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Asymptotic Series of the Associated Legendre Function with Respect to Seismic Surface Waves

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Abstract

The asymptotic series of the associated Legendre function is investigated for the angular degrees n=2-13 in the context of the amplitude and wavelength of seismic surface waves. The approximate formula for the associated Legendre function is used to determine the wavelength and phase velocity of free oscillations of the Earth and long-period surface waves. The approximate formula is derived from the first term of the asymptotic series and its error increases with decreasing n. In the present study, the effect of higher terms on the approximate formula is studied with respect to the amplitudes of the 1st term (a1), the 2nd term (a2), and the 3rd term (a3). For the colatitude angle θ of $\pi/3 \leqq \theta \leqq 2\pi/3$, the amplitude ratios a2/a1 are approximately 3.5%-1.0% for n=2-11 and are less than 1.0% for n $\geqq$ 12, while the amplitude ratios a3/a1 are approximately 0.4% - 0.03% for n=2-13. The effect of amplitudes of higher terms on the 1st term increases as the colatitude angle approaches the pole or the antipode. The wavelengths of the 1st term ($\lambda 1$), the 2nd term ($\lambda 2$), and the 3rd term ($\lambda 3$) are in the sequence $\lambda 1 > \lambda 2 > \lambda 3$. The wavelength ratios $\lambda 2 / \lambda 1$ and $\lambda 3 > \lambda 1$ for n=2-13 are 0.71%-0.93% and 0.55%-0.87%, respectively. The relationship of the wavelengths between the different angular degrees may be expressed as $\lambda_n^{\,k} = \lambda_{n-1}^{\,k+1}$, where n and k are respectively the angular degree and the ordinal number of the asymptotic series.

Keywords: Asymptotic series; Approximate formula; Associated legendre function; Seismic surface waves.

1. Introduction

In the spherical coordinate system, the equation of wave motion for free oscillation of the Earth for the colatitude component can be represented by the associated Legendre's differential equation [1, 2]. The phase velocities of an Earth model were theoretically determined using the approximate formula for the associated Legendre function [3] and the theoretical phase velocities were compared with the observed ones [4]. For the study of seismic surface waves the approximate formula has been conventionally implemented for small and large angular degrees [5-8] with the threshold size of the angular degrees n $\geqq$2 for the colatitude angle $\pi/6 \leqq \theta \leqq 5\pi/6$ [9]. The approximate formula is derived from the first term of the asymptotic series of the associated Legendre function for mathematical conditions n>>1 and n>>m [10]. The error of the approximate formula increases with decreasing n [3, 11]. However, the characteristics of the higher terms of the asymptotic series have not been studied yet. Therefore, the accuracy of the approximate formula for small angular degrees is not mathematically elucidated. The present analysis aims to calculate the higher terms of the asymptotic series and to clarify the effect of the higher terms on the first term.

2. Associated Legendre Function

The wave equation of the free oscillation of the Earth for the colatitudinal component in the spherical coordinate system can be represented by the associated Legendre's differential equations [5, 6, 10, 12, 13]:

$$\frac{d}{d\vartheta}\left(sin\vartheta\,\frac{d\Theta}{d\vartheta}\right) - \left[\frac{m^2}{sin^2\vartheta} - \mathrm{n(n+1)}\right](sin\vartheta)\Theta = 0. \qquad (1)$$

$$\frac{\mathrm{d}^2\Theta}{d\vartheta^2} + cot\vartheta\,\frac{d\Theta}{d\vartheta} + \left[n(n+1) - \frac{m^2}{sin^2\vartheta}\right]\Theta = 0 \qquad (2)$$

$$(1-x)^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-\ x^2}\right\}y = 0 \qquad (3)$$

where $\cos\theta = x$ and $\Theta = y$. For $m\neq 0$ the solutions are given by the associated Legendre functions $P_n^m(x)$ and $Q_n^m(x)$ [3, 6]:

$$\Theta = P_n^m(x) = (1-x^2)^{m/2}\, d^m P_n(x)/dx^m \quad (4)$$

$$\Theta = Q_n^m(x) = (1-x^2)^{m/2}\, d^m Q_n(x)/dx^m\ . \quad (5)$$

The first kind of the associated Legendre function $P_n^m(x)$ and the second kind of the associated Legendre function $Q_n^m(x)$ give independent solutions for $-1<x<1$ [10].

The solutions of the free oscillations of the Earth have the following form [3, 14]:

$$u = U_n(r)\, Y_n(\theta,\phi)\, e^{i\omega t}, \quad (6)$$

where r is the radius of the Earth, θ and ϕ are respectively the colatitude and azimuthal angles. ω is the angular frequency. $Y_n(\theta,\phi)$ represents a surface spherical function of degree n such that

$$Y_n(\theta,\phi) = P_n^m(\cos\theta)\, e^{im\phi}. \quad (7)$$

The parameters n and m are the angular and azimuthal degrees, respectively.

3. Asymptotic Series

When n is very large [10], the associated Legendre function is represented as follows:

$$P_n^m(\cos\theta) = (2e^{m\pi i}/\pi^{3/2})\cos(m\pi)\,\Gamma(n+m+1)\,\Sigma_{k=0}^{\infty}\{[\Gamma(k+m+(1/2))$$
$$\times\ \Gamma(k-m+(1/2))/(k!\ \Gamma(n+k+(3/2))(2\sin\theta)^{k+(1/2)})]$$
$$\times\ \cos[(n+k+(1/2))\theta + (2m-2k-1)\pi/4]\}$$
$$[\,n+m \neq \text{negative integer},\ 2n\neq -3, -5, -7, \ldots\,] \quad (8)$$

In the above formula the right-hand series converges for $\pi/6 < \theta < 5\pi/6$, and the series provides an asymptotic expansion for $\varepsilon \leqq \theta \leqq \pi-\varepsilon$; $[\varepsilon>0]$ if n and m are real numbers with n>>m and n>>1. In the present analysis mathematical conditions $n\geqq 2$ and $\pi/6 \leqq \theta \leqq 5\pi/6$ are assumed, as was considered in the study of the approximate formula [9].

Equation (8) represents that the wave number of the spherical waves is n+k+1/2; accordingly, the wavelength (λ) and phase velocity (C) are determined from:

$$\lambda = 2\pi a/(n+k+1/2), \quad (9)$$

$$C = \lambda/T = 2\pi a/[(n+k+1/2)T], \quad (10)$$

where a is the radius of the Earth (a=6371 km) and T is the eigenperiod.

In the calculation of the higher terms of the asymptotic series, Eq. (8) was separated into two functions, W(n, m) and Z(k, n, m):

$$P_n^m(\cos\theta) = W(n, m)\, Z(k, n, m) \quad (11)$$

$$W(n, m) = (2e^{m\pi i}/\pi^{3/2})\cos(m\pi)\,\Gamma(n+m+1) \quad (12)$$

$$Z((k, n, m) = \Sigma_{k=0}^{\infty}[\ \Gamma(k+m+(1/2))\,\Gamma(k-m+(1/2))/$$
$$\times\quad (k!\ \Gamma(n+k+(3/2))\,(2\sin\theta)^{k+(1/2)})$$
$$\times\quad \cos[(n+k+(1/2))\theta + (2m-2k-1)\pi/4]\ \}$$
$$= A1(k=0) + A2(k=1) + A3(k=2) + \ldots \quad (13)$$

It is noted that the function W(n, m) depends only on the parameters n and m, while the function Z(k, n, m) depends on the parameters n, m, and k. From Eq. (13) we determine the coefficients of the 1st term (A1), the 2nd term (A2), and the 3rd term (A3).

4. Coefficients of the Higher Terms

4.1. Coefficient of the 1st Term (k=0)

From Eq. (11),

$$A1 = [\ \Gamma(0+m+(1/2))\,\Gamma(0-m+(1/2))/(0!\,\Gamma(n+0+(3/2))(2\sin\theta)^{0+(1/2)})]$$
$$\times\ \cos[(n+0+(1/2))\theta + (2m-2\cdot 0-1)\pi/4]. \quad (14)$$

We separate A1 into an amplitude term a1 and a phase angle term b1 as follows:

$$A1(k=0) = a1\cdot b1 \quad (15)$$

$$a1 = \Gamma(m+1/2)\,\Gamma(-m+1/2)/(\Gamma(n+(3/2))(2\sin\theta)^{0+(1/2)}) \quad (16)$$

$$b1 = \cos[(n+1/2)\theta + (2m-1)\pi/4]. \quad (17)$$

4.2. Coefficient of the 2nd Term (k=1)

From Eq. (13),

$$A2 = [\ \Gamma(1+m+(1/2))\,\Gamma(1-m+(1/2))/(1!\,\Gamma(n+1+(3/2))(2\sin\theta)^{1+(1/2)})]$$
$$\times\ \cos[(n+1+(1/2))\theta + (2m-2\cdot 1-1)\pi/4]. \quad (18)$$

We separate A1 into an amplitude term a1 and a phase angle term b1 as follows:

$$A2(k=1) = a2\cdot b2 \quad (19)$$

$$a2 = \Gamma(m+(3/2))\,\Gamma(-m+(3/2))/(\Gamma(n+(5/2))(2\sin\theta)^{3/2})$$
$$= (m+(1/2))(-m+(1/2))/\{(n+(3/2))(2\sin\theta)\}\cdot a1 \quad (20)$$

$$b2 = \cos[(n+(3/2))\theta + (2m-3)\pi/4]. \quad (21)$$

4.3. Coefficient of the 3rd Term (k=2)

From Eq. (13),

$$A3 = [\Gamma(2+m+(1/2))\Gamma(2-m+(1/2))/(2!\Gamma(n+2+(3/2))(2\sin\theta)^{2+(1/2)})] \times \cos[(n+2+(1/2))\theta + (2m-2\cdot 2-1)\pi/4]. \quad (22)$$

We separate A3 into an amplitude term a3 and a phase angle term b3 as follows:

$$A3(k=2) = a3 \cdot b3 \quad (23)$$

$$a3 = \Gamma(m+(5/2))\Gamma(-m+(5/2))/\{2!\Gamma(n+(7/2))(2\sin\theta)^{5/2)}\} = \{(m+(3/2))(m+(1/2))/\{(-m+(3/2))(-m+(1/2))\}/ \times \{2!(n+(5/2))(n+(3/2))(2\sin\theta)^2\} \cdot a1 \quad (24)$$

$$b3 = \cos[(n+(5/2))\theta + (2m-5)\pi/4]. \quad (25)$$

The amplitude terms a1, a2, and a3 are calculated for three Zones of the colatitude angle θ:

$$\text{Zone 1:} \quad \text{for } \pi/3 \leqq \theta \leqq 2\pi/3, \quad (26)$$

$$\text{Zone 2:} \quad \text{for } \theta = \pi/4 \text{ or } 3\pi/4, \quad (27)$$

$$\text{Zone 3:} \quad \text{for } \theta = \pi/6 \text{ or } 5\pi/6. \quad (28)$$

For Zone 1, the variable s is defined as:

$$s = 1/\sin\theta = 1.154 \sim 1.0 \fallingdotseq 1.0;\ s^2 \fallingdotseq 1.0. \quad (29)$$

For Zone 2, the variable s is defined as:

$$s = 1/\sin\theta = \sqrt{2};\ s^2 = 2. \quad (30)$$

For Zone 3, the variable s is defined as:

$$s = 1/\sin\theta = 2;\ s^2 = 4. \quad (31)$$

5. Results

In the calculation of amplitudes and phase angles of the higher terms of the asymptotic series, the parameter m= 0 is assumed because the present analysis assumes a spherically symmetric Earth. The calculated amplitudes and wavelengths of the 1st term, the 2nd term, and the 3rd term for small n (n=0-13) are shown in Table 1, Table 2, and Table 3. Although the calculations are done for n=0 and n=1, they are merely shown as a reference, because they do not satisfy the mathematical conditions of n≧2, as mentioned in Section 1.

Table-1. Amplitudes of the higher terms of the asymptotic series. The amplitudes of the 1st term, the 2nd term, and the 3rd term are denoted by a1, a2, and a3, respectively. For the function W(n,m) see the text

n	W(n, m)	a1	a2	a3
0	p • 0!	q	s • a1/12	s^2 • a1/52
1	p • 1!	q • 2/3!!	s • a1/20	s^2 • a1/124
2	p • 2!	q • 2^2/5!!	s • a1/28	s^2 • a1/224
3	p • 3!	q • 2^3/7!!	s • a1/36	s^2 • a1/352
4	p • 4!	q • 2^4/9!!	s • a1/44	s^2 • a1/508
5	p • 5!	q • 2^5/11!!	s • a1/52	s^2 • a1/693
6	p • 6!	q • 2^6/13!!	s • a1/60	s^2 • a1/907
7	p • 7!	q • 2^7/15!!	s • a1/68	s^2 • a1/1,148
8	p • 8!	q • 2^8/17!!	s • a1/76	s^2 • a1/1,419
9	p • 9!	q • 2^9/19!!	s • a1/84	s^2 • a1/1,717
10	p • 10!	q • 2^{10}/21!!	s • a1/92	s^2 • a1/2,044
11	p • 11!	q • 2^{11}/23!!	s • a1/100	s^2 • a1/2,400
12	p • 12!	q • 2^{12}/25!!	s • a1/108	s^2 • a1/2,784
13	p • 13!	q • 2^{13}/27!!	s • a1/116	s^2 • a1/3,196

$p=2/\pi^{3/2}$, $q=2\pi^{1/2}/(2\sin\theta)^{1/2}$, $s=1/\sin\theta$

Table-2. Amplitude ratios a2/a1 and a3/a1 for Zone 1 ($\pi/3 \leqq \theta \leqq 2\pi/3$), Zone 2 ($\theta = \pi/4$ or $3\pi/4$), and Zone 3 ($\theta = \pi/6$ or $5\pi/6$)

	Zone 1		Zone 2		Zone 3	
n	a2/a1	a3/a1	a2/a1	a3/a1	a2/a1	a3/a1
0	0.08333	0.01923	0.11785	0.03846	0.16666	0.07692
1	0.05	0.00806	0.07071	0.01612	0.10	0.03224
2	0.03571	0.00446	0.05050	0.00892	0.07142	0.01784
3	0.02777	0.00284	0.03927	0.00568	0.05554	0.01136
4	0.02272	0.00196	0.03213	0.00392	0.04544	0.00784
5	0.01923	0.00144	0.02719	0.00288	0.03846	0.00576
6	0.01666	0.00110	0.02356	0.00220	0.03332	0.00440
7	0.01470	0.00087	0.02078	0.00174	0.02940	0.00348
8	0.01315	0.00070	0.01859	0.00140	0.02630	0.00280
9	0.01190	0.00058	0.01682	0.00116	0.02380	0.00232

10	0.01086	0.00049	0.01535	0.00098	0.02172	0.00196
11	0.01	0.00041	0.01414	0.00082	0.02	0.00164
12	0.00926	0.00036	0.01309	0.00072	0.01852	0.00144
13	0.00862	0.00031	0.01219	0.00062	0.01724	0.00124

Table-3. Wavelengths of the higher terms of the asymptotic series. The wavelengths of the 1st term, the 2nd term, and the 3rd term are denoted by λ1, λ2, and λ3

n	λ1 (km)	λ2 ((km)	λ3 ((km)	λ2 /λ1	λ3 /λ1
0	80,060	26,686	16,012	0.33332	0.2
1	26,686	16,012	11,437	0.60001	0.42857
2	16,012	11,437	8,895	0.71427	0.55552
3	11,437	8,895	7,278	0.77773	0.63635
4	8,895	7,278	6,158	0.81821	0.69229
5	7,278	6,158	5,337	0.84611	0.73330
6	6,158	5,337	4,709	0.86667	0.76469
7	5,337	4,709	4,214	0.88239	0.78958
8	4,709	4,214	3,812	0.89488	0.80951
9	4,214	3,812	3,481	0.90460	0.82605
10	3,812	3,481	3,202	0.91316	0.83997
11	3,481	3,202	2,965	0.91985	0.85176
12	3,202	2,965	2,761	0.92598	0.86227
13	2,965	2,761	2,583	0.93119	0.87116

6. Discussion

The function W(n, m) expressed by Eq. (12) shows that it is proportional to the factorial of the angular degree n and increases monotonously with increasing n (Table 1). The function Z(k, n, m) expressed by Eq. (13) shows that the amplitudes of the 1st term a1, the 2nd term a2, and the 3rd term a3 expressed by Eq. (16), Eq. (20), and Eq. (24), respectively, depend on $1/(\sin\theta)^{1/2}$, $1/(\sin\theta)^{3/2}$, and $1/(\sin\theta)^{5/2}$, respectively. These amplitude terms decrease monotonously with increasing n, forming the sequence a1>a2>a3 (Table 1). The ratios a2/a1 and a3/a1 decrease with increasing n for Zone 1, Zone 2, and Zone 3 (Table 2). As mentioned in Section 1, the approximate formula is derived from the 1st term of the asymptotic series. Considering the above amplitude sequence, the approximate formula may be mostly influenced by the 2nd term.

6.1. Characteristics of the Amplitude Terms

The amplitude ratios a2/a1 (Table 2) are:

For Zone 1, for n=2-11, approximately 3.5% - 1.0&%,
for n ≧12, approximately < 1.0%,
For Zone 2, for n=2-13, approximately 5.0% – 1.2%,
For Zone 3, for n=2-13, approximately 7.1% - 1.7%.

The amplitude ratios a3/a1 (Table 2) are:

For Zone 1, for n=2-13, approximately 0.44% - 0.03%,
For Zone 2, for n=2-13, approximately 0.89% – 0.06%,
For Zone 3, for n=2-3, approximately 1.7% - 1.1%.
for n=4-13, approximately 0.7% - 0.1%.

These results are summarized as follows:

(A) The amplitude ratios of both a2/a1 and a3/a1 increase in the following order:
for Zone 1 < for Zone 2 < for Zone 3.

(B) The effect of the 2nd term on the 1st term is strong, when the colatitude angle θ approaches the pole and the antipode.

(C) The effect of the 2nd term on the 1st term is less than 1.0% for n≧12 for only Zone 1.

(D) For the amplitude ratio a3/a1, the effect of the 3rd term on the 1st term is:
(d-1) extremely weaker than that for the ratio a2/a1,
(d-2) less than 1.0% for n=2-13, for three Zones except for n=2-3 for Zone 3.

6.2. Characteristics of the Phase Angle Terms

The wavenumbers of the travelling waves for the 1st term, the 2nd term, and the 3rd term of the asymptotic series are determined from Eq. (17), Eq. (21), and Eq. (25), respectively. The wavelengths of the 1st term λ1, the 2nd term λ2, and the 3rd term are calculated by the use of Eq. (9) (Table 3). The wavelength characteristics are summarized as follows:

(A) The wavelengths decrease in the order λ1 > λ2 > λ3.

(B) The wavelength ratios λ2 / λ1 are approximately 0.71%-0.77% for n=2-3 , 0.81%-0.89% for n=4-8, and 0.90%-0.93% for n=9-13.

(C) The wavelength ratios $\lambda 3 / \lambda 1$ are approximately 0.55% for n=2, 0.63%-0.69% for n=3-4, and 0.73%-0.78% for n=5-7, and 0.80%-0.87% for n=8-13.

(D) The wavelength of 8895 km of $\lambda 1$ for n=4 corresponds to that of $\lambda 2$ for n=3 and to that of $\lambda 3$ for n=2.

(E) The relationship of the wavelengths shown between different angular degrees n=4, n=3, and n=2 is recapitulated through angular degrees (n=2-13) such that

$$\lambda_n^k = \lambda_{n-1}^{k+1} \quad (32)$$

where n is the angular degree and k is the ordinal number of the higher terms of the asymptotic series.

Equations (6) and (11) show that the travelling waves of higher terms for the angular degree n have the same eigenperiod. Therefore,

$$\lambda 1 / C1 = \lambda 2 / C2 = \lambda 3 / C3 = T \quad (33)$$

where C1, C2, and C3 are the phase velocities of the 1st term, the 2nd term, and the 3rd term, respectively. Considering the characteristic (A) and (33):

(F) The phase velocities are in the sequence:

$$C1 > C2 > C3. \quad (34)$$

The characteristic (F) shows that the travelling waves of higher terms have a series of wavelets with slower phase velocities and smaller amplitudes than those of the 1st term for three Zones (Table 2).

6.3. Seismic Surface Waves on the Angular Distance

When seismic surface waves travel the pole and antipode, the waves have a phase advance of $\pi/2$ due to a polar phase shift [3]. Normal mode studies of long-period surface waves and free oscillation of the Earth including the pole and antipode have been conducted by several authors [15-21].

Systematic fluctuations of fundamental spheroidal mode eigenfrequency measurement are shown as a function of angular degree, which depend on the angular distance [20]. In their examples the fluctuations are observed for different epicentral distances, with the periodicity as predicted by Dahlen's tan $(k\Delta - \pi/4)$ term (2 for Δ = close to 90°, approximately corresponding to Zone 1; 4 for Δ = close to 45° or 135°, corresponding to Zone 2; 8 for Δ close to 21° …, approximately corresponding to Zone 3), where Δ is the epicentral distance and k=n+1/2. The wavenumber k=n+1/2 corresponds to the one expressed in the phase angle term b1 of the 1st term (Eq. (17)). Multiplet amplitude anomalies and frequency shifts on an aspherical earth model are studied using an asymptotic method [16]. In this study the multiplet amplitude perturbations near the antipode are represented using several methods, in which a singularity at the source antipode caustic is included.

The multiplet amplitude anomalies and frequency shifts are considered to be highly influenced by lateral heterogeneous structures in the neighborhood of the source station great circle path [15-17, 19-21].

7. Conclusion

Higher terms of the asymptotic series of the associated Legendre function for small angular degrees (n=2-13) were calculated separating the asymptotic series into the amplitude terms and the phase angle terms. The calculation was carried out for three colatitude angles of Zone 1 ($\pi/3 \leqq \theta \leqq 2\pi/3$), Zone 2 ($\theta = \pi/4$ or $3\pi/4$), and Zone 3 ($\theta = \pi/6$ or $5\pi/6$). Higher terms of the amplitudes of the 1st term (a1), the 2nd term (a2), and the 3rd term (a3) are characterized as follows:

(C-1) All the amplitude terms decrease with increasing n.

(C-2) The amplitude terms are in the sequence: a1>a2>a3.

(C-3) The amplitude ratios of both a2/a1 and a3/a1 are in the following sequence: for Zone 1 < for Zone 2 < for Zone 3.

(C-4) The amplitude ratios a2/a1 for Zone 1 are approximately 3.5%-1.0% for n=2-11, and < 1.0% for n>12.

The wavelengths $\lambda 1$, $\lambda 2$, and $\lambda 3$, which are calculated from the phase angle terms of the 1st term (b1), the 2nd term (b2), and the 3rd term (b3), respectively, are characterized as follows:

(C-5) The wavelengths are in the sequence: $\lambda 1 > \lambda 2 > \lambda 3$.

(C-6) The wavelength ratios $\lambda 2/\lambda 1$ are 0.71%-0.93% for n=2-13.

(C-7) The wavelength ratios $\lambda 3/\lambda 1$ are 0.55%-0.87% for n=2-13.

(C-8) The relationship of the wavelengths between different angular degrees is expressed as $\lambda_n^k = \lambda_{n-1}^{k+1}$, where n and k are respectively the angular degree and the ordinal number of the asymptotic terms.

(C-9) The phase velocities C1, C2, and C3 determined from the 1st term (b1), the 2nd term (b2), and the 3rd term (b3), respectively, by the use of Eq. (33), are in the sequence: C1 > C2 > C3.

From the conclusion above, it is possible to conjecture the following: for n=2, the amplitudes of the 2nd terms are 3.5%, 5.0%, and 7.1% of those of the 1st terms for Zone 1, Zone 2, and Zone 3, respectively, while the phase velocities of the 2nd terms are 71% of those of the 1st terms for the three Zones. For n=11, the amplitudes of the 2nd terms are 1.0%, 1.4%, and 2.0% of those of the 1st terms for Zone 1, Zone 2, and Zone 3, respectively, while the phase velocities of the 2nd terms are 91% of the 1st terms for the three Zones. For n≧12, the amplitude of the 2nd term is less than 1.0% of that of the 1st term for Zone 1.

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