\frac{W_{i,t+1}L_{i,t+1}(1-t_{i,t+1})}{P_{i,t+1}(1+r_{i,t})\dots(1+r_{i,T})}]<\infty \quad (2)$$

With, in the country (i) in period (t): ($C_{i,t}$): real consumption; ($M_{i,t}$): monetary base held; ($P_{i,t}$): level of prices; ($W_{i,t}$): nominal hourly wage; ($t_{i,t}$): taxation rate on personal income; ($L_{i,t}$): hours worked by the household; ($\pi_{i,t}$): inflation rate; (i_t): nominal interest rate; [$r_{i,t}=i_t-E(\pi_{i,t+1})$]: real interest rate.

Current consumption and real money balances as well as anticipated consumption levels and real money balances after the current period (T), mustn't exceed current real activity revenues and anticipated revenues for all future periods after the current period (T). Therefore, in this model, we allow for the possibility to borrow from one period to another, but we limit anticipated future revenues in order to avoid the possibility of Ponzi schemes.

Furthermore, Woodford (2003) demonstrates that if the interest rate on the monetary base is null, frictions justifying the demand for money are necessary, in order to obtain an equilibrium path for the price level. Otherwise, without a positive interest rate on the monetary base, it would be impossible to derive the inflationary consequences of monetary policy. Therefore, for example, Wolman (1998) explicitly models money demand, using a shopping-time technology: consumers hold money in order to economize on transactions time. Holding the medium of exchange provides transaction services that reduce the time and other resources needed in 'shopping' for the numerous distinct consumption goods; it economizes on the use of credit which is costly. In order to justify money demand, the author assumes that ignoring money demand would also mean ignoring the welfare cost of positive nominal interest rates. So, in our model, we suppose that the opportunity cost of holding money is equal to ($\frac{i}{1+i}$), where (i) is the interest rate on a risk free investment, supposing that no interest rate is paid on the monetary base.

We suppose that the utility function of a representative household has the form:

$$U_{i,t}=\alpha_c\frac{\theta}{(\theta-1)}(C_{i,t})^{\frac{(\theta-1)}{\theta}}+\alpha_m f\left(\frac{M_{i,t}}{P_{i,t}}\right)-\alpha_l\frac{1}{(1+\varphi)}L_{i,t}^{(1+\varphi)} \quad (3)$$

The indices ($0<\alpha_c<1$), ($0<\alpha_m<1$) and ($0<\alpha_l<1$) are the respective weights given to consumption, detention of money balances and leisure in the utility function.

Utility is an increasing and concave function of ($C_{i,t}$), an index of the household's consumption of all goods that are supplied; ($\theta>0$) is the elasticity of intertemporal substitution. Utility is also a decreasing and convex function of hours worked ($L_{i,t}$), and ($\varphi\geq0$). Furthermore, for example, Woodford (1996), Woodford (2003) or Eggertsson and Woodford (2003) also introduce real money balances ($m_{i,t}=M_{i,t}/P_{i,t}$) held by households in period (t) to carry over in period (t+1) in their utility function, as they obtain liquidity services from the detention of a fraction of the real monetary base, facilitating transactions. Utility increases with real money balances until a given satiation level (m_m). Utility is then an increasing and concave function of money balances held by the representative household: $f'(m_{i,t})>0$ and $f''(m_{i,t})<0$, at least before the satiation level for real money balances is reached. However, for (i=0), the maximal level of money balances (m_m) is held, and $\partial U_{i,t}/\partial m_{i,t}=0$. Besides, (Woodford, 2003)Woodford mentions that we can justify the additive separability, in the utility function, between consumption and real money balances, by the weakness of the transactions made in cash (monetary base) in comparison with total national income or households' consumption levels. The economy would be a 'cashless limiting economy'.

In this context, the result of the maximization of equation (1) under the constraint (2) implies the following first order Euler conditions, regarding timing of expenditure decisions and inter-temporal substitution, for whatever period (T):

$$\frac{\partial U_{i,T}}{\partial C_{i,T}}=\beta(1+r_{i,T})\frac{\partial E_T(U_{i,T+1})}{\partial C_{i,T+1}}$$

$$\frac{\partial U_{i,T}}{\partial(M_{i,T}/P_{i,T})}=\frac{\beta(1+i_{T+1})i_T(1+r_{i,T})}{i_{T+1}(1+i_T)}\frac{\partial E_T(U_{i,T+1})}{\partial(M_{i,T+1}/P_{i,T+1})}$$

$$\frac{\partial U_{i,T}}{\partial L_{i,T}}=\frac{\beta(1+r_{i,T})W_{i,T}(1-t_{i,T})P_{i,T+1}}{W_{i,T+1}(1-t_{i,T+1})P_{i,T}}\frac{\partial E_T(U_{i,T+1})}{\partial L_{i,T+1}} \quad (4)$$

Furthermore, we can mention that according to equation (3), $\frac{\partial U_{i,T}}{\partial C_{i,T}}=\alpha_c(C_{i,T})^{-\frac{1}{\theta}}$, and therefore: equation (4) implies, (□T):

$$C_{i,T}=[\frac{1}{\beta(1+r_{i,T})}]^{\theta}E_T(C_{i,T+1}) \quad (5)$$

So, in logarithms, with $\log(1+r_{i,T})\sim(r_{i,T})$ provided ($r_{i,T}$) is sufficiently small, we have:

$$c_{i,T}=-\theta\log\beta-\theta r_{i,T}+E_T(c_{i,T+1}) \quad (6)$$

Besides, for the representative agent in the country (i), we obtain the following optimal substitution between consumption, money balances and working time, if we exclude the case of the corner solution with zero money balances:

$$\frac{\partial U_{i,T}}{\partial C_{i,T}}=\left(\frac{1+i_T}{i_T}\right)\frac{\partial U_{i,T}}{\partial(M_{i,T}/P_{i,T})}=-\frac{P_{i,T}}{W_{i,T}(1-t_{i,T})}\frac{\partial U_{i,T}}{\partial L_{i,T}} \quad (7)$$

Therefore, a higher nominal interest rate reduces the marginal utility of money balances. Furthermore, a higher real wage net of taxes reduces the marginal utility of leisure and increases the one of labour. So, regarding the labour demand, we have:

$$\frac{\partial U_{i,T}}{\partial L_{i,T}} = -\alpha_l L_{i,T}^{\varphi} = -\frac{W_{i,T}(1-t_{i,T})}{P_{i,T}}\frac{\partial U_{i,T}}{\partial C_{i,T}} = -\frac{W_{i,T}(1-t_{i,T})}{P_{i,T}}\alpha_c\left(C_{i,T}\right)^{-\frac{1}{\theta}} \quad (8)$$

In differential and in logarithms, we have:

$$l_{i,T} = \frac{1}{\varphi}\left(w_{i,T}-p_{i,T}\right) - \frac{1}{\varphi}t_{i,T} + \frac{1}{\varphi}\log\left(\frac{\alpha_c}{\alpha_l}\right) - \frac{1}{\varphi\theta}c_{i,T} \quad (9)$$

So, labour supply increases with the real wage, but it decreases with the taxation rate ($t_{i,T}$) and with the disutility of working time (φ).

As explained in McCallum (2000) or McCallum and Nelson (1999), the resource constraint in the economy is that the GDP growth rate can be divided in four components: growth rates of consumption (c_t), investment (inv_t), public expenditure (g_t) and net exportations (exp_t), whose average relative shares in GDP are (z_1), (z_2), (z_3) and (z_4). So, we have:

$$y_{i,T} = z_1 c_{i,T} + z_2 inv_{i,T} + z_3 g_{i,T} + z_4 exp_{i,T} \qquad with\text{: } \textstyle\sum_{i=1}^{4} z_i = 1 \quad (10)$$

Therefore, equations (6) and (10) imply:

$$y_{i,T} = E_t\left(y_{i,T+1}\right) + z_2\left[inv_{i,T} - E_t\left(inv_{i,T+1}\right)\right] + z_3\left[g_{i,T} - E_t\left(g_{i,T+1}\right)\right] + z_4\left[exp_{i,T} - E_t\left(exp_{i,T+1}\right)\right] - z_1\theta\log\beta - z_1\theta r_{i,T} \quad (11)$$

So, in differential with respect to a potential output level, equation (11) becomes:

$$\left(y_{i,T} - y_{i,T}^{p}\right) = E_T\left(y_{i,T+1} - y_{i,T+1}^{p}\right) + z_2\left[inv_{i,T} - E_T\left(inv_{i,T+1}\right)\right] + z_3\left[g_{i,T} - E_t\left(g_{i,T+1}\right)\right] + z_4\left[exp_{i,T} - E_t\left(exp_{i,T+1}\right)\right] - z_1\theta\log\beta - z_1\theta\left[i_T - E_T\left(\pi_{i,T+1}\right)\right] - y_{i,T}^{p} + E_t\left(y_{i,T+1}^{p}\right) \quad (12)$$

Therefore, with ($\sigma = z_1\theta$), and with: $log\beta = -\log(\frac{1}{\beta}) = -\log\left[1 + \left(\frac{1-\beta}{\beta}\right)\right] \sim -\frac{(1-\beta)}{\beta}$ as (1-β) is small, we have the final equation:

$$\boldsymbol{x_{i,T} = E_T\left(x_{i,T+1}\right) - \sigma\left[i_T - E_T\left(\pi_{i,T+1}\right) - \overline{r_{i,T}}\right]} \qquad \mathbf{(13)}$$

- ($x_{i,t}$): output-gap, differential between effective output and its potential level ($y_{i,t} - y_{i,t}^{p}$);
- (σ): real interest rate elasticity of the output-gap, 'inter-temporal elasticity of substitution' of household expenditure.
- ($\overline{r_{i,t}}$): equilibrium or natural real interest rate, which corresponds to the steady-state real rate of return if prices and wages were fully flexible. It is the real rate of interest required to keep aggregate demand equal at all times to the natural rate of output. It includes all non-monetary disturbances: shock on preferences (β), shock on long-run potential output and on productivity, exogenous change in short run public expenditure, etc. Indeed, even if we want to insist on the implications of nominal rigidities for monetary policy, we must take into account real disturbances. This natural interest rate is as follows:

$$\overline{r_{i,T}} = \frac{(1-\beta)}{\beta} + \frac{z_2}{\sigma}\left[inv_{i,T} - E_T\left(inv_{i,T+1}\right)\right] + \frac{z_3}{\sigma}\left[g_{i,T} - E_T\left(g_{i,T+1}\right)\right] + \frac{z_4}{\sigma}\left[exp_{i,T} - E_T\left(exp_{i,T+1}\right)\right] - \frac{1}{\sigma}\left[y_{i,T}^{p} - E_T\left(y_{i,T+1}^{p}\right)\right] \quad (14)$$

Therefore the equilibrium real interest rate decreases with (β), and it increases with the impatience of households to consume; it also increases with the temporary increase in government expenditure, in investment or in net exports, and it decreases with the temporary increase in productivity and in potential output.

In the economic literature, a positive demand shock ($\sigma\overline{r_{i,t}}$) on the equilibrium real interest rate is often modelled as a white noise. For example, such a shock could follow a path like the following: $\sigma\overline{r_{i,t}} = \rho\sigma\overline{r_{i,t-1}} + \varepsilon_i^d$ $\varepsilon_i^d \sim idd(0, \sigma_{\varepsilon^d}^2)$ $\quad 0 < |\rho| < 1$.

So, according to equation (13), higher future expected output increases current output and consumption, because households prefer to smooth consumption, and then higher future revenues raise their current consumption and current output. Current output is also a decreasing function of the excess of the real interest rate above its natural level, because of the inter-temporal substitution of consumption.

2.2. The Supply Equation

We suppose a continuum of firms indexed by (i). The representative firm (i) produces a differentiated good in a monopolistic competition setting. It defines prices in order to maximize its profit, taking other variables as given. Capital is supposed to be fixed in the short run, whereas labour is defined according to the maximization program of households in (9). It is the only factor of production which is variable in the short run. Monopolistic competition gives to goods suppliers a market power regarding price-setting, while at the same time fitting the empirical evidence of a large number of firms in the economy. However, each firm is supposed to use the same technology. So, the production function has the following form, for the representative, competitive firm (i):

$$Y_{i,t} = A_{i,t}L_{i,t}^{1-v} \qquad 0<v<1 \qquad (15)$$

Where ($A_{i,t}$): technology or productivity shock, common to all firms in a given country, and evolving exogenously over time; ($L_{i,t}$): number of hours worked; (v) represents the decreasing returns of the production function.

Let's consider a Calvo-type framework of staggered priced, where a fraction (0<α<1) of goods prices remain unchanged each period, whereas prices are adjusted for the remaining fraction (1-α) of goods. Monopolistically competitive firms choose their nominal prices to maximize profits subject to constraints on the frequency of future price adjustments, and taking into account that prices may be fixed for many periods. So, they minimize the loss function:

$$Min_{\widetilde{p_{i,t}}} \sum_{k=0}^{\infty} (\alpha\beta)^k E_t(\widetilde{p_{i,t}} - p^*_{i,t+k})^2 \qquad (16)$$

Where ($p^*_{i,t}$) is the logarithm of the optimal price that a firm (i) will set in period (t) if there were no price rigidity.

The firm minimizes expected losses in profit for all future periods (t+k) due to the fact that it will not be able to set a frictionless optimal price in this period (t+k).These losses are subject to the actualization rate (β), as closer profits are given a higher weight than more distant ones. Besides, the probability that the price (p_t) will be fixed for (k) periods, until the period (t+k), is (α^k). Thus, by deriving in function of the reset price ($\widetilde{p_{i,t}}$), we have:

$$\sum_{k=0}^{\infty} (\alpha\beta)^k\, \widetilde{p_{i,t}} = \frac{1}{(1-\alpha\beta)} \widetilde{p_{i,t}} = \sum_{k=0}^{\infty} (\alpha\beta)^k E_t(p^*_{i,t+k}) \qquad (17)$$

$$\widetilde{p_{i,t}} = (1-\alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t(p^*_{i,t+k}) \qquad (18)$$

Therefore, the firm (i) tries to set the optimal reset price ($\widetilde{p_{i,t}}$) to the level of a weighted average of the prices that it would have expected to reset in the future if there weren't any price rigidities.

The optimal strategy of the firm (i) is to fix prices at marginal costs: $p^*_{i,t} = mc_{i,t}$, where ($mc_{i,t}$) is the nominal marginal production cost of the firm (i).

Prices in period (t) are an average of past prices and reset prices:

$$p_{i,t} = \alpha p_{i,t-1} + (1-\alpha)\widetilde{p_{i,t}} \qquad (19)$$

So, by combining equations (18) and (19), we obtain[1]:

$$\widetilde{p_{i,t}} = \frac{1}{(1-\alpha)} p_{i,t} - \frac{\alpha}{(1-\alpha)} p_{i,t-1} = (\alpha\beta) E_t(\widetilde{p_{i,t+1}}) + (1-\alpha\beta) mc_{i,t}$$

$$= \frac{\alpha\beta}{(1-\alpha)} E_t(p_{i,t+1}) - \frac{\alpha^2\beta}{(1-\alpha)} p_{i,t} + (1-\alpha\beta) mc_{i,t} \qquad (20)$$

Therefore, we obtain the following equation:

$$\pi_{i,t} = \beta E_t(\pi_{i,t+1}) + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}(mc_{i,t} - p_{i,t}) \qquad (21)$$

Inflation then depends on expected future inflation, and on the gap between the frictionless optimal price level and the current price level, i.e.: on the real marginal cost. Indeed, inflationary pressures can be due to the fact that prices which can be reset by firms are increased.

Now, we have to clarify the expression of the real marginal production cost for the firm (i) [see for example: Woodford (1996) Woodford (2003) or Galí (2008)]. According to equation (15), the number of hours worked is: $L_{i,t} = (\frac{Y_{i,t}}{A_{i,t}})^{\frac{1}{1-v}}$, and the variable production cost of the quantity ($Y_{i,t}$) is: $W_{i,t}L_{i,t} = W_{i,t}(\frac{Y_{i,t}}{A_{i,t}})^{\frac{1}{1-v}}$. So, differentiating this expression, the nominal marginal production cost of the quantity ($Y_{i,t}$) is:

$$\frac{\partial(W_{i,t}L_{i,t})}{\partial Y_{i,t}} = \frac{W_{i,t}}{A_{i,t}^{\frac{1}{1-v}}(1-v)} (Y_{i,t})^{\frac{v}{1-v}} \qquad (22)$$

So, in logarithms and in variations, we obtain the following real marginal production cost:

$$mc_{i,t} - p_{i,t} = w_{i,t} - p_{i,t} - \frac{1}{1-v} a_{i,t} - \log(1-v) + \frac{v}{1-v} y_{i,t} \qquad (23)$$

Afterwards, the expression of the real wage in equation (9) and the equilibrium condition in (10) imply:

$$w_{i,t} - p_{i,t} = \varphi l_{i,t} + t_{i,t} - \log\left(\frac{\alpha_c}{\alpha_l}\right) + \frac{1}{\sigma} y_{i,t} - \frac{z_2}{\sigma} inv_{i,t} - \frac{z_3}{\sigma} g_{i,t} - \frac{z_4}{\sigma} exp_{i,t} \quad (24)$$

Besides, the real production in equation (15) implies: $l_{i,T} = \frac{1}{1-v} y_{i,t} - \frac{1}{1-v} a_{i,t}$

So, with (23) and (24), we obtain the following real marginal production cost:

$$(mc_{i,t} - p_{i,t}) = \left(\frac{\varphi + v}{1-v} + \frac{1}{\sigma}\right) y_{i,t} - \frac{(1+\varphi)}{(1-v)} a_{i,t} + t_{i,t} - \log\left(\frac{\alpha_c}{\alpha_l}\right) - \frac{z_2}{\sigma} inv_{i,t} - \frac{z_3}{\sigma} g_{i,t} - \frac{z_4}{\sigma} exp_{i,t} - \log(1-v) \qquad (25)$$

[1] Equation (18) implies: $\widetilde{p_{i,t}} = (\alpha\beta)E_t(\widetilde{p_{i,t+1}}) + (1-\alpha\beta)p^*_{i,t}$.
Equation (19) implies: $\widetilde{p_{i,t}} = \frac{1}{(1-\alpha)} p_{i,t} - \frac{\alpha}{(1-\alpha)} p_{i,t-1}$.

Therefore, we can mention that real marginal production costs are pro-cyclical, and are an increasing function of the output gap: indeed, a production higher than its potential level increases marginal production costs faster than prices, with more competition for the available factors of production.

Finally, by combining equations (21) and (25), we have the following supply function for the country (i) in a given period (T):

$$\boldsymbol{\pi_{i,T} = \beta E_T(\pi_{i,T+1}) + k_1 k_2 x_{i,T}} \qquad \mathbf{(26)}$$

$$with: \quad k_1 = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \qquad k_2 = \left(\frac{\varphi+v}{1-v} + \frac{1}{\sigma}\right)$$

$$y_{i,T}^p = \frac{(1+\varphi)}{k_2(1-v)} \mathrm{a}_{i,T} - \frac{1}{k_2} t_{i,T} + \frac{1}{k_2} \log[\frac{(1-v)\alpha_c}{\alpha_l}] + \frac{z_2}{k_2\sigma} inv_{i,T} + \frac{z_3}{k_2\sigma} g_{i,T} + \frac{z_4}{k_2\sigma} exp_{i,T}$$

So, the inflation rate depends on its anticipated value for the future period. It is also proportional to the log deviation of output from its flexible counterpart (output gap), to the excess of the production supply in comparison with the natural level of output if prices were flexible. Equation (26) is then the simplest form of the New-Keynesian Phillips curve. However, this equation could still be complicated by the introduction of inflation inertia, or the distinction between nominal wages and prices stickiness.

- ($k_1 k_2$) is an indicator of price flexibility. This parameter decreases with the indicator of price-stickiness and the parameter (α), the longer the average time interval between price changes. It increases with the measure of decreasing returns in the production function (v). In the same way, price flexibility decreases with the inter-temporal elasticity of substitution in private consumption (σ), and with the time discount factor (β). Finally, it increases with the dis-utility, for households, of labour supply (φ), as labour supply is then elastic to the level of the real wage.
- ($y_{i,t}^p$) is a positive productivity or deflationary supply shock increasing the potential output. It corresponds to a mark-up shock unrelated to variations in economic activity, decreasing marginal costs. It can depend on frictions in the wage contracting process, pushing real wages away from their equilibrium values. It increases with positive productivity shocks, with positive shocks on investment, on public expenditure or on net exports, but it decreases with taxation rates. For example, such a supply shock could follow a path like the following: $y_{i,t}^p = \xi y_{i,t-1}^p + \varepsilon_i^s \qquad \varepsilon_i^s \sim idd(0, \sigma_{\varepsilon^s}^2) \quad 0 < |\xi| < 1$.

2.3. Preferences of the Central Bank

The preferences of the central bank can be given micro-foundations. The objective function is then derived as a second-order Taylor series approximation to the level of expected life-time utility function of the representative household. As mentioned by Svensson (1999), the corresponding optimal interest rate will then be derived from the assignment of a loss function to minimize, and we will define an optimal targeting rule, and not an instrument rule expressing the interest rate as a prescribed function of predetermined and/or forward-looking variables of the model. The optimal interest rate will be the endogenous result of the optimization of a loss function, and many useful information can intervene as intermediate variables. Therefore, we will consider in this paper that the welfare objective is a decreasing function of quadratic variations in the inflation rate or in economic activity in comparison with their optimal values Woodford (2003).

We suppose that the central bank's inter-temporal loss minimization problem is realized in the framework of a quadratic loss function with 'flexible inflation targeting'. This term was first mentioned by Svensson (1999), in order to define a monetary policy which would not only be concerned with inflation stabilization, but which also attaches a non-negligible weight to output stabilization. Svensson (2000) mentions that 'flexible inflation targeting' is usually more efficient than a monetary policy of strict inflation targeting, as it induces less variability in variables other than inflation, by effectively targeting inflation at a longer horizon. Such a modelling is adopted in Billi (2007) or Jung *et al.* (2005), for example, also taking into account the non-negativity constraint on nominal interest rates. The instrument of the central bank is the short term nominal interest rate, and it chooses the path of all future nominal interest rates $\{i_0, i_1, \ldots\}$ in order to minimize the following loss function:

$$Min \sum_{t=0}^{\infty} \beta^t E_t[(\pi_{i,t} - \pi^*)^2 + \lambda(x_{i,t} - x^*)^2 + \mu(i_t - i_{t-1})^2] \qquad (27)$$

Where: (0<β<1) is the time discount factor,

(λ>0) and (μ>0) are the respective weights given to the goals of stabilizing the output gap and smoothing variations in interest rates in comparison with the goal of targeting an inflation level normalized to unity.

(π^*) is the optimal inflation target chosen by the monetary authority. We suppose that this target must be above zero, and high enough in order that the central bank can reach this goal with a minimized risk of violating the Zero Lower Bound constraint. The target must be high enough to avoid the risk that a bad shock would push the economy in a deflationary spiral which could exacerbate welfare losses for representative households, instead of making the economy revert to a stable equilibrium. Therefore, there is a kind of 'inflation bias' necessary in the context of the ZLB.

Besides, the central bank tries to stabilize the output-gap, to limit under-utilization of resources and deviations of real economic activity from its natural level, which is the efficient level corresponding to the productive potential of the economy. However, Woodford (1999) mentions that the output-gap target (x^*) is then also usually positive. Indeed, the natural rate of output can be considered as inefficiently low, for example because of the small amount of market power held by producers of differentiated goods, or because of the delays in prices adjustments. So, the social optimum regarding the output level may exceed its natural level because of distortions such as imperfect competition or taxes.

Finally, we suppose that the central bank smoothes variations in interest rates, in conformity with empirical observations of interest rate smoothing by central banks. This smoothing allows a kind of approximation of a commitment solution, according to Billi (2007) or Jung *et al.* (2005). In the same way, Woodford (1999) mentions that an inertial monetary policy can be a substitute to commitment for the monetary authority, while it is at the same time highly representative of the empirical behaviour of central banks. Indeed, the author provides evidence that interest rate changes by the FED are only moderate, and that adjustment of these rates to new economic conditions are only gradually and progressively realized ('gradualism'). This is due to the fact that a central bank cannot directly influence private-agents expectations. However, in the absence of commitment, an inertial monetary policy rule is the best way in order that private expectations of future policy actions adjust in an appropriate way in response to shocks. Svensson and Woodford (2003) or Giannoni and Woodford (2005) also underline that history-dependence and inertia in monetary policy are efficient ways to be closer to the optimal monetary policy. In the same way, Clarida *et al.* (1999) mention that uncertainty about structural parameters and about the efficiency of monetary policy may also explain the limited variations in interest rates.

The last condition to integrate in our modelling is: ($i_t \geq 0$), the Zero Lower Bound on the nominal interest rate, the non-negativity constraint on short term nominal interest rates, also introduced in Adam and Billi (2007), Billi (2007), Jung *et al.* (2005), Eggertsson and Woodford (2003), Woodford (1999) or Nakov (2008) Indeed, nobody would choose to hold assets bearing a negative return whereas they can hold money bearing a zero nominal return. This constraint only means that the marginal utility of money holding and of real monetary balances cannot be negative for a representative consumer. It results from the availability of cash as a riskless, perfectly liquid zero-return asset. This ZLB condition is widely admitted in the economic literature, even when theoretically, it is an implication of transaction and storage cost properties of the medium of exchange, as mentioned and theoretically discussed by McCallum (2000). The validity of this condition depends upon the assumption that it is costless at the margin to store money, the economy's medium of exchange. While some central banks, such as the European Central Bank (ECB) and the Swiss National Bank (SNB) have been able to introduce slightly negative nominal rates, there is clearly a limit to how negative the nominal rate can be before savers turn to cash. Hence, while the true bound might not be exactly zero, it is likely to be some small negative number.

This additional constraint is not taken into account in most papers on DSGE New-Keynesian models. That's why the main contribution of the current paper is to consider explicitly this constraint in the resolution of the model derived from the previous equations, which implies a major non-linearity in the model. We can also mention that according to equation (14), the average real interest rate is positive according to the time preference of the representative household ($\beta<1$), at least in the long run. So, the nominal interest rate should remain positive with small enough disturbances, and the ZLB should not be binding. However, the real interest rate can become negative if public expenditure, investment or exports are temporary very small, or if the productivity temporary increases.

2.4. Calibration of the Parameters of the Model

Before turning to the implications of our model, we have to give estimations of the parameters mentioned in the previous sections.

- The time discount factor (β) is unanimously calibrated at 0.99 in the economic literature.
- The real-rate elasticity of the output-gap (σ) is calibrated at 6.25 by Billi (2007), Adam and Billi (2007), Jung *et al.* (2005) or Woodford (1999). It is calibrated at 1 by Clarida *et al.* (2000) or Galí (2008), at 4 by Nakov (2008). We will retain the following parameter: σ=6.25.
- The slope of the Phillips curve, the demand elasticity of prices (k_1k_2), is calibrated at 0.024 by Billi (2007), Adam and Billi (2007), Jung *et al.* (2005) or Woodford (1999). It is calibration at 0.11 by Christiano *et al.* (2010). We will retain the following value: k_1k_2=0.024.
- The weight of stabilizing the output gap (λ) is calibrated at 0.003 by Billi (2007), Adam and Billi (2007) or Jung *et al.* (2005). It is calibrated at 0.048 by Woodford (1999) or Giannoni and Woodford (2002). We will retain the following value: λ=0.01.
- The weight of interest rate smoothing (μ) is calibrated at 0.236 by Woodford (1999) or Giannoni and Woodford (2002). We will retain the following value: μ=0.236.

3. Resolution of the Model

We differentiate the Lagrangian in order to obtain the first-order Kuhn-Tucker conditions, as in Jung *et al.* (2005), Woodford (1999), Eggertsson and Woodford (2003) or Nakov (2008). In this context, the contribution of the current paper is to define analytically the solution of the previous model by introducing explicitly the ZLB constraint for the interest rate, which is hardly the case in the economic literature. The analytical solution of the model is then much more detailed in our paper than in many previous contributions (see the Appendixes).

So, using equations (13), (26) and (27), in period (T), the central bank chooses a path for the short term interest rate minimizing the following loss function:

$$Lagrangian = \mathcal{L}_{i,T} = E_T \sum_{t=T}^{\infty} \beta^t \{[(\pi_{i,t} - \pi^*)^2 + \lambda(x_{i,t} - x^*)^2 + \mu(i_t - i_{t-1})^2]$$
$$+z_{i1,t}[x_{i,t} - x_{i,t+1} + \sigma(i_t - \pi_{i,t+1} - \overline{r_{i,t}})]$$
$$+z_{i2,t}[\pi_{i,t} - \beta\pi_{i,t+1} - k_1 k_2 x_{i,t}] + z_{i3,t}[i_t - 0]\} \quad (28)$$

3.1. Steady State Values and Long-Term Equilibrium

The steady state values of the endogenous variables of the model are the values to which the economy converges under optimal policy if there were no uncertainty. They are the constant values that would satisfy the basic equations of the model in all periods.

$$Lagrangian = \mathcal{L}_{i,\infty} = (\pi_{i,\infty} - \pi^*)^2 + \lambda(x_{i,\infty} - x^*)^2 + z_{i1,\infty}\sigma(i_\infty - \pi_{i,\infty} - \overline{r_{i,\infty}})$$
$$+z_{i2,\infty}[(1-\beta)\pi_{i,\infty} - k_1 k_2 x_{i,\infty}] + z_{i3,\infty} i_\infty \quad (29)$$

The first order conditions of optimality then imply:

$$\begin{cases} \frac{\partial \mathcal{L}_{i,\infty}}{\partial \pi_{i,\infty}} = 2(\pi_{i,\infty} - \pi^*) - \sigma z_{i1,\infty} + (1-\beta) z_{i2,\infty} = 0 \\ \frac{\partial \mathcal{L}_{i,\infty}}{\partial x_{i,\infty}} = 2\lambda(x_{i,\infty} - x^*) - k_1 k_2 z_{i2,\infty} = 0 \\ \frac{\partial \mathcal{L}_{i,\infty}}{\partial i_\infty} = \sigma z_{i1,\infty} + z_{i3,\infty} = 0 \end{cases}$$
$$\begin{cases} (\pi_{i\infty} - i_\infty + \overline{r_{i,\infty}}) z_{i1,\infty} = 0 \\ x_{i,\infty} = \frac{(1-\beta)\pi_{i,\infty}}{k_1 k_2} \\ z_{i3,\infty} i_\infty = 0 \qquad i_\infty \geq 0 \end{cases} \quad (30)$$

Therefore, there is not a unique but two Rational Expectations Equilibriums.

- If the Zero Lower Bound is binding in the long term equilibrium, we have (i_∞ =0) and ($\sigma z_{i1,\infty} = -z_{i3,\infty} \neq 0$). $\pi_{i,\infty} = -\overline{r_{i,\infty}} = -\frac{(1-\beta)}{\beta} < 0$ is then negative, as well as ($x_{i,\infty}$). It is the so-called 'Friedman equilibrium'. However, this situation is not optimal. It is a deflationary situation, and the loss function of the central bank (28) is then strictly positive, even if the inflation and activity targets are null, as soon as the long term real interest rate is non null. The deflationary process is then self-fulfilling: economic agents anticipate a decrease in economic activity and in prices, the interest rate is brought to zero in order to avoid this deflationary spiral, but this monetary policy is not sufficient to escape the deflationary situation.

Therefore, we will suppose that the long term real interest rate is sufficiently large to avoid the convergence towards the former solution. On the contrary, we will concentrate on the long term equilibrium where the ZLB is not binding.

- If the Zero Lower Bound is not binding in the long term equilibrium, we have (i_∞>0) and ($z_{i3,\infty} = z_{i1,\infty} = 0$). Therefore, the system (30) implies:

$$\pi_{i,\infty} = \frac{k_1 k_2 [k_1 k_2 \pi^* + \lambda(1-\beta) x^*]}{[\lambda(1-\beta)^2 + k_1^2 k_2^2]}$$
$$x_{i,\infty} = \frac{(1-\beta)\pi_{i,\infty}}{k_1 k_2} = \frac{(1-\beta)[k_1 k_2 \pi^* + \lambda(1-\beta) x^*]}{[\lambda(1-\beta)^2 + k_1^2 k_2^2]} \quad (31)$$

The loss function is then the following:

$$L_{i,\infty} = (\pi_{i,\infty} - \pi^*)^2 + \lambda(x_{i,\infty} - x^*)^2 = \lambda \frac{[\lambda(1-\beta)^2 + k_1^2 k_2^2][k_1 k_2 x^* - (1-\beta)\pi^*]^2}{[\lambda(1-\beta)^2 + k_1^2 k_2^2]^2} \quad (32)$$

So, this loss function is smaller than if the ZLB is binding, and therefore, it is the first best outcome. This loss function can even be minimized and null if:

$$\pi^* = \frac{k_1 k_2}{(1-\beta)} x^* \quad (33)$$

The inflation target must then be positive to reach the desired activity target, in order to increase output to a more efficient level. All long term optimal plans must verify the previous equation; therefore, the optimal inflation target depends on the desired level of output.

3.2. Equilibrium with Discretion

Firstly, we suppose that the central bank cannot commit to future policies, and chooses the current interest rate by re-optimizing every period. In choosing its optimal monetary policy, the central bank takes private sector expectations as given. The economic policy is then ‘time consistent’, as rational expectations imply that the central bank has no incentive to change its plans in an unexpected way. Future expectations about inflation, output and interest rate cannot be manipulated by the central bank; they are independent from current actions. So, the discretionary problem is reduced to a sequence of static optimization problems in which the central bank minimizes current period losses. As mentioned by Clarida *et al.* (1999), this case is the closest to reality, as no major central bank makes any type of binding commitment over the future course of its monetary policy. However, in the loss function (27) of our model, smoothing of interest rates constrains future short term interest rates. Besides, discretion usually results in a ‘stabilization bias’, as private sector expectations cannot be manipulated in order to improve the short-run trade-off between stabilizing output and inflation.

The optimal first order conditions of the system (28) are then as follows:

$$\begin{cases} \frac{\partial \mathcal{L}_{i,0}}{\partial \pi_{i,t}} = 2(\pi_{i,t} - \pi^*) + z_{i2,t} = 0 \\ \frac{\partial \mathcal{L}_{i,0}}{\partial x_{i,t}} = 2\lambda(x_{i,t} - x^*) + z_{i1,t} - k_1 k_2 z_{i2,t} = 0 \\ \frac{\partial \mathcal{L}_{i,0}}{\partial i_{i,t}} = 2\mu(i_t - i_{t-1}) + \sigma z_{i1,t} + z_{i3,t} = 0 \end{cases}$$

$$\begin{cases} \{x_{i,t} - E_t(x_{i,t+1}) + \sigma[i_t - E_t(\pi_{i,t+1}) - \overline{r_{i,t}}]\}\, z_{i1,t} = 0 \\ \pi_{i,t} - \beta E_t(\pi_{i,t+1}) - k_1 k_2 x_{i,t} = 0 \\ z_{i3,t}\mathrm{i}_t = 0 \qquad z_{i3,t} \geq 0 \qquad i_t \geq 0 \end{cases} \quad (34)$$

We can mention that we could neglect the constraint related to the demand equation (13) and the parameter ($z_{i1,t}$). Indeed, as soon as the supply equation (26) is verified and implies a link between the variation in the inflation rate and in the output gap, the demand equation can provide the interest rate corresponding to these levels of inflation and economic activity. For example, Evans and Honkapohja (2006), Clarida *et al.* (1999) or Svensson and Woodford (2003)avoid the demand equation (13) and the non-negativity constraint for the interest rate in the Lagrange function. Indeed, in the modelling of these authors, equation (13) is only used afterwards to define the interest rate rule and the optimal monetary reaction function. However, in the current paper, we have chosen to include these constraints and to distinguish between cases where the ZLB is or is not binding.

We suppose that the ZLB is binding for the (N-1) first periods (section 4), but that it is no longer binding afterwards (section 5).

4. Optimal Equilibrium When the ZLB is binding

4.1. Optimal Levels of Economic Activity and Inflation

When the ZLB is binding, we have: (i_T=0) and ($z_{i3,T}\geq 0$), for the T<N first periods. As the ZLB is binding, monetary policy is constrained and it is not optimal regarding demand. So, we have ($z_{i1,T}$>0). Besides, in a given period (T), first order conditions (34) are reduced to:

$$\begin{cases} z_{i1,T} = -2k_1 k_2(\pi_{i,T} - \pi^*) - 2\lambda(x_{i,T} - x^*) \\ z_{i2,T} = -2(\pi_{i,T} - \pi^*) \\ z_{i3,T} = -\sigma z_{i1,T} = 2k_1 k_2 \sigma(\pi_{i,T} - \pi^*) + 2\lambda\sigma(x_{i,T} - x^*) \end{cases}$$

$$\begin{cases} x_{i,T} = E_T(x_{i,T+1}) + \sigma E_T(\pi_{i,T+1}) + \sigma\overline{r_{i,T}} \\ \pi_{i,T} = \beta E_T(\pi_{i,T+1}) + k_1 k_2 x_{i,T} \end{cases} \quad (35)$$

The two last equations of the system (35) also imply:

$$\pi_{i,T} = (\beta + \sigma k_1 k_2) E_T(\pi_{i,T+1}) + k_1 k_2 E_T(x_{i,T+1}) + \sigma k_1 k_2 \overline{r_{i,T}}$$

Therefore, the system to solve is the following:

$$\begin{pmatrix} x_{i,T} \\ \pi_{i,T} \end{pmatrix} = \begin{pmatrix} 1 & \sigma \\ k_1 k_2 & (+\sigma k_1 k_2) \end{pmatrix} \begin{pmatrix} E_T(x_{i,T+1}) \\ E_T(\pi_{i,T+1}) \end{pmatrix} + \sigma \begin{pmatrix} 1 \\ k_1 k_2 \end{pmatrix} \overline{r_{i,T}} \quad (36)$$

Solving this equation (see Appendix A), we obtain:

$$x_{i,T} = a_{LB,N-T} E_T(x_{i,N}) + b_{LB,N-T} E_T(\pi_{i,N}) + \sigma \sum_{n=T}^{N-1} (a_{LB,n-T} + k_1 k_2 b_{LB,n-T}) \overline{r_{i,n}}$$

$$\pi_{i,T} = \frac{k_1 k_2}{\sigma} b_{LB,N-T} E_T(x_{i,N}) + d_{LB,N-T} E_T(\pi_{i,N}) + k_1 k_2 \sum_{n=T}^{N-1} (b_{LB,n-T} + \sigma d_{LB,n-T}) \overline{r_{i,n}} \quad (37)$$

$$with: \quad b_{LB,n} = \frac{\sigma}{\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} (r_{LB,1}^n - r_{LB,2}^n)$$

$$a_{LB,n} = \left[\frac{(1 - \beta - \sigma k_1 k_2)}{2\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} + \frac{1}{2}\right] r_{LB,1}^n + \left[\frac{(-1 + \beta + \sigma k_1 k_2)}{2\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} + \frac{1}{2}\right] r_{LB,2}^n$$

$$d_{LB,n} = \left[\frac{(-1+\beta+\sigma k_1 k_2)}{2\sqrt{(1+\beta+\sigma k_1 k_2)^2-4\beta}}+\frac{1}{2}\right] r_{LB,1}^{n} + \left[\frac{(1-\beta-\sigma k_1 k_2)}{2\sqrt{(1+\beta+\sigma k_1 k_2)^2-4\beta}}+\frac{1}{2}\right] r_{LB,2}^{n}$$

$$r_{LB,1} = \frac{(1+\beta+\sigma k_1 k_2)}{2}+\frac{\sqrt{(1+\beta+\sigma k_1 k_2)^2-4\beta}}{2}$$

$$r_{LB,2} = \frac{(1+\beta+\sigma k_1 k_2)}{2}-\frac{\sqrt{(1+\beta+\sigma k_1 k_2)^2-4\beta}}{2}$$

The values of output and inflation at date (N) of exit of the ZLB then influence the current inflation and economic activity at date (T); these expectations can sustain current economic growth and inflation. In the same way, postponing the date of exit of the ZLB (increasing N) also sustains current economic activity and inflation. Indeed, the factors ($a_{LB,N-T}$), ($b_{LB,N-T}$) and ($d_{LB,N-T}$) increase with the horizon and the date (N).

4.2. Stabilization of Demand Shocks

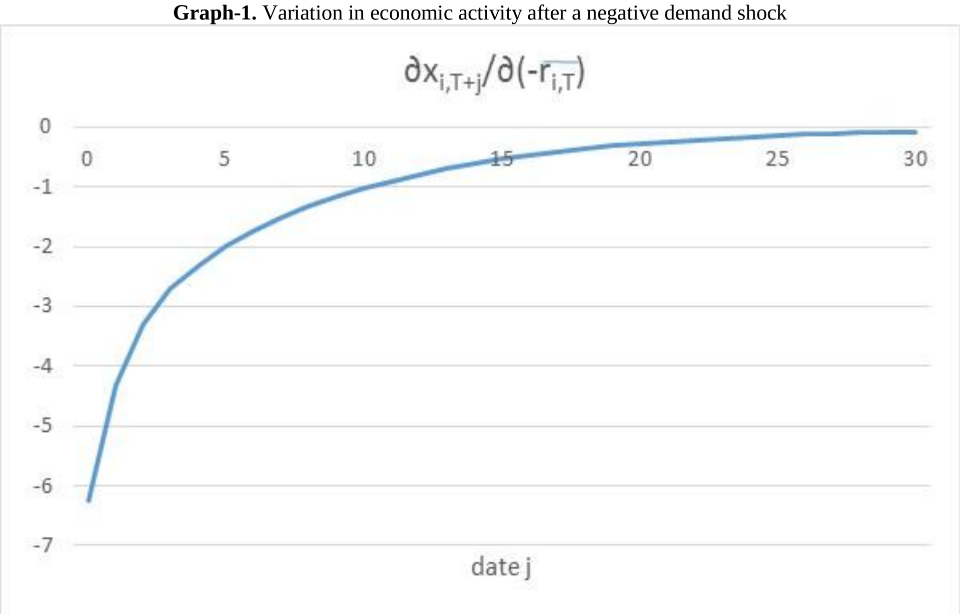
The ZLB is binding regarding the stabilization of negative demand shocks if ($\overline{r_{i,t}} < 0$)[2]. Furthermore, in case of negative demand shocks, with: $\sigma\overline{r_{i,t}} = \sigma\rho\overline{r_{i,t-1}} + \varepsilon_i^d$, we obtain the following relations:

$$\frac{\partial x_{i,T+j}}{\partial(-\overline{r_{i,T}})} = -\sigma\left(a_{LB,j}+k_1 k_2 b_{LB,j}\right)\rho^j \qquad j = 0, 1, \dots, N-T-1$$

$$\frac{\partial \pi_{i,T+j}}{\partial(-\overline{r_{i,T}})} = -k_1 k_2\left(b_{LB,j}+\sigma d_{LB,j}\right)\rho^j \qquad j = 0, 1, \dots, N-T-1 \qquad (38)$$

With the basic calibration of our model (σ=6.25, k_1k_2=0.024, β=0.99) and a persistence of demand shocks: ρ=0.6, we have therefore the following graphs:

Graph-1. Variation in economic activity after a negative demand shock

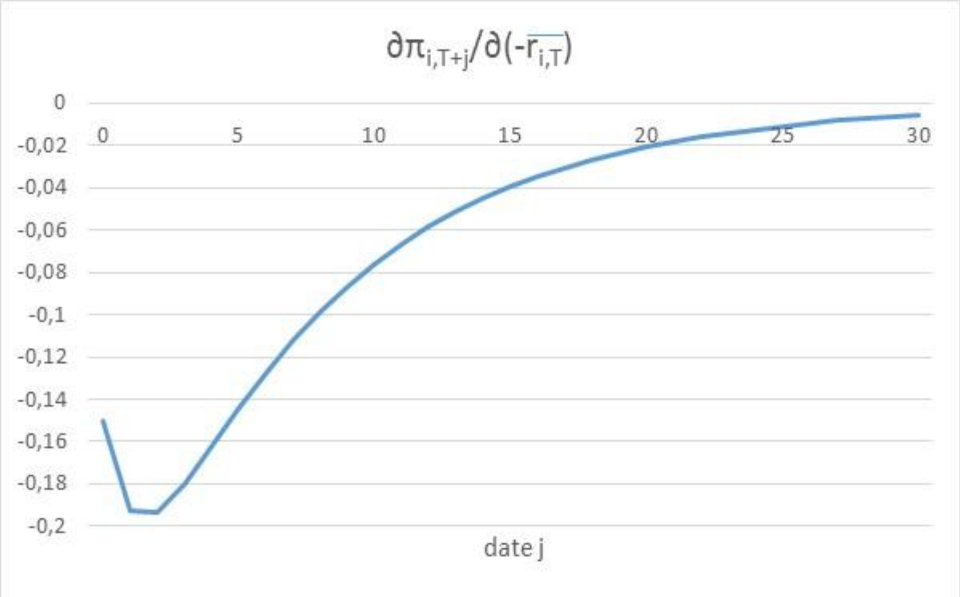


[2] We can mention that equations (14) and (26) imply:

$$\overline{r_{i,T}}$$

$$+\frac{(1-\beta)}{\beta}-\frac{1}{(\varphi\sigma+\nu\sigma+1-\nu)}\{(1+\varphi)[a_{i,T}-E_T(a_{i,T+1})]-(1-\nu)[t_{i,T}$$

Graph- 2. Variation in inflation after a negative demand shock



So, demand shocks cannot perfectly be stabilized by monetary policy, as the interest rate is null because of the ZLB. After the recession and the negative demand shock, economic activity and inflation progressively increase, provided the persistence of demand shocks (ρ) is not too high. Indeed, negative demand shocks can lead to unlimited decrease in activity and in prices if $\rho \geq 0.69$, with the basic calibration of our parameters. Besides, in case of negative demand shocks, the levels of economic activity and inflation are the weakest if the intertemporal elasticity of substitution of household expenditure (σ) is high, if the time discount factor (β) is high, or if price flexibility ($k_1 k_2$) is high. So, quite counter-intuitively, price flexibility can exacerbate economic difficulties, whereas on the contrary, price rigidity can dampen deflation and mitigate the depression. In the same way, Adam and Billi (2007) mention that if the equilibrium real interest rate is very low (negative demand shock), the ZLB strongly increases welfare losses in case of discretionary monetary policy; therefore, taking into account the ZLB implies that the potential gains from a commitment policy could be very large.

To sum up, as mentioned by Nakov (2008), a negative equilibrium interest rate ($\overline{r_{i,T}}$) and a negative demand shock imply a decrease in the current economic activity level and deflationary tensions, which cannot be avoided by monetary policy as the nominal interest rate can no longer decrease. The deflationary situation then becomes independent of current policy actions. If private agents continue to anticipate this decrease in activity and in prices, nothing can avoid this deflationary spiral. However, as soon as private agents' expectations about the equilibrium real interest rate become positive, current economic activity and prices can be stabilized again, as monetary policy (interest rates increases) becomes again efficient.

5. Optimal Equilibrium When the ZLB Is Not Binding

When the ZLB is not binding, we have: ($i_T > 0$) and ($z_{i3,T} = 0$), for the periods ($T \geq N$). As monetary policy is not constrained, this policy is optimal regarding the demand equation. So, first order conditions (34) are reduced to:

$$\begin{cases} z_{i1,T} = -2k_1k_2(\pi_{i,T} - \pi^*) - 2\lambda(x_{i,T} - x^*) = -\dfrac{2\mu}{\sigma}(i_T - i_{T-1}) \\ z_{i2,T} = -2(\pi_{i,T} - \pi^*) \end{cases}$$

$$\begin{cases} 0 < i_T = i_{T-1} + \dfrac{\sigma k_1 k_2}{\mu}(\pi_{i,T} - \pi^*) + \dfrac{\sigma\lambda}{\mu}(x_{i,T} - x^*) \\ x_{i,T} - E_T(x_{i,T+1}) + \sigma[i_{i,T} - E_t(\pi_{i,T+1}) - \overline{r_{i,T}}] = 0 \\ \pi_{i,T} - \beta E_T(\pi_{i,T+1}) - k_1 k_2 x_{i,T} = 0 \end{cases} \quad (39)$$

The three last equations of the system (39) and equation (33) imply:

$$\pi_{i,T} = (\beta + \sigma k_1 k_2)E_T(\pi_{i,T+1}) + k_1 k_2 E_T(x_{i,T+1}) - \sigma k_1 k_2 i_T + \sigma k_1 k_2 \overline{r_{i,T}}$$

$$i_T = i_{T-1} + \frac{\sigma k_1 k_2}{\mu}\pi_{i,T} - \frac{\sigma[k_1^2 k_2^2 + \lambda(1-\beta)]}{\mu(1-\beta)}x^* + \frac{\sigma\lambda}{\mu}x_{i,T}$$

$$(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)i_T = \mu i_{T-1} + \sigma(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda)E_T(\pi_{i,T+1})$$
$$+\sigma(k_1^2 k_2^2 + \lambda)E_T(x_{i,T+1}) + \sigma^2(k_1^2 k_2^2 + \lambda)\overline{r_{i,T}} - \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{(1-\beta)}x^*$$

The system to solve is then the following:

$$\begin{pmatrix} x_{i,T} \\ \pi_{i,T} \\ i_{i,T-1} \end{pmatrix} = \begin{pmatrix} 1 & \sigma & -\sigma \\ k_1 k_2 & (\beta + \sigma k_1 k_2) & -\sigma k_1 k_2 \\ -\frac{\sigma}{\mu}(k_1^2 k_2^2 + \lambda) & -\frac{\sigma}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda) & \frac{1}{\mu}(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2 \lambda) \end{pmatrix} \begin{pmatrix} E_T(x_{i,T+1}) \\ E_T(\pi_{i,T+1}) \\ i_T \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ \sigma k_1 k_2 & 0 \\ -\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) & \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \end{pmatrix} \begin{pmatrix} \overline{r_{i,T}} \\ x^* \end{pmatrix} \quad (40)$$

5.1. Determination of an Optimal Targeting Rule

Solving the previous system (40), we obtain (see Appendix B):

$$x_{i,T} = a_k E_T(x_{i,T+k}) + b_k E_T(\pi_{i,T+k}) + c_k E_T(i_{T+k-1}) + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=T}^{T+k-1} c_{n-T}\, x^* + \sum_{n=T}^{T+k-1} \left[a_{n-T}\sigma + b_{n-T}\sigma k_1 k_2 - c_{n-T}\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) \right] \overline{r_{i,n}} \quad (41)$$

$$\pi_{i,T} = d_k E_T(x_{i,T+k}) + e_k E_T(\pi_{i,T+k}) + f_k E_T(i_{T+k-1}) + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=T}^{T+k-1} f_{n-T}\, x^* + \sum_{n=T}^{T+k-1} \left[d_{n-T}\sigma + e_{n-T}\sigma k_1 k_2 - f_{n-T}\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) \right] \overline{r_{i,n}} \quad (42)$$

$$i_{T-1} = g_k E_T(x_{i,T+k}) + h_k E_T(\pi_{i,T+k}) + j_k E_T(i_{T+k-1}) + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=T}^{T+k-1} j_{n-T}\, x^* + \sum_{n=T}^{T+k-1} \left[g_{n-T}\sigma + h_{n-T}\sigma k_1 k_2 - j_{n-T}\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) \right] \overline{r_{i,n}} \quad (43)$$

Where the values of the parameters (a_n) until (j_n) are mentioned in Appendix B.

So, for example, if we choose a one-year horizon, equation (43) implies[3]:

$$i_T = \frac{\mu}{(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)} i_{i,T-1} + \frac{\sigma(k_1^2 k_2^2 + \lambda)}{(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)} E_T(x_{i,T+1}) + \frac{\sigma^2(k_1^2 k_2^2 + \lambda)}{(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)} \overline{r_{i,T}} + [1 + \frac{(\sigma\beta k_1 k_2 - \mu)}{(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)}] E_T(\pi_{i,T+1}) - \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)(1-\beta)} x^* \quad (44)$$

So, as mentioned by Giannoni and Woodford (2002), the simple Taylor rule is not fully optimal. A robustly optimal policy rule is almost inevitably an implicit rule, which requires the central bank to use a structural model to project the economy's evolution under the contemplated policy action. Svensson (1999) argues that according to ‘flexible inflation targeting’, the instrument of the central bank, the interest rate, should not only respond to current main targeted variables, output and inflation, as in the Taylor rule. A targeting rule must use all available information, in particular regarding expected future inflation or economic activity, as intermediate targets, contrary to an instrument rule. He qualifies of ‘inflation-forecast targeting’ a rule like the one that we have found above, where expectations are considered as intermediate targets for the central bank. However, we have here only an implicit reaction function or targeting rule, without link between anticipated future variables and current variables.

As in Clarida *et al.* (1999) or Evans and Honkapohja (2003b), equation (44) shows that in response to a rise in expected future inflation, the nominal interest rate should rise more than proportionately in order to increase real interest rates [the coefficient for expected inflation is above unity; ‘Taylor prescription’], and to decrease global demand. This reaction more than proportional allows an increase also in the real interest rate, and the better stabilization of prices and economic activity levels. It avoids the existence of self-fulfilling sunspot equilibriums, with a decline in the real interest rate and an outburst in inflation. This can explain the success of the US monetary policy after 1979, in the Volcker-Greenspan era. Indeed, econometrical estimations show that the coefficient of expected inflation then became superior to one (around 2), whereas it was below unity before 1979, when monetary policy was too accommodative Clarida *et al.* (2000). Indeed, when the Taylor principle is not satisfied, an increase

[3] For k=1, equation (43) implies:

$$i_T = \frac{1}{j_1} i_{i,T-1} - \frac{g_1}{j_1} E_T(x_{i,T+1}) - \frac{h_1}{j_1} E_T(\pi_{i,T+1}) - \frac{1}{j_1}\left[g_0\sigma + h_0\sigma k_1 k_2 - j_0\frac{\sigma^2}{\mu}\left(k_1^2 k_2^2 + \lambda\right)\right] \overline{r_{i,T}} - \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{j_1 \mu(1-\beta)} j_0 x^*.$$

in the nominal interest rate can reduce the real interest rate, and increase both the output gap and self-fulfilling inflationary expectations Christiano *et al.* (2010).

Besides, according to equation (44), the nominal interest rate should also increase in proportion to the increase in the equilibrium real interest rate ($\overline{r_{i,T}}$) in order to stabilize demand shocks. It should decrease with the excess of the expected future economic activity level above its target (x^*) and with the targeted inflation rate (π^*).

5.2. Stabilization of Demand Shocks

The ZLB is not binding regarding the stabilization of positive demand shocks ($\overline{r_{i,t}} > 0$). More precisely, in case of positive demand shocks, with: $\sigma\overline{r_{i,T}} = \sigma\rho\overline{r_{i,t-1}} + \varepsilon_i^d$, we obtain the following relations (see Appendix B):

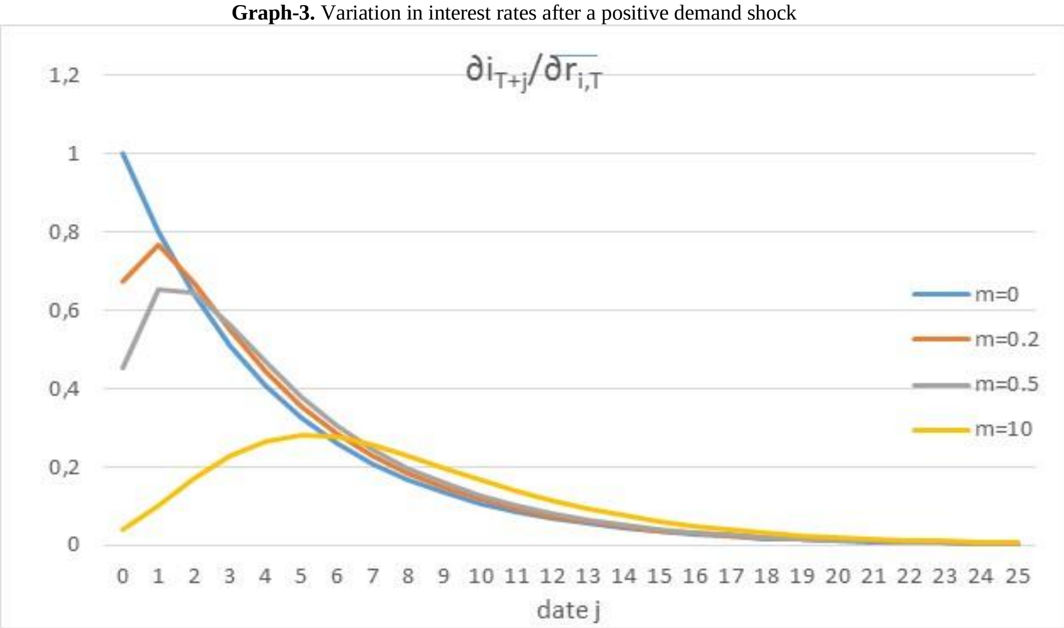
$$\frac{\partial i_{T+k-1}}{\partial \overline{r_{i,T}}} = -\frac{\sigma}{\mu j_k}\sum_{n=0}^{k-1}[\mu g_n + \mu k_1 k_2 h_n - j_n \sigma(k_1^2 k_2^2 + \lambda)]\,\rho^n \qquad for\ k \geq 1 \qquad (45)$$

$$\frac{\partial x_{i,T+k-1}}{\partial \overline{r_{i,T}}} = \frac{\rho^{k-1}\sigma}{\mu}\sum_{n=0}^{k-1}[\mu a_n + \mu k_1 k_2 b_n - c_n \sigma(k_1^2 k_2^2 + \lambda)]\rho^n - \frac{c_k \rho^{k-1}\sigma}{\mu j_k}\sum_{n=0}^{k-1}[\mu g_n + \mu k_1 k_2 h_n - j_n \sigma(k_1^2 k_2^2 + \lambda)]\,\rho^n \qquad (46)$$

$$\frac{\partial \pi_{i,T+k-1}}{\partial \overline{r_{i,T}}} = \frac{\rho^{k-1}\sigma}{\mu}\sum_{n=0}^{k-1}[\mu d_n + \mu k_1 k_2 e_n - f_n \sigma(k_1^2 k_2^2 + \lambda)]\rho^n - \frac{f_k \rho^{k-1}\sigma}{\mu j_k}\sum_{n=0}^{k-1}[\mu g_n + \mu k_1 k_2 h_n - j_n \sigma(k_1^2 k_2^2 + \lambda)]\,\rho^n \qquad (47)$$

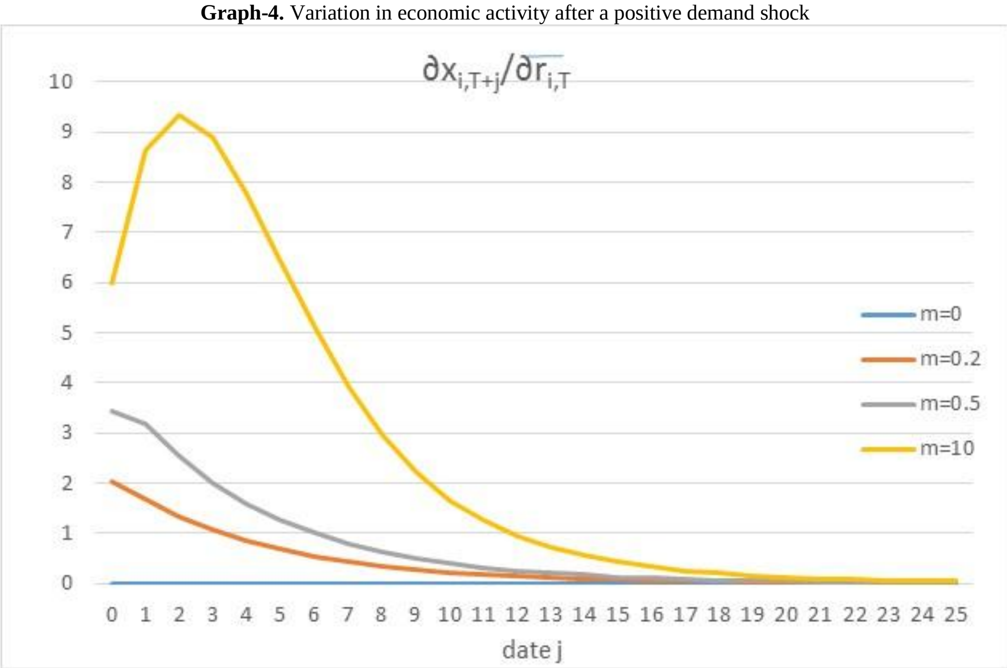
With the basic calibration of our model (σ=6.25, k_1k_2=0.024, β=0.99, λ=0.01), and a persistence of demand shocks corresponding to: ρ=0.8, we have therefore the following graphs:

Graph-3. Variation in interest rates after a positive demand shock

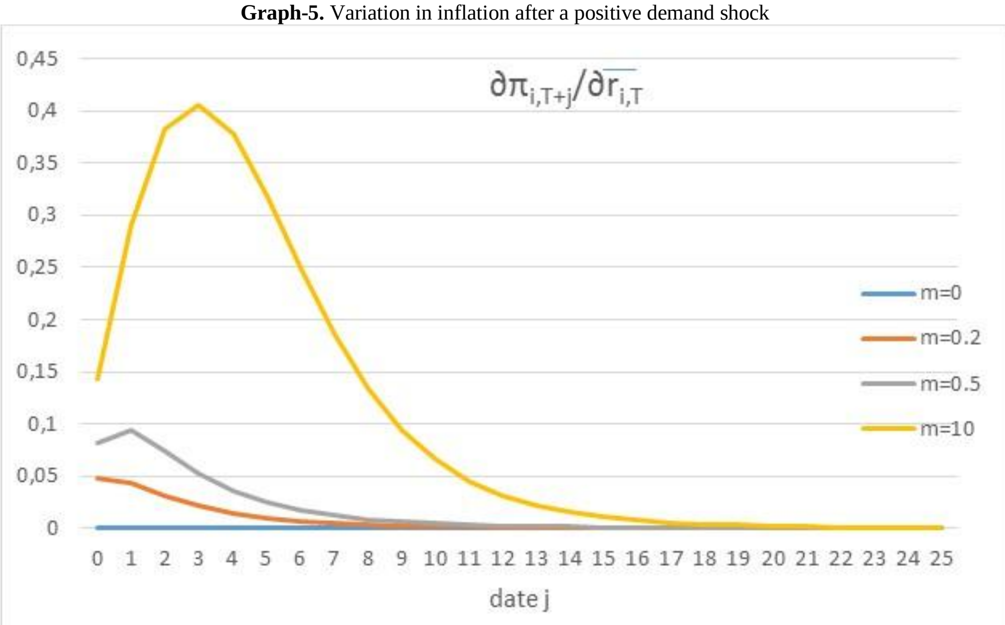


Therefore, obviously, variations in interest rates are all the more accentuated as interest rate smoothing is a more negligible goal for the governments (μ is small). Besides, the increase in interest rates is all the more accentuated as the inter-temporal elasticity of substitution of household expenditure (σ) is high, and as monetary policy is then more efficient in stabilizing economic activity. It is also all the more accentuated as the weight given by the central bank to output stabilization (λ) is high. The progressive decrease in interest rates is afterwards the fastest as the shock persistence (ρ) is small. In these conditions, the stabilization of economic variables depends on the importance of interest rate smoothing (μ) for the governments. Indeed, graphs are as follows:

Graph-4. Variation in economic activity after a positive demand shock



Graph-5. Variation in inflation after a positive demand shock



So, demand shocks can perfectly be stabilized by monetary policy only if variations in interest rates are without costs for the governments (μ=0). Nevertheless, interest rates smoothing (μ>0) implies an imperfect stabilization of economic variables. Economic growth and inflation are then all the more accentuated as the intertemporal elasticity of substitution of household expenditure (σ) is high. Inflation also increases with price flexibility (k_1k_2). Furthermore, economic growth and inflation are also more accentuated if stabilizing the output gap has a weakest weight (λ) for the central bank. Finally, persistence of the demand shock (ρ) extends the duration of the disequilibrium in economic variables.

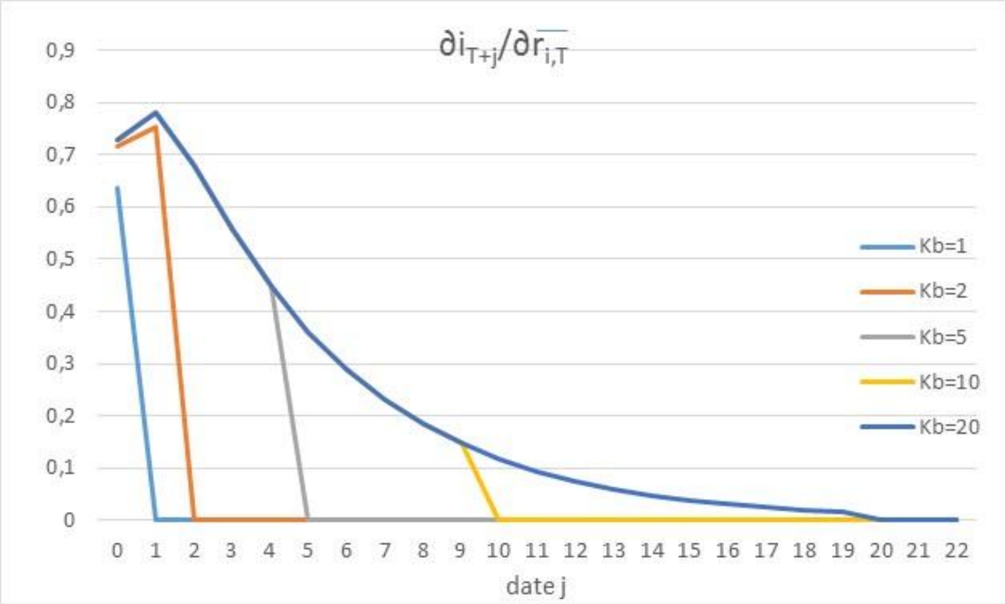
5.3. Consequences of the Dangers of a Future Binding ZLB

Another contribution of the analytical modelling of our paper is to the show the consequences of the following situation: the Zero Lower Bound is currently not binding in period (T), but it is anticipated to be binding from a given period (T+K_b). Indeed, in this situation, we have (see Appendix B):

$$\frac{\partial i_{T+K_b-k}}{\partial \overline{r_{i,T}}} = \begin{cases} -\dfrac{\sigma\rho^{1-k}}{\mu j_{K_b}} \displaystyle\sum_{n=k}^{K_b} [\mu g_{n-1} + \mu k_1 k_2 h_{n-1} - j_{n-1}\sigma(k_1^2 k_2^2 + \lambda)]\rho^{n-1} & for\ 1 \le k \le K_b \\ 0\ \ for\ k \le 0 \end{cases} \qquad (48)$$

With the basic calibration of our parameters (σ=6.25, k_1k_2=0.024, β=0.99, λ=0.01, μ=0.236), and a shock persistence; ρ=0.8, we obtain the following graph:

Graph-6. Effect of the dangers of a future ZLB on optimal interest rate



So, nominal interest rates should be lowered faster in response to positive demand shocks than in the absence of any constraint and if the nominal interest rates were allowed to become negative, in the case where the ZLB is a danger for the efficiency of the future monetary policy. As mentioned by Adam and Billi (2007), such 'preemptive easing' is optimal because expectations of a possibly binding bound in the future create deflationary and recessionary expectations which amplify the effects of adverse shocks on current economic variables ('deflationary bias'). Nakov (2008) also underlines that in case of a negative shock on the equilibrium real interest rate, the 'deflationary bias' can be very large with a discretionary monetary policy. The ZLB then implies a monetary policy which is both too aggressive (limited smoothing) and too expansionary (too low interest rate).

Besides, according to our modelling, monetary policy is looser when the inter-temporal elasticity of substitution of household expenditure (σ) or when the weight given by the central bank to prices stabilization (λ) are weak: the effect of the danger of a future ZLB to faster the decrease in the interest rate is then accentuated. According to Graph 6, the contribution of our modelling is then to show that the decrease in the interest rate must be the fastest as the period where the ZLB is supposed to be binding (K_b) is the nearest.

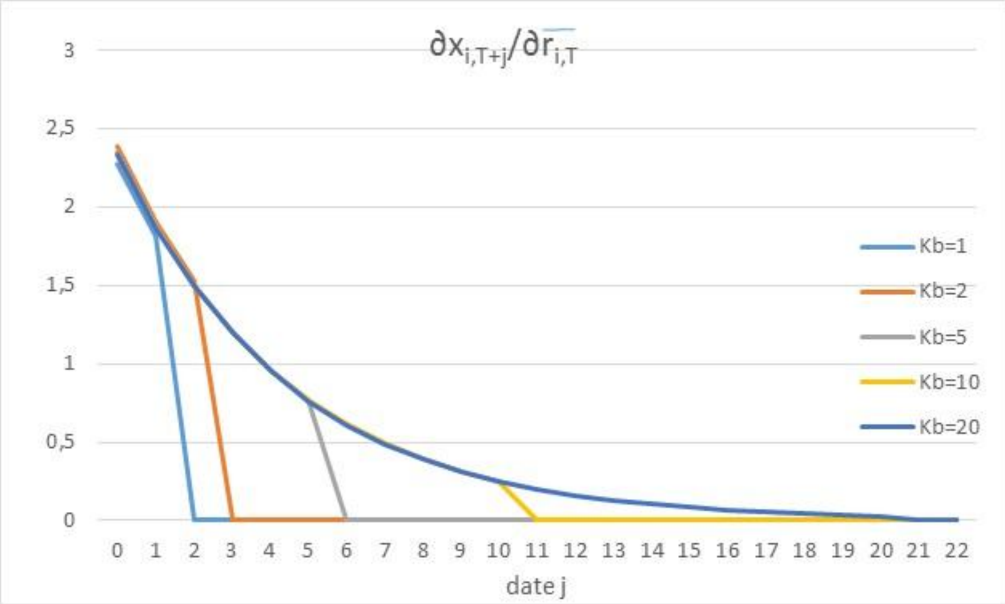
Regarding stabilization of economic variables, we obtain:

$$\frac{\partial x_{i,T+k-1}}{\partial \overline{r_{i,T}}} = \begin{cases} \frac{\sigma\rho^{k-1}}{\mu}\sum_{n=0}^{K_b-1}[\mu a_n + \mu k_1 k_2 b_n - c_n \sigma(k_1^2 k_2^2+\lambda)]\rho^n \\ \quad -\frac{\sigma c_{K_b}\rho^{k-1}}{\mu j_{K_b}}\sum_{n=0}^{K_b-1}[\mu g_n + \mu k_1 k_2 h_n - j_n \sigma(k_1^2 k_2^2+\lambda)]\rho^n \quad for\ k \le K_b+1 \\ 0 \qquad for\ \ k > K_b+1 \end{cases} \qquad (49)$$

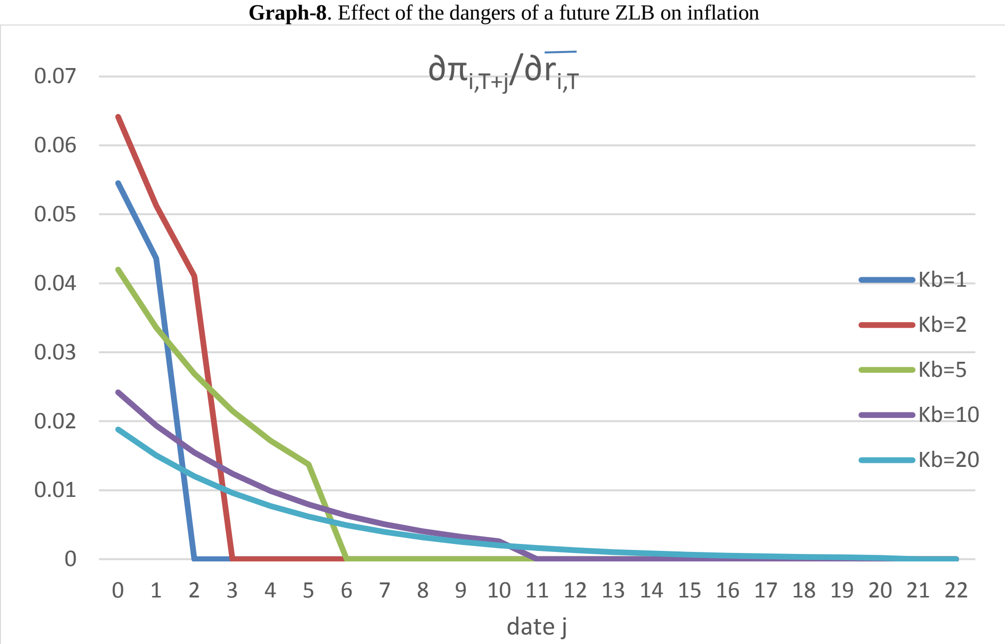
$$\frac{\partial \pi_{i,T+k-1}}{\partial \overline{r_{i,T}}} = \begin{cases} \frac{\sigma\rho^{k-1}}{\mu}\sum_{n=0}^{K_b-1}[\mu d_n + \mu k_1 k_2 e_n - f_n \sigma(k_1^2 k_2^2+\lambda)]\rho^n \\ \quad -\frac{\sigma f_{K_b}\rho^{k-1}}{\mu j_{K_b}}\sum_{n=0}^{K_b-1}[\mu g_n + \mu k_1 k_2 h_n - j_n \sigma(k_1^2 k_2^2+\lambda)]\rho^n \quad for\ k \le K_p+1 \\ 0 \qquad for\ \ k > K_b+1 \end{cases} \qquad (50)$$

So, graphs regarding variations in economic variables are as follows:

Graph-7. Effect of the dangers of a future ZLB on economic activity



Graph-8. Effect of the dangers of a future ZLB on inflation



So, the interest rate is lowered faster in order to offset the deflationary and recessionary consequences of a potential ZLB tomorrow. According to Graphs 7 and 8, the stabilization of economic activity and prices is then obviously speeded-up. Indeed, the dangers of a future ZLB and the lower interest rate reduce expectations of future economic activity and inflation, and this leads to lower current inflation and output.

Therefore, our model shows that in case of a negative demand shock, monetary policy should reduce nominal interest rates more aggressively than suggested by a model without Zero Lower Bound. Rational agents anticipate the possibility of reaching the lower bound in the future, which implies downward pressure on expected output and inflation, and which amplifies the effects of adverse shocks well before the bound is reached. So, the interest rate must be decreased more aggressively and should reach sooner the ZLB. The ZLB also modifies optimal monetary policy in case of non-binding shocks: the interest rate should be increased less strongly (or more reduced) than it would be optimal without ZLB. Such a 'preemptive' monetary policy strongly reduces deflationary and recessionary tensions and welfare losses. Indeed, creating credible future inflationary expectations allows the current inflation rate to be positive even if the real equilibrium interest rate is negative.

6. Conclusion

In a discretionary framework, when the ZLB is binding, a negative demand shock implies a decrease in the current economic activity level and deflationary tensions, which cannot be avoided by monetary policy as the nominal interest rate can no longer decrease. If private agents continue to anticipate this decrease in activity and in prices, nothing can then avoid this deflationary spiral. On the contrary, provided the ZLB is not binding, monetary policy can efficiently contribute to the stabilization of economic activity and inflation in case of demand shocks. The variation in interest rates is then all the more accentuated as interest rate smoothing is a more negligible goal for the central bank. In absence of interest rate smoothing, economic activity and prices could even perfectly be stabilized by monetary policy.

The contribution of our paper is to provide a clear analytical New-Keynesian framework sustaining these results. Besides, our analytical modelling also shows that even if the ZLB is currently not binding, the central bank should take into account the dangers of a potential future binding ZLB. Indeed, the interest rate should be decreased the fastest as a negative demand shock and the possibility to reach the ZLB is anticipated for a nearest future period.

The economic literature has studied the advantages of a commitment monetary policy in order to affect private agents' expectations and to avoid an excessive deflationary spiral when the ZLB risks to be binding. However, our paper shows that the necessity of a 'pre-emptive' active monetary policy can be demonstrated even in a discretionary framework, which has the advantage to be time-consistent and to be in conformity with the empirical practices of independent central banks.

Appendix A: Equilibrium when the ZLB is binding

In a given period T<N, the system to solve is the following:

$$\begin{pmatrix} x_{i,T} \\ \pi_{i,T} \end{pmatrix} = \begin{pmatrix} 1 & \sigma \\ k_1 k_2 & \beta + \sigma k_1 k_2 \end{pmatrix} \begin{pmatrix} E_T(x_{i,T+1}) \\ E_T(\pi_{i,T+1}) \end{pmatrix} + \sigma \begin{pmatrix} 1 \\ k_1 k_2 \end{pmatrix} \overline{r_{i,T}} \qquad (A1)$$

$$\begin{pmatrix} x_{i,T} \\ \pi_{i,T} \end{pmatrix} = \begin{pmatrix} 1 & \sigma \\ k_1 k_2 & \beta + \sigma k_1 k_2 \end{pmatrix}^{N-T} \begin{pmatrix} E_T(x_{i,N}) \\ E_T(\pi_{i,N}) \end{pmatrix} + \sigma \sum_{n=T}^{N-1} \begin{pmatrix} 1 & \sigma \\ k_1 k_2 & \beta + \sigma k_1 k_2 \end{pmatrix}^{n-T} \begin{pmatrix} 1 \\ k_1 k_2 \end{pmatrix} \overline{r_{i,n}}$$

We have then to find the solution of the following matrix:

$$\begin{pmatrix} 1 & \sigma \\ k_1 k_2 & \beta + \sigma k_1 k_2 \end{pmatrix}^{n} = \begin{pmatrix} a_{LB,n} & b_{LB,n} \\ c_{LB,n} & d_{LB,n} \end{pmatrix} \qquad (A2)$$

with: $u_{LB,n} = (1 + \beta + \sigma k_1 k_2) u_{LB,n-1} - \beta u_{LB,n-2}$ as characteristic equation of the matrix.
Besides, we obtain:

$$x_{i,T} = a_{LB,N-T} E_T(x_{i,N}) + b_{LB,N-T} E_T(\pi_{i,N}) + \sigma \sum_{n=T}^{N-1} (a_{LB,n-T} + k_1 k_2 b_{LB,n-T}) \overline{r_{i,n}} \qquad (A3)$$

$$\pi_{i,T} = c_{LB,N-T} E_T(x_{i,N}) + d_{LB,N-T} E_T(\pi_{i,N}) + \sigma \sum_{n=T}^{N-1} (c_{LB,n-T} + k_1 k_2 d_{LB,n-T}) \overline{r_{i,n}} \qquad (A4)$$

$r^2 - (1 + \beta + \sigma k_1 k_2) r + \beta = 0$

The two solutions of this equation are the following:

$$r_{LB,1} = \frac{(1 + \beta + \sigma k_1 k_2)}{2} + \frac{\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}}{2}$$

$$r_{LB,2} = \frac{(1 + \beta + \sigma k_1 k_2)}{2} - \frac{\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}}{2} \qquad (A5)$$

So, solutions of the above recurrent sequence take the following form:

$u_{LB,n} = (x) r_{LB,1}^n + (y) r_{LB,2}^n$

- $a_{LB,0} = a_{LB,1} = 1 \qquad a_{LB,2} = 1 + \sigma k_1 k_2$

$$a_{LB,n} = \left[\frac{(1 - \beta - \sigma k_1 k_2) + \sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}}{2\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} \right] r_{LB,1}^n + \left[\frac{(-1 + \beta + \sigma k_1 k_2) + \sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}}{2\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} \right] r_{LB,2}^n$$

- $b_{LB,0} = 0 \quad b_{LB,1} = \sigma \quad b_{LB,2} = (1 + \beta + \sigma k_1 k_2)\sigma$ etc….

$$b_{LB,n} = \frac{\sigma}{\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} (r_{LB,1}^n - r_{LB,2}^n)$$

- $c_{LB,0} = 0 \quad c_{LB,1} = k_1 k_2 \quad c_{LB,2} = (1 + \beta + \sigma k_1 k_2) k_1 k_2 \qquad c_{LB,n} = \frac{k_1 k_2}{\sigma} b_{LB,n}$

- $d_{LB,0} = 1 \quad d_{LB,1} = \beta + \sigma k_1 k_2 \quad d_{LB,2} = (\beta + 2\sigma k_1 k_2)\beta + (1 + \sigma k_1 k_2)\sigma k_1 k_2$ etc….

$$d_{LB,n} = \left[\frac{(-1 + \beta + \sigma k_1 k_2) + \sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}}{2\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} \right] r_{LB,1}^n + \left[\frac{(1 - \beta - \sigma k_1 k_2) + \sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}}{2\sqrt{(1 + \beta + \sigma k_1 k_2)^2 - 4\beta}} \right] r_{LB,2}^n$$

Appendix B: Equilibrium when the ZLB is not binding

For T≥N, the system to solve is the following:

$$\begin{pmatrix} x_{i,T} \\ \pi_{i,T} \\ i_{T-1} \end{pmatrix} = A \begin{pmatrix} E_T(x_{i,T+1}) \\ E_T(\pi_{i,T+1}) \\ i_T \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ \sigma k_1 k_2 & 0 \\ -\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) & \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \end{pmatrix} \begin{pmatrix} \overline{r_{i,T}} \\ x^* \end{pmatrix}$$

$$\begin{pmatrix} x_{i,T} \\ \pi_{i,T} \\ i_{T-1} \end{pmatrix} = A^k \begin{pmatrix} E_T(x_{i,T+k}) \\ E_T(\pi_{i,T+k}) \\ E_T(i_{i,T+k-1}) \end{pmatrix} + \sum_{n=T}^{T+k-1} A^{n-T} \begin{pmatrix} \sigma & 0 \\ \sigma k_1 k_2 & 0 \\ -\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) & \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \end{pmatrix} \begin{pmatrix} \overline{r_{i,n}} \\ x^* \end{pmatrix} \quad (B1)$$

We have then to find the solution of the following matrix:

$$A^n = \begin{pmatrix} 1 & \sigma & -\sigma \\ k_1 k_2 & (\beta + \sigma k_1 k_2) & -\sigma k_1 k_2 \\ -\frac{\sigma}{\mu}(k_1^2 k_2^2 + \lambda) & -\frac{\sigma}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda) & \frac{1}{\mu}(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda) \end{pmatrix}^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & j_n \end{pmatrix}$$

With, as characteristic equation of the matrix:

$$u_n = \left(2 + \beta + \sigma k_1 k_2 + \frac{\sigma^2 k_1^2 k_2^2}{\mu} + \frac{\sigma^2\lambda}{\mu}\right) u_{n-1} - \left(1 + 2\beta + \sigma k_1 k_2 + \frac{\beta\sigma^2\lambda}{\mu}\right) u_{n-2} + \beta u_{n-3} \quad (B2)$$

Besides, we obtain:

$$x_{i,T} = a_k E_T(x_{i,T+k}) + b_k E_T(\pi_{i,T+k}) + c_k E_T(i_{T+k-1}) + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=T}^{T+k-1} c_{n-T}\, x^* + \sum_{n=T}^{T+k-1} \left[a_{n-T}\sigma + b_{n-T}\sigma k_1 k_2 - c_{n-T}\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) \right] \overline{r_{i,n}} \quad (B3)$$

$$\pi_{i,T} = d_k E_T(x_{i,T+k}) + e_k E_T(\pi_{i,T+k}) + f_k E_T(i_{T+k-1}) + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=T}^{T+k-1} f_{n-T}\, x^* + \sum_{n=T}^{T+k-1} \left[d_{n-T}\sigma + e_{n-T}\sigma k_1 k_2 - f_{n-T}\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) \right] \overline{r_{i,n}} \quad (B4)$$

$$i_{T-1} = g_k E_T(x_{i,T+k}) + h_k E_T(\pi_{i,T+k}) + j_k E_T(i_{T+k-1}) + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=T}^{T+k-1} j_{n-T}\, x^* + \sum_{n=T}^{T+k-1} \left[g_{n-T}\sigma + h_{n-T}\sigma k_1 k_2 - j_{n-T}\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) \right] \overline{r_{i,n}} \quad (B5)$$

$$u_n = \left(2 + \beta + \sigma k_1 k_2 + \frac{\sigma^2 k_1^2 k_2^2}{\mu} + \frac{\sigma^2\lambda}{\mu}\right) u_{n-1} - \left(1 + 2\beta + \sigma k_1 k_2 + \frac{\beta\sigma^2\lambda}{\mu}\right) u_{n-2} + \beta u_{n-3}$$

- $a_0 = a_1 = 1 \qquad a_2 = 1 + \sigma k_1 k_2 + \frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)$ etc…
- $b_0 = 0 \quad b_1 = \sigma \quad b_2 = \sigma(1 + \beta + \sigma k_1 k_2) + \frac{\sigma^2}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda)$ *etc …*
- $c_0 = 0 \quad c_1 = -\sigma \quad c_2 = -2\sigma - \sigma^2 k_1 k_2 - \frac{\sigma^3}{\mu}(k_1^2 k_2^2 + \lambda)$ etc.…
- $d_0 = 0 \quad d_1 = k_1 k_2 \qquad d_2 = k(1 + \beta + \sigma k_1 k_2) + \frac{\sigma^2 k_1 k_2}{\mu}(k_1^2 k_2^2 + \lambda)$ etc.…
- $e_0 = 1 \quad e_1 = \beta + \sigma k_1 k_2$
  $e_2 = k_1 k_2 \sigma + (\beta + \sigma k_1 k_2)^2 + \frac{\sigma^2 k_1 k_2}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda)$ etc…
- $f_0 = 0 \quad f_1 = -\sigma k_1 k_2 \quad f_2 = -\sigma k_1 k_2(2 + \beta + \sigma k_1 k_2) - \frac{\sigma^3 k_1 k_2}{\mu}(k_1^2 k_2^2 + \lambda)$ etc…
- $g_0 = 0 \quad g_1 = -\frac{\sigma}{\mu}(k_1^2 k_2^2 + \lambda)$
  $g_2 = -\frac{2\sigma}{\mu}(k_1^2 k_2^2 + \lambda) - \frac{\sigma k_1 k_2}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda) - \frac{\sigma^3}{\mu^2}(k_1^2 k_2^2 + \lambda)^2$ etc.…
- $h_0 = 0 \quad h_1 = -\frac{\sigma}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda)$
  $h_2 = -\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) - \frac{\sigma}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda)\left(1 + \beta + \sigma k_1 k_2 + \frac{\sigma^2 k_1^2 k_2^2}{\mu} + \frac{\sigma^2\lambda}{\mu}\right)$
- $j_0 = 1 \quad j_1 = 1 + \frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)$

$$j_2 = \frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda) + \frac{\sigma^2 k}{\mu}(\beta k_1 k_2 + \sigma k_1^2 k_2^2 + \sigma\lambda) + \frac{1}{\mu^2}(\mu + \sigma^2 k_1^2 k_2^2 + \sigma^2\lambda)^2 \quad \text{etc}\ldots$$

We can also obtain backward looking equations.
Indeed, the previous equation (B5) implies:

$$i_{T+k-1} = \frac{1}{j_k} i_{T-1} - \frac{g_k}{j_k} x_{i,T+k} - \frac{h_k}{j_k}\pi_{i,T+k} - \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{j_k \mu(1-\beta)} \sum_{n=0}^{k-1} j_n x^* - \frac{1}{j_k}\sum_{n=0}^{k-1}\left[g_n\sigma + h_n\sigma k_1 k_2 - j_n \frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)\right]\overline{r_{\iota,T+n}} \qquad (B6)$$

Putting this equation (B6) in equations (B3) and (B4), we have:

$$x_{i,T} = (a_k - \frac{c_k g_k}{j_k}) x_{i,T+k} + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)} \sum_{n=0}^{k-1}(c_n - \frac{c_k j_n}{j_k}) x^* + \frac{c_k}{j_k} i_{T-1} + (b_k - \frac{c_k h_k}{j_k})\pi_{i,T+k} + \sum_{n=0}^{k-1}\left[a_n\sigma + b_n\sigma k_1 k_2 - \frac{c_k g_n \sigma}{j_k} - \frac{c_k h_n \sigma k_1 k_2}{j_k} + (\frac{c_k j_n}{j_k} - c_n)\frac{\sigma^2(k_1^2 k_2^2 + \lambda)}{\mu}\right]\overline{r_{\iota,T+n}} \quad (B7)$$

$$\pi_{i,T} = (d_k - \frac{f_k g_k}{j_k}) x_{i,T+k} + (e_k - \frac{f_k h_k}{j_k})\pi_{i,T+k} + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)}(\sum_{n=0}^{k-1}(f_n - \frac{f_k j_n}{j_k}) x^* + \frac{f_k}{j_k} i_{T-1} + \sum_{n=0}^{k-1}\left[d_n\sigma + e_n\sigma k_1 k_2 - \frac{f_k g_n \sigma}{j_k} - \frac{f_k h_n \sigma k_1 k_2}{j_k} + (\frac{f_k j_n}{j_k} - f_n)\frac{\sigma^2(k_1^2 k_2^2 + \lambda)}{\mu}\right]\overline{r_{\iota,T+n}} \quad (B8)$$

If the ZLB is anticipated to be binding from a given period (T+K$_b$):
$E_T(i_{T+K_b}) = 0$. So, for 1⩽k⩽K$_b$, equation (B6) implies:

$$i_{T+K_b-k} = \frac{1}{j_{K_b}} i_{T-k} - \frac{g_{K_b}}{j_{K_b}} x_{i,T+K_b-k+1} - \frac{h_{K_b}}{j_{K_b}}\pi_{i,T+K_b-k+1} - \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{j_{K_b}\mu(1-\beta)} \sum_{n=0}^{K_b-1} j_n x^* - \frac{1}{j_{K_b}}\sum_{n=0}^{K_b-1}\left[g_n\sigma + h_n\sigma k_1 k_2 - j_n\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)\right]\overline{r_{\iota,T+n-k+1}} \qquad (B9)$$

and $i_{T+K_b-k} = 0 \; for \; k \le 0$.

Putting this equation (B9) in equations (B3) and (B4) then implies:

$$x_{i,T+k-1} = (a_{K_b} - \frac{c_{K_b} g_{K_b}}{j_{K_b}}) x_{i,T+K_b+k-1} + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)}(\sum_{n=0}^{K_b-1} c_n - \frac{c_{K_b}}{j_{K_b}}\sum_{n=0}^{K_b-1} j_n) x^* + (b_{K_b} - \frac{c_{K_b} h_{K_b}}{j_{K_b}})\pi_{i,T+K_b+k-1} + \frac{c_{K_b}}{j_{K_b}} i_{T+k-2} + \sum_{n=0}^{K_b-1}\left[a_n\sigma + b_n\sigma k_1 k_2 - c_n\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)\right]\overline{r_{\iota,n+T+k-1}} - \frac{c_{K_b}}{j_{K_b}}\sum_{n=0}^{K_b-1}\left[g_n\sigma + h_n\sigma k_1 k_2 - j_n\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)\right]\overline{r_{\iota,T+n+k-1}} \quad \text{for } 1 \le k \le K_b + 1 \qquad (B10)$$

$$\pi_{i,T+k-1} = \left(d_{K_b} - \frac{f_{K_b} g_{K_b}}{j_{K_b}}\right) x_{i,T+K_b+k-1} + \frac{\sigma(k_1^2 k_2^2 + \lambda - \lambda\beta)}{\mu(1-\beta)}\left(\sum_{n=0}^{K_b-1} f_n - \frac{f_{K_b}}{j_{K_b}}\sum_{n=0}^{K_b-1} j_n\right) x^* + (e_{K_b} - \frac{f_{K_b} h_{K_b}}{j_{K_b}})\pi_{i,T+K_b+k-1} + \frac{f_{K_b}}{j_{K_b}} i_{T+k-2} + \sum_{n=0}^{K_b-1}\left[d_n\sigma + e_n\sigma k_1 k_2 - f_n\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)\right]\overline{r_{\iota,n+T+k-1}} - \frac{f_{K_b}}{j_{K_b}}\sum_{n=0}^{K_b-1}\left[g_n\sigma + h_n\sigma k_1 k_2 - j_n\frac{\sigma^2}{\mu}(k_1^2 k_2^2 + \lambda)\right]\overline{r_{\iota,T+n+k-1}} \quad \text{for } 1 \le k \le K_b + 1 \quad (B11)$$

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