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A Classification of the Mathematical Expressions of the Asymptotic Series Expansion of the Associated Legendre Function

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Abstract

The mathematical expressions of the asymptotic series expansion of the associated Legendre function, which have been widely described in the mathematical literature, are here shown to be classifiable into two types according to their manner of expression. The Type-A mathematical expressions are represented by the first three higher terms of the asymptotic series, while the Type-B expressions are represented by the infinite sequence. As a result, the Type-B mathematical expressions differ slightly from one another, despite having identical computation results. The distinct representation of the Type-B expressions arises from (1) the treatment of the gamma functions including the colatitude angle in the spherical coordinate system and the azimuthal order number of the associated Legendre function and (2) the method of expressing the phase angle term. Moreover, one of the mathematical expressions of the Type-B is derived using a hypergeometric formula different from other usual one.

Keywords: Asymptotic series; Associated legendre function; Gamma function; Hypergeometric function.

1. Introduction

The associated Legendre function plays important roles for the study of the free oscillation of a heterogeneous Earth model and long-period surface waves [1, 2]. The wavelength and phase velocity are determined using an approximate formula for the associated Legendre function [3-5]. The approximate formula has been investigated by numerous authors using a range of mathematical methods [6-9]. Several studies have discussed the optimal mathematical conditions for the use of the approximate formula [10, 11], while others contributed a formula calculating the threshold size of the angular degree [12, 13]. Moreover, theoretical methods for computing synthetic seismograms for aspherical global earth models have been developed for the spherical coordinates by various authors [14-25].

The Legendre functions of the first and second kind, $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$, respectively, are the solutions of the associated Legendre's differential equation.

Through variable transformation formulas, the associated Legendre's differential equation reduces to the Gauss' equation of hypergeometric type [26]. The solutions of the Gauss' equation are also given by $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$, which are expressible in the various asymptotic series by means of the transformation formulas of the Gauss hypergeometric function. The asymptotic series expansion of the associated Legendre functions is represented by various mathematical expressions [26-31]. Each of the above-discussed mathematical expressions is slightly different, which can engender confusion when attempting to use the asymptotic series expansion.

The present study aims to classify these mathematical expressions into two simple categories. Factors giving rise to the differences between the mathematical expressions of Type-B are also discussed.

2. Mathematical Expressions of the Asymptotic Series Expansion

The mathematical expressions of the asymptotic series expansion of the associated Legendre function may be classified into two types according to their manner of expression. The first type is Type-A, which replaces the asymptotic series expansion with the first three higher terms. The second type is Type-B, which forms the infinite sequence of the asymptotic series expansion. The asymptotic series expansion includes the Gamma function and four mathematical parameters of n, m, θ, and k. The parameter n is the angular degree and the parameter m is the azimuthal order number of the associated Legendre function. The parameters θ and k are the colatitude angle in the spherical coordinate system and the ordinal number of the asymptotic series, respectively.

The following expressions are classified as Type-A:

2.1. Type-A

2.1.1. Type-A1 [29, p. 352]

$P_n^m(\cos\theta) = A1\ [\ P1 + A2 \cdot P2 + A3 \cdot P3 + \ .\ .\ .\ ],$ (1)

$A1 = 2\,\Gamma(n+m+1) / (\pi^{1/2}\Gamma(n+3/2))$

$P1 = \cos[(n+1/2)\theta - \pi/4+m\pi/2]/(2\sin\theta)^{1/2}$

$A2 = (1^2-4m^2)/\{2(2n+3)\}$

$P2 = \cos[(n+3/2)\theta-3\pi/4+ m\pi/2]/(2\sin\theta)^{3/2}$

$A3 = (1^2-4m^2)(3^2-4m^2)/[2 \cdot 4 \cdot (2n+3)(2n+5)]$

$P3 = \cos[(n+5/2)\theta-5\pi/4+ m\pi/2]/(2\sin\theta)^{5/2}$

2.1.2. Type-A2 [27, p. 239]

$P_n^m(\cos\theta) = A1\ [\ P1 - A2 \cdot P2 + A3 \cdot P3 - \ .\ .\ .\ ],$ (2)

$A1 = [(n+m)! /(n+1/2)!]\ (2/\pi\sin\theta)^{1/2}$

$P1 = \cos[(n+1/2)\theta + (m-1/2)\pi/2]$

$A2 = (4m^2-1^2) /[2 \cdot 1!\ (2n+3)]$

$P2 = \cos[(n+3/2)\theta- (m-3/2)\pi/2]/ (2\sin\theta)$

$A3 = (4m^2-1^2)(4m^2-3^2)/[2^2 \cdot 2!(2n+3)(2n+5)]$

$P3 = \cos[(n+5/2)\theta+ (m-5/2)\pi/2]/(2\sin\theta)^2$

2.2. Type-B

The mathematical expressions of Type-B are similar to one another. However, they are slightly different in terms of their manner of expressions. Therefore they are expanded into the first three higher terms of the asymptotic series as for the Type-A mathematical expressions. This process may be useful for comparing the accordance between the respective mathematical expressions.

The following expressions are classified as Type-B.

2.2.1. Type-B1 [31, p. 125]

$P_n^m(\cos\theta) = (2e^{m\pi i} /\pi^{3/2})\cos(m\pi)\Gamma(n+m+1)\Sigma_{k=0}^{\infty}\ \{[\Gamma(k+m+1/2)$

$\times\ \Gamma(k-m+1/2)/(\ k\ !\ \Gamma(n+k+3/2)(2\sin\theta)^{k+(1/2)})]$

$\times\ \cos[(n+k+1/2)\theta+ (2m-2k-1)\pi/4\ ]\}$

$[\ n+m \neq$ non-negative integer, $2n\neq -3, -5, -7, \ldots\ ]$ (3)

The expression is convergent if $\pi/6 < \theta < \pi 5/6$.

For the expansion of the asymptotic series into the first three higher terms, Eq. (3) is separated into two functions, W(n, m) and Z(k, n, m). The former depends on the parameters n and m, while the latter depends on the parameters k, n, and m.

$W(n, m) = (2e^{m\pi i}/\pi^{3/2})\cos(m\pi)\Gamma((n+m+1)$ (4)

$Z(k, n, m) = \Sigma_{k=0}^{\infty}\{[\Gamma(k+m+1/2)\Gamma(k-m+1/2)/(k!$

$\times\ \ \Gamma(n+k+3/2)(2\sin\theta)^{k+(1/2)})]$

$\times\ \ \cos[\ (n+k+1/2)\theta+ (2m-2k-1)\pi/4\ ]\}$ (5)

(i) B1-a: First term (k=0)

$Z(0, n, m) =\Gamma(m+1/2)\Gamma(-m+1/2)/[0!\ \Gamma(n+3/2)(2\sin\ )^{1/2}]$

$\times\ \ \cos[(n+1/2)\theta-\pi/4 +m\pi/2]$ (6)

$\therefore W(n, m)\ Z(0, n, m) = (2\ e^{m\pi i} /\pi^{1/2})(\Gamma(n+m+1)/\Gamma(n+3/2))$

$\times\ \ (\cos(m\pi)/\pi)\ (-1)^m\pi/(2\sin\theta)^{1/2}$

$\times\ \ \cos[(n+1/2)\theta-\pi/4+m\pi/2]$

$= [\ 2\,\Gamma(n+m+1)\ e^{m\pi i} / (\pi^{1/2}\Gamma(n+3/2)\ ]$

$\times\ \ \cos[(n+1/2)\theta-\pi/4+m\pi/2] / (2\sin\theta)^{1/2}$ (7)

(ii) B1-b: Second term (k=1)

$Z(1, n, m) =\Gamma(1+m+1/2)\Gamma(1-m+1/2)/[1!\Gamma(n+1+3/2)(2\sin\theta)^{3/2}]$

$\times\ \ \cos[(n+3/2)\theta-3\pi/4 +m\pi/2]$

$=\{\ (m+1/2)\Gamma(m+1/2)\ (-m+1/2)\Gamma(-m+1/2) /$

$\times\ \ \ [(n+3/2)\Gamma(n+3/2)\ \ (2\sin\theta)^{3/2}\ ]\}$

$\times\ \ \ \cos[(n+3/2)\theta-3\pi/4 +m\pi/2]$

$= (1^2-4m^2)\ (-1)^m\pi/[2(2n+3)\Gamma(n+3/2)\ \ (2\sin\theta)^{3/2}\ ]$

$\times\ \ \ \cos[(n+3/2)\theta-3\pi/4 +m\pi/2]$ (8)

$\therefore W(n, m)\ Z(1, n, m) = (2\ e^{m\pi i} /\pi^{1/2})(\Gamma(n+m+1)/\Gamma((n+3/2))$

$\times\ \ (\cos(m\pi)/\pi)\ [\ (1^2-4m^2)/2(2n+3)]\ (-1)^m\pi$

$\times\ \ \cos[(n+3/2)\theta- 3\pi/4+m\pi/2] / (2\sin\theta)^{3/2}$

$= [\ 2\,\Gamma(n+m+1)\ e^{m\pi i} / (\pi^{1/2}\Gamma(n+3/2)\ )\ ]$

$\times\ \ [(1^2-4m^2)/2(2n+3)]$

$\times\ \ \cos[(n+3/2)\theta- 3\pi/4+m\pi/2] / (2\sin\theta)^{3/2}$ (9)

(iii) B1-c: Third term (k=2)

$Z(2, n, m) = [\Gamma(2+m+1/2)\Gamma(2-m+1/2)/(2!\ \Gamma(n+2+3/2)(2\sin\theta)^{5/2})]$
x $\cos [(n+5/2)\theta-5\pi/4 +m\pi/2]$
$= ((3+2m)/2)\Gamma(3/2+m)\ ((3-2m)/2)\Gamma(3/2-m)\ /$
x $(2 \cdot (5+2n)/2)\Gamma(n+5/2)\ (2\sin\theta)^{5/2})$
x $\cos [(n+5/2)\theta- 5\pi/4 +m\pi/2]$
$= (9-4m^2)(1/2+m)\Gamma(1/2+m)\ (1/2-m)\Gamma(1/2-m)\ /$
x $[2 \cdot 2 \cdot (5+2n)(n+3/2)\Gamma(n+3/2)\ (2\sin\theta)^{5/2}]$
x $\cos [(n+5/2)\theta- 5\pi/4 +m\pi/2]$
$= (9-4m^2)(1^2-4m^2)(-1)^m\pi/[2 \cdot 4 \cdot (5+2n)$
x $(3+2n)\ \Gamma(n+3/2)(2\sin\theta)^{5/2}]$
x $\cos[(n+5/2)\theta-5\pi/4 +m\pi/2]$ (10)
$\therefore W(n, m)\ Z(2, n, m) = (2\ e^{m\pi i} /\pi^{1/2})(\Gamma(n+m+1)/\Gamma(n+3/2\ ))$
x $(\cos (m\pi)/\pi)\ \{[(1^2-4m^2)/(3^2-4m^2)]\ (-1)^m\pi/$
x $[2 \cdot 4 \cdot (2n+3)(2n+5)]\}$
x $\cos[(n+5/2)\theta- 5\pi/4 +m\pi/2]/ (2\sin\theta)^{5/2}$
$= (2\ e^{m\pi i} /\pi^{1/2})(\Gamma(n+m+1)/\Gamma(n+3/2))$
x $\{ (1^2-4m^2)(3^2-4m^2)/[2 \cdot 4 \cdot (2n+3)(2n+5)] \}$
x $\cos[(n+5/2)\theta-5\pi/4 +m\pi/2]/ (2\sin\theta)^{5/2}$ (11)

From Eq. (7), Eq. (9), and Eq. (11), Eq. (3) may be expressed as follows:

$P_n^m(\cos\theta) = (2\ e^{m\pi i} /\pi^{1/2})(\Gamma(n+m+1)/\Gamma(n+3/2))$
x $\{\ \cos[(n+1/2)\theta-\pi/4+m\pi/2] / (2\sin\theta)^{1/2}$
x $+ ((1^2-4m^2)/2(2n+3))\cos[(n+3/2)\theta-3\pi/4+m\pi/2] / (2\sin\theta)^{3/2}$
x $+ ((1^2-4m^2)(3^2-4m^2)/(2 \cdot 4 \cdot (2n+3)(2n+5)))$
x $\cos[(n+5/2)\theta-5\pi/4 +m\pi/2]/ (2\sin\theta)^{5/2}\quad + \dots \}$ (12)

2.2.2. Type-B2 [30, pp. 1742-1743]

$P_n^m(\cos\theta) = (2\Gamma(n+m+1) / (\pi^{1/2}\Gamma(n+3/2))$
x $\Sigma_{k=0}^{\infty} [\Gamma(1/2+m+k)\ \Gamma(1/2-m+k)\ \Gamma(n+3/2) /$
x $(\Gamma(1/2+m)\ \Gamma(1/2-m)\ \Gamma(n+k+3/2)\ k!)\ ]$
x $\cos[(n+(2k+1)/2)\theta- (2k+1)\pi/4+m\pi/2] / (2\sin\theta)^{k+1/2}$ (13)
($m+n \neq$ non-negative integer, $n+1/2 \neq$ non-negative integer)
Equation (13) is convergent if $\pi/6<\theta< 5\pi/6$.
As for Type-B1, Eq. (13) was separated into the two functions W(n, m) and Z(k, n, m):
$W(n, m) = 2\Gamma(n+m+1) / (\pi^{1/2}\Gamma(n+3/2))$ (14)
$Z(k, n, m) = \Sigma_{k=0}^{\infty} [\Gamma(1/2+m+k)\ \Gamma(1/2-m+k)\ \Gamma(n+3/2) /$
x $(\Gamma(1/2+m)\ \Gamma(1/2-m)\ \Gamma(n+k+3/2)\ k!)\ ]$
x $\cos[(n+(2k+1)/2)\theta- (2k+1)\pi/4+m\pi/2] /(2\sin\theta)^{k+1/2}$ (15)

In the function Z(k, n, m), the Gamma and sine functions which don't include the ordinal number k are moved to W(n, m):

$W(n, m) = [\ 2\Gamma(n+m+1) / (\pi^{1/2}\Gamma((n+3/2))\ ]$
x $\Gamma(n+3/2) / [\Gamma(1/2+m)\ \Gamma(1/2-m)\ (2\sin\theta)^{1/2}\ ]$
$= 2^{1/2}/ (\pi\sin\theta)^{1/2}\ (1/(-1)^m) \cdot \Gamma(n+m+1)/\pi$ (16)
$Z(k, n, m) =\Sigma_{k=0}^{\infty} [\Gamma(1/2+m+k)\ \Gamma(1/2-m+k) / (\Gamma(n+k+3/2)\ k!)\ ]$
x $\cos[(n+(2k+1)/2)\theta- (2k+1)\pi/4+m\pi/2] /(2\sin\theta)^k$ (17)

(i) B2-a: First term (k=0)
$W(n, m)\ Z(0, n, m) = [\ (2/(\pi\sin\theta))^{1/2}\ (1/(-1)^m)\ (\Gamma(n+m+1)/\pi)$
x $(-1)^m\pi/(\Gamma((n+3/2)\ 0!)]$
x $\cos[(n+1/2)\theta-\pi/4+ m\pi/2]$
$= [2(\Gamma(n+m+1)/(\pi^{1/2}\ \Gamma(n+3/2))\ ]$
x $\cos[(n+1/2)\theta-\pi/4+ m\pi/2] / (2\sin\theta)^{1/2}$ (18)

(ii) B2-b: Second term (k=1)
$W\ (n, m)\ Z(1, n, m) = [\ (2/(\pi\sin\theta))^{1/2}\ (1/(-1)^m)\ (\Gamma(n+m+1)/\pi)$
x $(1/2+m)\Gamma(1/2+m)\ (1/2-m)(\Gamma(1/2-m) / (1!(n+3/2)\Gamma(n+3/2)\ ]$
x $\cos[(n+3/2)\theta-3\pi/4+ m\pi/2] / (2\sin\theta)]$
$=[2\Gamma(n+m+1)/(\pi^{1/2}\Gamma(n+3/2))]\ [\ (1+2m)(1-2m)(1/4) /(\ (1/2)(2n+3))\ ]$
x $\cos[(n+3/2)\theta-3\pi/4+ m\pi/2] / (2\sin\theta)^{3/2}$
$=[2\Gamma(n+m+1)/(\pi^{1/2}\Gamma(n+3/2))]\ [\ (1^2-4^m) /(2(2n+3))\ ]$
x $\cos[(n+3/2)\theta-3\pi/4+ m\pi/2] / (2\sin\theta)^{3/2}$ (19)

(iii) B2-c: Third term (k=2)
$W\ (n, m)\ Z(2, n, m) = [\ (2/(\pi\sin\theta))^{1/2}\ (1/(-1)^m)\ (\Gamma(n+m+1)/\pi)$
x $\Gamma(1/2+m+2)\ (\Gamma(1/2-m+2) / (2!\Gamma(n+2+3/2)\ ]$
x $\cos[(n+5/2)\theta-5\pi/4+ m\pi/2] / (2\sin\theta)^2$
$=[2\Gamma(n+m+1)/(\pi^{1/2}(2\sin\theta)^{1/2})]\ \ (1/(-1)^m)(1/\pi)\ [\ (3/2+m)\Gamma(3/2+m)$
x $(3/2-m)\Gamma(3/2-m)/\ (2(n+5/2)\Gamma(n+5/2)\ )\ ]$

x $\cos[(n+5/2)\theta-5\pi/4+ m\pi/2] / (2\sin\theta)^2$
$=[2\Gamma(n+m+1)/(\pi^{1/2}(2\sin\theta)^{1/2})]\ (1/(-1)^m)\ (1/\pi)\ [(1/2)\ (3+2m)\ (1/2)$
x $(3-2m)\ (1/2+m)\Gamma(1/2+m)\ (1/2-m)\Gamma(1/2-m) / (2(n+5/2)(n+3/2)$
x $\Gamma(n+3/2)\ ]\ \cos[(n+5/2)\theta-5\pi/4+ m\pi/2] / (2\sin\theta)^2$
$=[2\Gamma(n+m+1)/(\pi^{1/2}\Gamma(n+3/2))]\ [\ (1^2-4^m)(3^2-4m^2) /$
x $(2\cdot 4\cdot (2n+3)\ (2n+5)\ )\ ]$
x $\cos[(n+5/2)\theta-5\pi/4+ m\pi/2] / (2\sin\theta)^{5/2}$ (20)

From Eq. (18), Eq. (19), and Eq. (20), Eq. (13) may be expressed as follows:
$P_n^m(\cos\theta) = (2/\pi^{1/2})(\Gamma(n+m+1)/\Gamma(n+3/2)\ )$
x $\{\ [\cos((n+1/2)\theta-\pi/4+m\pi/2) / (2\sin\theta)^{1/2}\ ]$
x $+\ [\ ((1^2-4m^2)/2(2n+3))\ \cos((n+3/2)\ -3\pi/4+m\pi/2)\ /(2\ \sin)^{3/2}\ ]$
x $+\ [\ (\ (1^2-4m^2)(3^2-4m^2)/(2\cdot 4\cdot (2n+3)(2n+5))\ )$
x $\cos((n+5/2)\theta-5\pi/4 +m\pi/2)/ (2\sin\theta)^{5/2}\ ]\ + \ldots \}$ (21)

2.2.3. Type-B3 [26, pp. 146-147]

$\Gamma(n+3/2)P_n^m(\cos\theta) =\ [2^{1/2}(\pi\sin\theta)^{-1/2}\ \Gamma(n+m+1)\ ]$
x $\Sigma_{k=0}^{\infty}\ (-1)^k\ [\ (1/2+m)_k\ (1/2-m)_k\ \ / (k!\ (2\sin\theta)^k\ (n+3/2)_k\ )\ ]$
x $\sin[(n+k+1/2)\theta+ (m/2+1/4)\pi+k\pi/2]$ (22)

As for Type-B1, Eq. (22) was separated into the two functions W(n, m) and Z(k, n, m):
$W(n, m) =\ 2^{1/2}(\pi\sin\theta)^{-1/2}\Gamma(n+m+1) /\Gamma(n+3/2)$ (23)
$Z(k, n, m) = \Sigma_{k=0}^{\infty}(-1)^k[(\Gamma(1/2+m+k)/\Gamma(1/2+m))\ \ (\Gamma(1/2-m+k)/$
x $\Gamma(1/2-m)\ )\ ]\ \cdot (1/k!)\ \ \cdot [\Gamma(n+3/2) /\Gamma(n+3/2+k)]$
x $(1/(2\sin)^k\ ) \cdot \sin[(n+k+1/2)\theta+(m\pi/2+\pi/4)+k\pi/2]$
$= \Gamma(n+3/2)/((-1)^m\pi)\Sigma_{k=0}^{\infty}\ (-1)^k\ [\Gamma(1/2+m+k)/\Gamma(1/2-m+k))\ /$
x $(k!\Gamma(n+3/2+k)\ )\ ]\ (1/(2\sin\theta)^k\ ) \cdot$
x $\sin[(n+k+1/2)\theta+ k\pi/2+m\pi/2+\pi/4]$ (24)

In the function Z(k, n, m), the Gumma and sine functions which don't include the ordinal number k are moved to W(n,m):
$W(n, m) =\ 2^{1/2}(\pi\sin\theta)^{-1/2}\Gamma(n+m+1) /((-1)^m\pi)$ (25)
$Z(k, n, m) =\Sigma_{k=0}^{\infty}(-1)^k[\Gamma(1/2+m+k)\ (\Gamma(1/2-m+k)/$
x $(k!\Gamma(n+3/2+k)\ )\ ]\ (1/(2\sin\theta)^k\ ) \cdot$
x $\cos[(n+k+1/2)\theta+(k+m)\pi/2\ -\pi/4]$ (26)

(i) B3-a: First term (k=0)
$W(n, m)\ Z(0, n, m) =\ (-1)^0\ (2 /(\pi\sin\theta))^{1/2}\ (\Gamma(n+m+1) /((-1)^m\pi))$
x $((-1)^m\pi/(0!\Gamma(n+3/2)\ ))\ \cos[(n+1/2)\theta+ m\pi/2\ -\pi/4]$
$=\ (-1)^0\ (2/\pi^{1/2}\ )\ (\Gamma(n+m+1)/\Gamma(n+3/2)\ )$
x $\cos[(n+1/2)\theta+ m\pi/2\ -\pi/4] / (2\sin\theta)^{1/2}$ (27)

(ii) B3-b: Second term (k=1)
$Z(1, n, m) =\ (-1)^1\ [(\Gamma(1/2+m+1)\Gamma(1/2-m+1)/(1!\Gamma(n+3/2+1)\ )]$
x $\sin[(n+1+1/2)\theta+ (1+m)\pi/2\ +\pi/4]$
$=\ (-1)^1\ [\ ((1+2m)/2)\ (1-2m)/2\ )\ (-1)^m\pi/\ ((2n+3)/2)\Gamma(n+3/2))]$
x $\sin[(n+3/2)\theta+3\pi/4+m\pi/2\ ] / (2\sin\theta)$ (28)
$\therefore W(n, m)\ Z(1, n, m) =\ (-1)^1\ (2\cdot 2 /(\pi\cdot 2\sin\theta))^{1/2}\ (\Gamma(n+m+1) /((-1)^m\pi))$
x $(\ (1-4m^2)\ (-1)^m\pi/(2(2m+3)\Gamma(n+(3/2)\ )$
x $\sin[(n+3/2)\theta+3\pi/4+m\pi/2\ ] / (2\sin\theta)$
$=\ \ (2/\pi^{1/2}\ )\ [(\Gamma(n+m+1)/\Gamma(n+3/2)\ )\ (1-4m^2)/(2(2n+3))\ ]$
x $\cos[(n+3/2)\theta\ -\ 3\pi/4\ +m\pi/2] / (2\sin\theta)^{3/2}$ (29)

(iii) B3-c: Third term (k=2)
$Z(2, n, m) =\ (-1)^2\ [(\Gamma(1/2+m+2)\Gamma(1/2-m+2)/(2!\Gamma(n+3/2+2)\ )]$
x $\sin[(n+2+1/2)\theta+ (2+m)\pi/2\ +\pi/4] / (2\sin\ \theta)^2$
$=\ [\ (3/2+m)\Gamma(3/2+m)\ (3/2-m)\Gamma(3/2-m) /$
x $(\ 2(n+5/2)\Gamma(n+5/2)\ )\ ]$
x $\sin[(n+5/2)\theta+5\pi/4+m\pi/2\ ] / (2\sin\theta)^2$
$=\ [(3/2+m)(1/2+m)\Gamma(1/2+m)\ (3/2-m)(1/2-m)\Gamma(1/2-m) /$
$2(n+5/2)(n+3/2)\Gamma(n+3/2)\ ]$
x $\cos[(n+5/2)\theta\ -\ 5\pi/4\ +m\pi/2] / (2\sin\theta)^2$
$= (1^2-4m^2)\ (3^2-4m^2)\ (-1)^m\pi/(2\cdot 4$
x $(2n+5)\ (2n+3)\Gamma(n+3/2)\ )$
x $\cos[(n+5/2)\theta\ -\ 5\pi/4\ +m\pi/2] /\ \ (2\sin\theta)^2$ (30)
$\therefore W(n, m)\ Z(2, n, m) =\ (2\cdot 2 /(\pi\cdot 2\sin\theta))^{1/2}\ (\Gamma(n+m+1) /((-1)^m\pi))$
x $(1^2-4m^2)(3^2-4m^2)(-1)^m\pi/(2\cdot 4(2n+5)(2n+3)\Gamma(n+3/2)\ )$
x $\cos[(n+5/2)\theta\ -\ 5\pi/4\ +m\pi/2] / (2\sin\theta)^{5/2}$
$= (2/\pi^{1/2}\ )\ [(\Gamma(n+m+1)/\Gamma(n+3/2)\ )\ (1^2-4m^2)(3^2-4m^2) /$
$(2\cdot 4(2n+3)\ (2n+5)\ )]$

x $\cos[(n+5/2)\theta - 5\pi/4 + m\pi/2] / (2\sin\theta)^{5/2}$ (31)

From Eq. (27), Eq. (29), and Eq. (31), Eq. (22) may be expressed as follows:

$P_n^m(\cos\theta) = (2/\pi^{1/2})(\Gamma(n+m+1)/\Gamma(n+3/2))$

x $\{ \cos[(n+1/2)\theta + m\pi/2 - \pi/4] / (2\sin\theta)^{1/2}$

x $+ ((1-4m^2)/(2(2n+3))) \cos[(n+3/2)\theta - 3\pi/4 + m\pi/2]/(2\sin\theta)^{3/2}$

x $+ ((1^2-4m^2)(3^2-4m^2) / (2 \cdot 4(2n+3)(2n+5)))$

x $\cos[(n+5/2)\theta - 5\pi/4 + m\pi/2] / (2\sin\theta)^{5/2}$

x $+ \ldots \}$ (32)

2.2.4. Type-B4 [28, p. 325]

$P_n^m(\cos\theta) = \pi^{-1/2}\ 2^{m+1}(\sin\theta)^m (\Gamma(n+m+1) / \Gamma(n+3/2))$

x $\Sigma_{k=0}^{\infty} [(m+1/2)_k (n+m+1)_k / (k!\ (n+3/2)_k)]$

x $\sin[(n+m+2k+1)\theta]$

$(0 < \theta < \pi)$ (33)

As in the case of Type-B1, Eq. (33) was separated into the two functions W(n, m) and Z(k, n, m):

$W(n, m) = (2/\pi^{-1/2})(2\sin\theta)^m\ \Gamma(n+m+1) / \Gamma(n+3/2)$ (34)

$Z(k, n, m) = \Sigma_{k=0}^{\infty} [(\Gamma(m+1/2+k)/\Gamma(m+1/2))$

x $(\Gamma(n+m+2k)/\Gamma(n+m+k))(1/k!)$

x $(\Gamma(n+3/2)/\Gamma(n+3/2+k)) \sin((n+m+2k+1)\theta]$ (35)

The function W(n, m) expressed by Eq. (34) is different from that expressed by Eq. (23) of Type-B3. Similarly, the function Z(k, n, m) expressed by Eq. (35) is different from that expressed by Eq. (24) of Type-B3.

However, Eq. (33) is identical to 3.5 Eq. (2) of [26, p. 146] in which the expression was obtained using hypergeometric functions different from those used for the derivation of Eq. (22) of Type-B3 (see Appendix). The mathematical characteristics of Eq. (22) are accordant with those of Eq. (1), Eq. (2), Eq. (3), and Eq. (13). Therefore, Eq. (33) is not expanded into first three higher terms in the present analysis.

3. Discussion

Equations (1) and (2) of Type-A are completely identical by way of the asymptotic series of the associated Legendre function. The difference exists only in the formation of $(2\sin\theta)^{-1/2}$ in the sequence. The two equations have a mathematical limitation; namely, the computation accuracy is limited within the first three terms of the asymptotic series expansion. However, the approximate value for the associated Legendre function can be easily obtained if the parameters n, m, and θ are assigned.

Equations (3), (13), and (22) of Type-B perfectly consist of the sequence of the asymptotic series of the associated Legendre function. Through the analysis of the expansion of the asymptotic series, which is conducted in Section 2, it is found that Eq. (3), Eq. (13), and Eq. (22) can be expressed as Eq. (1) or Eq. (2), as long as the asymptotic series is limited within the first three terms. However, the coefficient $e^{m\pi i}$ of Eq. (3) seems to be redundant, as shown in Eq. (12). It should be noted here that the coefficient $e^{m\pi i}$ is forced on $P_n^m((\cos\theta)$ in Eq. (3) [31, p. 120] . There are some articles in which, for the definition of $P_n^m((\cos\theta)$, (i) the coefficient $e^{m\pi i}$ is added; (ii) the coefficient $e^{m\pi i}$ is deleted; and (iii) the coefficient $[\sin(n+m)\pi] / \sin(n\pi)$ or $\Gamma(n+m+1)$ is replaced, instead of $e^{m\pi i}$ [31, p.85; p. 120].

It is also noted that Eq. (16) of Type-B2 and Eq. (25) of Type-B3 have the same function W(n, m), which does not depend on the ordinal number k of the asymptotic series. The conspicuous point of Eq. (22) of Type-B3 is that the phase angle term is expressed by both the sine function and the ordinal number $(-1)^k$. On the other hand, the phase angle terms of Eq. (3) of Type-B1 and Eq. (13) of Type-B2 are expressed by only a cosine function.

4. Conclusion

The asymptotic series of the associated Legendre function may be classified into two types according to the manner of expression. The first type (Type-A) [30, 31] is the mathematical expression in which the asymptotic series is directly expanded into the first three higher terms. If the colatitude degree n, the azimuthal order m, and the colatitude angleθ are given in the formula, the associated Legendre function can be easily calculated. However, information on additional terms beyond these three is not involved.

The second type (Type-B) [26-29] is the mathematical expression in which the asymptotic series is expressed by the infinite series as the sum of the sequence of the ordinal numbers from 0 to∞ . In the present study, three formulas [26, 28, 29] belonging to Type-B were analyzed. Their mathematical expressions are slightly different from one another because the treatments of the Gamma function and the trigonometric function including the parameters m and θ are different. If the three formulas are expanded into the first three higher terms, they become equivalent to those of the asymptotic series of Type-A. The formula of Type-B [27] is different from the other formulas [26, 28, 29] because of the use of a different transformation formula in the hypergeometric equation [26].

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Appendix: Difference between Type-B3 and Type-B4

The Legendre functions are solutions of Legendre's differential equation:

$$(1-z^2)\, d^2w/dz^2 - 2z\, dw/dz + \{\nu(\nu+1) - \mu^2/(1-z^2)\}w = 0 \qquad \text{(A1)}$$

z, ν, μ unrestricted.

Under the substitution $w = (z^2 - 1)^{1/2\,\mu}\, v$, with $\zeta=1/2 - 1/2\, z$ as the independent variable, Eq. (A1) becomes [26, pp. 121-122]:

$$\zeta(1-\zeta)\, d^2v/d\zeta^2 + (\mu+1)(1-2\zeta)\, dv/d\zeta + (\nu-\mu)(\nu+\mu+1)\, v = 0 \qquad \text{(A2)}$$

This is Gauss' equation (the hypergeometric equation) with $a = \mu - \nu$, $b = \nu+\mu+1$, and $c = \mu+1$. The functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are solutions of Eq. (A2).

The hypergeometric equation is represented as:

$$z(1-z)\, d^2u/dz^2 + [c-(a+b+1)z]\, du/dz - abu = 0 \qquad \text{(A3)}$$

The solution of Eq. (A3) is:

$$u_1 = \Sigma_{n=0}^{\infty} (a)_n (b)_n z^n / [(c)_n\, n!] \equiv {}_2F_1(a, b; c; z) \equiv F(a, b; c; z). \qquad \text{(A4)}$$

where $(a)_n = \Gamma(a+n) / \Gamma(a)$.

By means of the transformation formula of the hypergeometric function, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are expressible in several ways in the forms

$$P_\nu^\mu(z) = A_1\, F(a_1, b_1; c_1; \zeta) + A_2\, F(a_2, b_2; c_2; \zeta) \qquad |\zeta| < 1 \qquad \text{(A5)}$$

$$e^{-\mu\pi} Q_\nu^\mu(z) = A_3\, F(a_3, b_3; c_3;\zeta) + A_4\, F(a_4, b_4; c_4; \zeta) \qquad |\zeta| < 1 \qquad \text{(A6)}$$

where ζ is a function of z and depends on the choice of the transformation. The various expansions (A5) and (A6) are shown to be the transformation formulas [26, pp. 124 – 139]. By the choices of transformation formulas 3.2 (44) and 3.2 (45) below, given in the expansions for $e^{-\mu\pi} Q_\nu^\mu(z)$ [26], Eq. (22) of Type-B3 [26] and Eq. (33) of Type-B4 [28], respectively, are obtained in the form of the hypergeometric series through several mathematical transaction processes [26, pp. 146-147].

The transformation formula 3.2 (44) [26, pp. 136-137] is:

$A_3 = (1/2\pi)^{1/2}\Gamma(1+\nu+\mu)\, (z^2-1)^{-1/4}\, [z-(z^2-1)^{1/2}]^{\nu+1/2} / \Gamma(\nu+3/2)$

$A_4 = 0$

$a_3 = 1/2+\mu$, $b_3 = 1/2 - \mu$, $c_3 = \nu+3/2$, $\zeta = -z + (z^2-1)^{1/2}$

$a_4 = \ldots$, $b_4 = \ldots$, $c_4 = \ldots$, $\zeta = 2\, (z^2-1)^{1/2}$.

The transformation formula 3.2 (45) [26, pp. 136-137] is:

$A_3 = \pi^{1/2} 2^\mu \Gamma(1+\nu+\mu)\, (z^2-1)^{1/2\mu}\, [z+(z^2-1)^{1/2}]^{-1-\nu-\mu} / \Gamma(\nu+3/2)$

$A_4 = 0$

$a_3 = \mu+1/2$, $b_3 = 1+\nu+\mu$, $c_3 = \nu+3/2$, $\zeta = z - (z^2-1)^{1/2}$

$a_4 = \ldots$, $b_4 = \ldots$, $c_4 = \ldots$, $\zeta = z + (z^2-1)^{1/2}$.