Modeling Kenya Domestic Radicalization like A Disease Incorporating Rehabilitation Centers

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Abstract: The study presents a deterministic model for radicalization process in Kenya and use the model to assess impact of rehabilitation centers to radicalization burden. The possibility of other drivers of radicalization to individuals who are not religious fanatics, and also individuals in rehabilitated subclass continuing being violent was considered. The model incorporated rehabilitation of the radicalized but peaceful individuals in subclass $R(t)$, and also radicalized but violent individuals in subclass $T(t)$, allowing recovery of individuals in subclass $R(t)$ from the intervention of good clergies. The stationary points were computed, their stabilities investigated and important thresholds determining the progression of the radicalization evaluated. The model sensitivity indices indicate that high intervention rates hold great promise to reduce the radicalization burden.

Keywords: Radicalization; Religious fanatics; Rehabilitation and clergies.

1. Introduction

According to Collins English Dictionary, radicalization is defined as a process by which a group and/or an individual comes to adopt increasingly extreme social, political or religious aspirations and ideals that undermine or reject the status quo, or undermine and/or reject contemporary expressions and ideas of freedom of choice [1].

Radicalization processes have a global impact but they do not automatically lead to violence [2]. Although it is widely believed all terrorists are religious fanatics, the available literature suggest some are not religious at all but religious based terrorism is on the increase and more lethal than other forms (3). Statistics points that religious terrorism was linked to suicide terrorism which accounted for every three of hundred terrorist activities from 1980 through 2003 [3]. The reports suggest terrorist mind set evolve as a result of psychological and social factors and calls for empathy in learning why vulnerable individuals engage in violent extremism [2, 4].

In Kenya, lethal terrorist attacks are associated with an innocent approach to broader international issues and the country's internal domestic processes [5]. The external radicalization is assumed to be driven by the following: global Interest and power, internet, global jihad, Somali Remittances, Islamic financial Institutions, flaws in anti-Jihad, dependency of international trends in radicalization and foreign failures to tap into Indigenous talent [1].

Terror attacks from within still pose a great challenge to Kenya even as Kenya Defense Forces and its allies effectively fight external aggressors Al-Shabaab in Somalia [1]. In Kenya, domestic radicalization is a complex and multifaceted and is assumed to be driven by the following factors:

i. Individual factors; Individuals could just crave for glorious status as they seek purpose in life.
ii. Social economic factors; these could be unemployment, in access to education and other services and/or facilities.
iii. Political factors; these could be crave to tackle impunity, corruption and injustice.
iv. Religious factors; these could be as a result of indoctrination.

According to study [6], new Mathematical models have a lot potential to give insight to terrorism, devise new approach method and aid to mitigate its threat. The study also points out that the complexity of interacting political, sociological and psychological factors are still challenge to mathematical modelers. The study [7], calls for more research on involvement of Al-shabaab and Mombasa Republican Council (MRC) in radicalization.

The research study [8], formulated and tested the implications of an economic model of terrorism using rich panel data set of 127 countries from 1968 to 1991. The result pointed an inverse relationship between economic activity and terrorism.

The research study [9], developed a theoretical assessment of how radicalization is reached involving determining factors of environment and an individual's motivations using a case studies Kosovo and Chechnya. The results stressed how intervening factors leads to different outcomes.

This research study, developed a deterministic model for Kenya domestic radicalization, taking in account how the aspect of religious fanaticism is being exploited by internal and external forces in radicalization process. Beside religious fanaticism, the possibility other drivers of radicalization were considered. Five compartment classes were

developed and nonlinear first order ordinary differential equations deduced to describe the dynamics of radicalization. The radicalization reproduction numbers were determined. Model analysis was carried out and analytical results obtained. Numerical simulation was carried out to confirm analytical results.

2. Model Development

The study described the model, stated assumption and deduced model equations.

2.1. Model Description

The total human population of Kenya is divided in compartments based on religion and the status of radicalization. Let $P(t)$ be the total human population which is sub-divided into five compartments: individuals who are religious fanatic in class M (t), individuals who are not religious fanatic in class N (t), individuals radicalized but peaceful R(t), individuals radicalized and engaging in violent activities T(t) and radicalized individuals who are undergoing rehabilitation H(t) .

The rate at which Kenyan convert from N (t) to M (t) is ω . The radicalized individuals recover after rehabilitation to M (t), at the rate τ. The constant natural death rate is given by μ. The birth rate is given by π. The radicalized individuals progress to engage in violent activities in class T(t) at a rate of θ. The rate at which individuals in subclasses R(t) and T(t) enroll for rehabilitation is γ and ρ respectively. Death or eloping to foreign countries due to terrorist related activities occur at the rates of δ_1 and δ_2 in sub-classes T(t) and H(t) respectively. Radicalized individuals recover subclass R(t) to M(t) at a rate of Ω while individuals fall back to T(t) from H(t) is α and the radicalization rate is given by β. η_1 and η_2, are proportions between zero and one. λ (t), is the force of radicalization, is given by, $\lambda(t) = \beta(R + \eta_1 T + \eta_2 H)$.

We further assumes: that there is homogeneous mixing of the Kenyan population, it is not possible for individuals in subclass R(t) to recover to subclass N(t), the effect of immigration and emigration is not significant. For simplicity, this study assumes: the effect of the foreign radicalization is not significant in this study, conversion from M(t) to N(t) is negligible and it is not possible for individuals in subclass to recover to N(t) . Although foreign individuals in subclasses R(t) and T(t) usually enters Kenya territory illegally, their numbers is assumed to be not significant in this study. The net number of individuals who elope to foreign countries due to terrorist related activities and later come back to Kenya is assumed to be not significant in this study. The individuals in subclass H(t) are assumed to be involved radicalization but at a lower rate than individuals in subclasses R(t) and T(t) because of counter radicalization efforts offered in rehabilitation centers and so they have reduced the levels of radicalization. The individuals in subclass R(t) are assumed to have more contacts and influence to individuals in subclasses M(t) and/or N(t) than individuals in subclass T(t) who are assumed to be terrorist on the run or stigmatized. The study assumed, $0 \leq \eta_2 < \eta_1 < 1$. Individuals in subclass R(t) are assumed to contribute highest in radicalization . It is also assumed that individuals in subclass R(t) are not likely to die or elope to foreign countries due to terrorist related activities. The terrorist induced death or eloping to foreign is much lower in the subclass H(t) because counter radicalization effort reduces the likelihood of individuals engaging in violent activities significantly. Hence, we assumed, $\delta_1 > \delta_2$. The coefficient κ is assumed to be κ ≥ 1, implying the ease of radicalizing individuals who are religious fanatics. We also assume that the radicalization is largely caused by religious indoctrination of fanatics and the other factors of radicalization in totality in our model.

2.2. Model Equations

We obtain the following system of first order nonlinear differential equations,

$$\frac{dM}{dt} = \sigma\pi + \tau H + \omega N + \Omega R - (\kappa\lambda + \mu)M \qquad (2.2.1),$$

$$\frac{dN}{dt} = (1 - \sigma)\pi - (\lambda + \omega + \mu)N \qquad (2.2.2),$$

$$\frac{dR}{dt} = \kappa\lambda M + \lambda N - \forall_1 R \qquad (2.2.3),$$

$$\frac{dT}{dt} = \theta R + \alpha H - \forall_2 T \qquad (2.2.4),$$

$$\frac{dH}{dt} = \rho T + \gamma R - \forall_3 H \qquad (2.2.5),$$

Where, $\forall_1 = (\theta + \Omega + \gamma + \mu), \forall_2 = (\rho + \delta_1 + \mu), \forall_3 = (\tau + \alpha + \delta_2 + \mu),$

$$P(t) = M(t) + N(t) + R(t) + T(t) + H(t) \text{ and } \lambda = \beta(R + \eta_1 T + \eta_2 H)$$

The initial conditions of the systems$[(2.2.1) - (2.2.5)]$ are represented by; M(0) = M_0,

N(0) = $N_0, R(0) = R_0, T(0) = T_0 \text{ and } H(0) = H_0$. The sum of the system of equations$[(2.3.1) - (2.3.7)]$, the rate of change of total population is given by,

$$\frac{dP}{dt} = \pi - \mu P - \delta_1 T - \delta_2 H.$$

3. Model Analysis

The radicalization model is analyzed by proving various theorems and carrying out algebraic computation dealing with different attributes.

3.1. Feasible Region and Bound of the Solution

We determine the feasible region and bound of the model by stating and proving the theorem below.
Theorem1. The region ξ is given by

$$\xi = \left\{ M(t), N(t), R(t), T(t), H(t) \varepsilon\ R_+^5, P \le \frac{\pi}{\mu} \right\}$$

is positively invariant and attracting with respect to model system$[(2.2.1) - (2.2.5)]$.

Proof

Let$\{ M(t), N(t), R(t), T(t), H(t)$ be any solutions of the system with non-negative initial conditions $\{M(0) \ge 0, N(0) \ge 0, R(0) \ge 0, T \ge 0, H \ge 0\}$.

Since,$\frac{dM}{dt} = \sigma\pi + \tau H + \omega N + \Omega R - (\kappa\lambda + \mu)M$, it follows that $\frac{dM}{dt} \ge -(\kappa\lambda + \mu)M$. On integration, we obtain, $\frac{d}{dt}[M(t)e^{\int_0^t -(\kappa\lambda+\mu)dM}] \ge 0$. Clearly, $M(t)e^{\int_0^t -(\kappa\lambda+\mu)dM}$ is a non-negative function of t, thus M (t) stays positive. The feasible region of $M(t), N(t), R(t), T(t)\ and\ H(t)$ is proved along the same lines as follows: $\frac{dN}{dt} = (1 - \sigma)\pi - (\lambda + \omega + \mu)N$, it follows that $\frac{dN}{dt} > -(\lambda + \omega + \mu)N$, which implies, $\frac{dN}{N} > -(\lambda + \omega + \mu)dt$. On integration, we obtain, $N(t) = C_1 e^{-(\lambda+\omega+\mu)t}$, where C_1 is a constant of integration, applying initial condition at $t = 0$, $C_1 = N(0)$. It follows that $N(t) > N(0)e^{-(\lambda+\omega+\mu)t} \ge 0$. Similarly, $R(t) > R(0)e^{-\forall_1 t} \ge 0$, $T(t) > T(0)e^{-\forall_2 t} \ge 0$ and $H(t) > H(0)e^{-\forall_3 t} \ge 0$.

The rate of change of total population is given by, $\frac{dP}{dt} = \pi - \mu P - \delta_1 T - \delta_2 H$, therefore,

$\frac{dP}{dt} + \mu P \le \pi$. On integration, we obtain $P(t) \le \frac{\pi}{\mu}\{1 + c_2 e^{-\mu t}\}$, where, c_2 is the constant of integration.

Hence, $\lim_{t\to\infty} N(t) \le \frac{\pi}{\mu}$. This proves the bound of the solutions inside ξ. This implies that all the solutions of our system of equations$[(2.2.1) - (2.2.5)]$, starting in ξ and remains in ξ for all $t \ge 0$. Thus ξ is positively invariant and attracting, and hence it is sufficient to consider the dynamics of our system in ξ. This completes the proof.

3.2. Radicalization Free Stationary Point (RFS)

Let $S^0 = (M^0, N^0, R^0, T^0, H^0)$ be the radicalization free stationary point of the system of equations$[(2.2.1) - (2.2.5)]$. Setting the subclasses R(t), T(t) and H(t) to zero, we obtain, $M^0 = \frac{(\omega+\mu\sigma)\pi}{\mu(\omega+\mu)}$ and $N^0 = \frac{(1-\sigma)\pi}{\omega+\mu}$. The RFS point$(S^0)$ for the system of equations$[(2.2.1) - (2.2.5)]$ is obtained as, $S^0 = (M^0, N^0, R^0, T^0, H^0) = \left\{ \frac{(\omega+\mu\sigma)\pi}{\mu(\omega+\mu)}, \frac{(1-\sigma)\pi}{\omega+\mu}, 0, 0, 0 \right\}$.

3.3. Control Reproduction Number and Basic Reproduction Number

In this study, the next-generation matrix method was used to determine the control reproduction number R_c of the radicalization model [10]. The other reproduction numbers such as basic reproduction number were derived from control reproduction number by setting some various parameters to zero. Using the notation f for a matrix of new radicals' recruits' terms and v for the matrix of the remaining transfer terms in our system, we get,

$$f = \begin{pmatrix} \kappa\lambda M + \lambda N \\ 0 \\ 0 \end{pmatrix}; v = \begin{pmatrix} \forall_1 R \\ -\theta R - \alpha H + \forall_2 T \\ -\rho T - \gamma R + \forall_3 H \end{pmatrix},$$

The matrices F and V are obtained by finding the Jacobian matrices of f and v evaluated at RFS point respectively we obtain:

$$F = \begin{pmatrix} \beta(kM^0 + N^0) & \eta_1\beta(\kappa M^0 + N^0) & \eta_2\beta(\kappa M^0 + N^0) \\ 0 & 0 & 0 \\ 0 & 0 & o \end{pmatrix}; V = \begin{pmatrix} \forall_1 & 0 & 0 \\ -\theta & \forall_2 & -\alpha \\ -\gamma & -\rho & \forall_3 \end{pmatrix}.$$

The inverse of V, is obtained as,

$$V^{-1} = \begin{pmatrix} \dfrac{1}{\forall_1} & 0 & 0 \\ \dfrac{\alpha\gamma + \theta\,\forall_3}{\forall_1(-\alpha\rho + \forall_2\,\forall_3)} & \dfrac{\forall_3}{-\alpha\rho + \forall_2\,\forall_3} & \dfrac{\alpha}{-\alpha\rho + \forall_2\,\forall_3} \\ \dfrac{\gamma\mu + \gamma\rho + \theta\rho + \gamma\delta_1}{\forall_1(-\alpha\rho + \forall_2\,\forall_3)} & \dfrac{\rho}{-\alpha\rho + \forall_2\,\forall_3} & \dfrac{(\rho + \delta_1 + \mu)}{-\alpha\rho + \forall_2\,\forall_3} \end{pmatrix}$$

The eigenvalues $[\chi_i, i = 1(1)3]$ of the Matrix FV^{-1} is obtained using Mathematica software, $\chi_1 = \chi_2 = 0$ and $\chi_3 = \beta(M^0 + \kappa N^0)\left\{\frac{1}{\forall_1} + \eta_1\left(\frac{\alpha\gamma + \theta\forall_3}{\forall_1(-\alpha\rho + \forall_2\forall_3)}\right) + \eta_2\left(\frac{\gamma\forall_2 + \theta\rho}{\forall_1(-\alpha\rho + \forall_2\forall_3)}\right)\right\}$. The control reproduction number R_C is given by the dominant eigenvalue of the matrix FV^{-1}, which is given by,

$$R_C = \beta(\kappa M^0 + N^0)\left\{\frac{1}{\forall_1} + \eta_1\left(\frac{\alpha\gamma + \theta\,\forall_3}{\forall_1(-\alpha\rho + \forall_2\,\forall_3)}\right) + \eta_2\left(\frac{\gamma\,\forall_2 + \theta\rho}{\forall_1(-\alpha\rho + \forall_2\,\forall_3)}\right)\right\},$$

__Definition 1:__ The control reproduction number R_C is the average number of M(t) and/or N(t), one radicalized individual in(R(t) or T(t) or H(t)) can radicalize when the counter radicalization rehabilitation efforts are already in place for individuals in subclasses R(t) and T(t). .

When radicalized individuals are not engaging in violent activities, little or no effort may be focused to those individuals to join rehabilitation centers and the corresponding reproduction number(R_γ) is obtained by setting γ to zero, we obtain,

$$R_\gamma = \beta(\kappa M^0 + N^0)\left\{\frac{1}{(\theta + \Omega + \mu)} + \eta_1\left(\frac{\theta\,\forall_3}{(\theta + \Omega + \mu)(-\alpha\rho + \forall_2\,\forall_3)}\right) + \eta_2\left(\frac{\theta\rho}{(\theta + \Omega + \mu)(-\alpha\rho + \forall_2\,\forall_3)}\right)\right\}.$$

__Definition 2:__ The control reproduction number R_γ is the average number of M(t) and/or N(t), one radicalized individual (in R(t) or T(t) or H(t)) can radicalize when the counter radicalization rehabilitation efforts are already in place for individuals in subclass T(t).

Sometimes policy makers may decide to focus on rehabilitating individuals in subclass R(t) and opt to deal with individuals in subclass T(t) using other means. The corresponding reproduction number(R_ρ) is obtained by setting ρ to zero, we obtain,

$$R_\rho = \beta(\kappa M^0 + N^0)\left\{\frac{1}{\forall_1} + \eta_1\left(\frac{\alpha\gamma + \theta\,\forall_3}{\forall_1((\delta_1 + \mu)\,\forall_3)}\right) + \eta_2\left(\frac{\gamma\mu + \gamma\delta_1}{\forall_1((\delta_1 + \mu)\,\forall_3)}\right)\right\}.$$

__Definition 3:__ The control reproduction number R_ρ is the average number of M(t) and/or N(t), one radicalized individual (in R(t) or T(t) or H(t)) can radicalize when the counter radicalization rehabilitation efforts are already in place for individuals in subclass T(t).

In absence of any intervention the basic reproduction number is obtained by setting ρ and γ to zero, we obtain,

$$R_0 = \beta(\kappa M^0 + N^0)\left\{\frac{1}{(\theta + \Omega + \mu)} + \eta_1\left(\frac{\theta\,\forall_3}{(\theta + \Omega + \mu)((\delta_1 + \mu)\,\forall_3)}\right)\right\}.$$

__Definition 4:__ The basic reproduction number R_0 is the average number of M(t) and/or N(t), one radicalized individual (in R(t) or T(t) or H(t)) can radicalize in absence of any counter radicalization effort.

3.4. Existence of the Radicalization Persistent Stationary Point (RPS)
Theorem 2
The radicalization persistence stationary point exist whenever $R_c^* > 1$.
Proof
To obtain the qualitative behavior of the existence of the radicalization persistence stationary point, the study proposes to add the system of equations$[(2.2.3) - (2.2.5)]$ to form one equation, for the system to tractable mathematically. Further, we let, $\phi_1 A = \Omega R, \phi_2 A = \delta_1 T, \phi_3 A = \tau H$,

$\phi_4 A = \delta_2 H, \phi_5 A = R + \eta_1 T + \eta_2 H$, RFS of the system of equations$[(3.4.1) - (3.4.3)]$ be denoted by $S^{00} = [M^{00}, N^{00}, A^{00}]$. and the RPS of the system of equations$[(3.4.1) - (3.4.3)]$ be denoted by $S^{***} = [M^{***}, N^{***}, A^{***}]$. The system of equations$[(2.2.1) - (2.2.5)]$ reduces to the system of equations below,

$$\frac{dM}{dt} = \sigma\pi + \omega N + (\phi_1 + \phi_3)A - (\kappa\lambda^{***} + \mu)M \qquad (3.4.1),$$

$$\frac{dN}{dt} = (1 - \sigma)\pi - (\lambda^{***} + \omega + \mu)N \qquad (3.4.2),$$

$$\frac{dA}{dt} = \lambda^{***}(\kappa M + N) - (\mu + \phi_1 + \phi_2 + \phi_3 + \phi_4)A \qquad (3.4.3),$$

where, $\lambda^{***} = \beta\phi_5 A \ and \ R_c^* = \frac{\beta\phi_5(\kappa M^{00} + N^{00})}{(\mu + \phi_1 + \phi_2 + \phi_3 + \phi_4)}$. At the radicalization persistence stationary point the system of equations$[(3.4.1) - (3.4.3)]$ are equal to zero therefore solving $M^{***}, N^{***} \ and \ A^{***}$ in terms of λ^{***} we obtain the following using Mathematica software,

$$M^{***} = (\pi\big(\mu(\lambda^{***}\sigma + \mu\sigma + \omega) + (\lambda^{***} + \mu\sigma + \omega)\phi_1 + (\lambda^{***}\sigma + \mu\sigma + \omega)\phi_2 + (\lambda^{***}\sigma + \mu\sigma + \omega)\phi_3$$
$$+ (\lambda^{***}\sigma + \mu\sigma + \omega)\phi_4\big))$$
$$/((\lambda^{***} + \mu + \omega)(\mu\phi_1 + (\kappa\lambda^{***} + \mu)\phi_2 + \mu(\kappa\lambda^{***} + \mu + \phi_3) + (\kappa\lambda^{***} + \mu)\phi_4)),$$

$$N^{***} = -\frac{\pi(-1 + \sigma)}{\lambda^{***} + \mu + \omega},$$

$$A^{***} = \frac{\pi\kappa\lambda^{***}(\mu\sigma + (\lambda^{***} + \mu - \mu\sigma) + \omega)}{(\lambda^{***} + \mu + \omega)(\mu\phi_1 + (\kappa\lambda^{***} + \mu)\phi_2 + \mu(\kappa\lambda^{***} + \mu + \phi_3) + (\kappa\lambda^{***} + \mu)\phi_4)}.$$

Substituting A^{***} in λ^{***} we obtain two cases,

Case 1; $\lambda^{***} = 0$, this correspond to the RFS(S^{00}) of the system$[(3.4.1) - (3.4.3)]$, given by,

$$S^{00} = [M^{00}, N^{00}, A^{00}] = \left[\frac{(\omega + \mu\sigma)\pi}{\mu(\omega + \mu)}, \frac{(1 - \sigma)\pi}{\omega + \mu}, 0\right].$$

Case 2; $a_1\lambda^{***2} + a_2\lambda^{***} + a_3 = 0$, this correspond to the RPS(S^{**}) of the system $[(3.4.1) - (3.4.3)]$, where,

$a_1 = \kappa\mu + \kappa\mu + \kappa\phi_4,$

$a_2 = \mu^2 + \kappa\mu^2 + \mu\omega + \kappa\mu\phi_1 + \mu\phi_2 + \kappa\mu\phi_2 + \omega\phi_2 + \kappa\mu\phi_3 + \mu\phi_4 + \kappa\mu\phi_4 + \omega\phi_4 - \pi\beta\kappa\phi_5,$

$a_3 = -\mu(\mu + \omega)(-1 + R_c^*)\big(\mu + \phi_1 + \phi_2 + \phi_3 + \phi_4\big).$

The condition necessary and sufficient for the real roots of quadratic equation in case 2 $a_3 < 0$, which follow that $R_c^* > 1$. This completes the proof.

The roots of the quadratic equation in case 2 are; $\lambda_1^{***} = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_1} > 0$ and $\lambda_2^{***} = \frac{-a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2a_1} < 0$. The root $\lambda_2^{***} < 0$ is not socially feasible hence the system has one radicalization persistent stationary point. The actual radicalization is determined by substituting λ^{***} in M^{***}, N^{***} and A^{***} with λ_1^{***} obtained above.

3.5. Stabilities of the Stationary Points
The study used the signs of eigenvalues, Routh-Hurwtz criteria for stability, bifurcation analysis and Lyapunov function to determine the stability of the stationary points.

3.5.1. Local Stability of the Radicalization Free Stationary Point (RFS).
To determine the local stability of the RFS, we shall state and proof the theorem below.
Theorem 3
The radicalization free stationary point is locally unstable in the sense of Routh-Hurwtz criteria for stability.
Proof
To establish the local stability of radicalization free equilibrium point(S^0), we use the Jacobian of the system of equations$[(2.2.1) - (2.2.5)]$ evaluated at S^0. Stability of this steady state is then determined based on the eigenvalues of the corresponding Jacobian matrix which are functions of the radicalization model parameters. The Jacobi matrix is obtained as

$$J(S^0) = \begin{pmatrix} -\mu & \omega & \Omega - \kappa\beta M^0 & -\eta_1\kappa\beta M^0 & \tau - \eta_2\kappa\beta M^0 \\ 0 & -(\omega + \mu) & -\beta N^0 & -\eta_1\beta N^0 & -\eta_2\beta N^0 \\ 0 & 0 & \beta(\kappa M^0 + N^0) - \forall_1 & \beta\eta_1(\kappa M^0 + N^0) & \beta\eta_2(\kappa M^0 + N^0) \\ 0 & 0 & \theta & -\forall_2 & \alpha \\ 0 & 0 & \gamma & \rho & -\forall_3 \end{pmatrix},$$

Clearly from the above matrix two of the eigenvalues are $-\mu$ and $-(\omega + \mu)$. The matrix reduces to,

$$\begin{pmatrix} \beta(\kappa M^0 + N^0) - \forall_1 & \beta\eta_1(\kappa M^0 + N^0) & \beta\eta_2(\kappa M^0 + N^0) \\ \theta & -\forall_2 & \alpha \\ \gamma & \rho & -\forall_3 \end{pmatrix},$$

It follows that the characteristic polynomial of the above matrix is $b_1\chi^3 + b_2\chi^2 + b_3\chi + b_4 = 0$, where $\chi_j, j = 1(1)3$, are eigenvalues and the constants b_1, b_2, b_3 and b_4 are obtained using Mathematica software as,

$$b_1 = -1; \quad b_2 = -(\forall_1 + \forall_2 + \forall_3) + \frac{R_C\forall_1(\alpha\rho - \forall_2\forall_3)}{-(\alpha\gamma + \theta\forall_3)\eta_1 - \forall_2(\forall_3 + \gamma\eta_2) + \rho(\alpha - \theta\eta_2)};$$

$$b_3 = (\forall_1((-1 + R_C)(\forall_2 + \forall_3)(-\alpha\rho + \forall_2\forall_3) - ((\forall_2 + \forall_3)(\alpha\gamma + \theta\forall_3) + \theta R_C(\alpha\rho - \forall_2\forall_3))\eta_1$$
$$- ((\theta\rho + \gamma\forall_2)(\forall_2 + \forall_3) + \gamma R_C(\alpha\rho - \forall_2\forall_3))\eta_2$$
$$- (\alpha\rho - \forall_2\forall_3)\big(\alpha\rho - (\alpha\gamma + \theta\forall_3)\eta_1 - \theta\rho\eta_2 - \forall_2(\forall_3 + \gamma\eta_2)\big))/(-\alpha\rho + (\alpha\gamma + \theta\forall_3)\eta_1 + \theta\rho\eta_2$$
$$+ \forall_2(\forall_3 + \gamma\eta_2));$$

$$b_4 = -(-1 + R_C)\forall_1(\alpha\rho - \forall_2\forall_3).$$

Using the Routh-Hurwitz criteria for stability, the first condition is the constants b_1, b_2, b_3 *and* b_4 should have the same sign. Since $b_1 < 0$, we determine necessary for b_2, b_3 *and* b_4 to be less than zero. Through inspection of the above expressions of b_2, b_3 *and* b_4, $b_2 < 0$ when $\alpha\rho > \forall_2 \forall_3$, $b_3 < 0$ when $R_C < 1$ and $b_4 < 0$ when $R_C < 1$. The condition $\alpha\rho > \forall_2 \forall_3$ is not feasible socially and therefore the first condition for stability in Routh-Hurwitz sense fails hence we conclude the RFS is locally unstable. This completes the proof.

3.5.2. Global Stability of the Radicalization Free Stationary Point

Theorem 4

The RFS is globally asymptotically stable in Lyapunov sense whenever the three conditions below are satisfied and unstable otherwise,

    i.   Either $\frac{N^0}{N} > \frac{M^0}{M}$ and $N < N^0 or \frac{N^0}{N} < \frac{M^0}{M}$ and $N > N^0$,

    ii. $\rho(\theta\tau - \alpha\Omega) + \forall_2(\gamma\tau + \Omega\forall_3) \leq 0$,

    iii. $R_C < 1$.

Proof

The study proposes the following Lyapunov function,

$$L(M, N, R, T, H) = M - M^0 - M^0 Ln\frac{M}{M^0} + N - N^0 - N^0 Ln\frac{N}{N^0} + Y_1 R + Y_2 T + Y_3 H \quad (i)$$

where Y_1, Y_2 and Y_3 are positive constants to be determined. The equation (i) satisfies the conditions; $L(M^0, N^0, R^0, T^0, H^0) = 0$ and $L(M, N, R, T, H) > 0$, therefore $L(M, N, R, T, H)$ is positive definite.

For $\frac{dL(M,N,R,T,H)}{dt}$ to be negative definite, it must satisfies $\frac{dL(M^0,N^0,R^0,T^0,H^0)}{dt} = 0$ and $\frac{dL(M,N,R,T,H)}{dt} < 0$

At DFE point for our system E^0 from equation (i),

$$\sigma\pi = -\omega N^0 + \mu M^0 \quad ; (1 - \sigma)\pi = (\omega + \mu)N^0$$

Calculating the derivative of equation (i), we get,

$$\frac{dL(M, N, R, T, H)}{dt} = \left(1 - \frac{M^0}{M}\right)\frac{dM}{dt} + \left(1 - \frac{N^0}{N}\right)\frac{dN}{dt} + Y_1\frac{dR}{dt} + Y_2\frac{dT}{dt} + Y_3\frac{dH}{dt},$$

$$= \left(1 - \frac{M^0}{M}\right)\{-\omega N^0 + \mu M^0 + \tau H + \omega N + \Omega R - (\lambda + \mu)M\} + \left(1 - \frac{N^0}{N}\right)\{(\omega + \mu)N^0 - (\kappa\lambda + \omega + \mu)N\}$$

$$+ Y_1\{\lambda M + \kappa\lambda N - \forall_1 R\} + Y_2\{\theta R + \alpha H - \forall_2 T\} + Y_3\{\rho T + \gamma R - \forall_3 H\},$$

,

$$\lambda = \beta(R + \eta_1 T + \eta_2 H)$$

$$= -\mu\frac{(M - M^0)^2}{M} - \mu\frac{(N - N^0)^2}{N} - \frac{M^0}{M}\{\tau H + \Omega R\} + \omega\left(\frac{N^0}{N} - \frac{M^0}{M}\right)\{N - N^0\} + \{-\beta + Y_1\beta\}RM$$

$$+ \{-\beta\eta_1 + Y_1\beta\eta_1\}TM + \{-\beta\eta_2 + Y_1\beta\eta_2\}HM + \{-\kappa\beta + \kappa Y_1\beta\}RN + \{-\kappa\beta\eta_1 + \kappa Y_1\beta\eta_1\}TN$$

$$+ \{-\kappa\beta\eta_2 + \kappa Y_1\beta\eta_2\}HN + \{\Omega - Y_1\forall_1 + Y_2\theta + Y_3\gamma + \beta M^0 + \kappa\beta N^0\}R$$

$$+ \{\tau + Y_2\alpha - Y_3\forall_3 + \eta_2\beta M^0 + \eta_2\kappa\beta N^0\}H + \{-Y_2\forall_2 + Y_3\rho + \eta_1\beta M^0 + \eta_1\kappa\beta N^0\}T$$

Setting the coefficients of RM, TM, HM, RN, TN, HN, R and H to zero and solving for Y_1, Y_2 and Y_3, we obtain,

$Y_1 = 1$,

$$Y_2 = (\forall_3{}^2(-(\Omega + (-1 + R_C)\forall_1)\forall_2 + \theta(-\Omega + \forall_1)\eta_1) + \gamma(\alpha\tau(\rho - \gamma\eta_1) - (\theta\rho\tau - \alpha\rho R_C\forall_1 + \gamma\tau\forall_2)\eta_2)$$

$$+ \forall_3(\alpha\rho\Omega + \alpha\rho(-1 + R_C)\forall_1 + \alpha\gamma(-\Omega + \forall_1)\eta_1 - \gamma\tau(\forall_2 + \theta\eta_1)$$

$$- (\theta\rho(\Omega - \forall_1) + \gamma(\Omega + (-1 + R_C)\forall_1)\forall_2)\eta_2))$$

$$/((\alpha\gamma + \theta\forall_3)(-\alpha\rho + (\alpha\gamma + \theta\forall_3)\eta_1 + \theta\rho\eta_2 + \forall_2(\forall_3 + \gamma\eta_2)))$$

$$Y_3 = \frac{1}{\alpha\gamma + \theta\forall_3}(\theta\tau - \alpha\Omega$$

$$+ \frac{\forall_1\left(R_C(\alpha\rho - \forall_2\forall_3)(\alpha - \theta\eta_2) + \alpha\left(-\alpha\rho + (\alpha\gamma + \theta\forall_3)\eta_1 + \theta\rho\eta_2 + \forall_2(\forall_3 + \gamma\eta_2)\right)\right)}{\left(-\alpha\rho + (\alpha\gamma + \theta\forall_3)\eta_1 + \theta\rho\eta_2 + \forall_2(\forall_3 + \gamma\eta_2)\right)}).$$

Substituting Y_1, Y_2, Y_3 and Y_4 in equation (ii) we obtain,

$$\frac{dL(M, N, R, T, H)}{dt} = -\mu\frac{(M - M^0)^2}{M} - \mu\frac{(N - N^0)^2}{N} - \frac{M^0}{M}\{\tau H + \Omega R\} + \omega\left(\frac{N^0}{N} - \frac{M^0}{M}\right)\{N - N^0\}$$

$$+ \left\{\frac{\rho(\theta\tau - \alpha\Omega) + \forall_2(\gamma\tau + \Omega\forall_3) + (-1 + R_C)\forall_1(-\alpha\rho + \forall_2\forall_3)}{\alpha\gamma + \theta\forall_3}\right\}T$$

The following are the conditions necessary for $\frac{dL(M,N,R,T,H)}{dt} < 0$, either $\frac{N^0}{N} > \frac{M^0}{M}$ and $N < N^0 or \frac{N^0}{N} < \frac{M^0}{M}$ and $N > N^0$, $\rho(\theta\tau - \alpha\Omega) + \forall_2(\gamma\tau + \Omega\forall_3) \leq 0$ *and* $R_C < 1$.

Condition $\rho(\theta\tau - \alpha\Omega) + \forall_2(\gamma\tau + \Omega\forall_3) \leq 0$, above does not make sense socially hence we conclude that the condition necessary for global stability of the radicalization free equilibrium point is $R_C < 1$ with either $\frac{N^0}{N} > \frac{M^0}{M}$ and $N < N^0 or \frac{N^0}{N} < \frac{M^0}{M}$ and $N > N^0$. This completes the proof.

3.5.3. Local Stability of the Radicalization Persistent Stationary Point (RPS)

The study determined the local stability of RPS by carrying bifurcation analysi [11, 12]. The study used the centre manifold theory to carry out the bifurcation analysis. To apply this theory, we use the following change of variables,

$$M = p_1, N = p_2, R = p_3, T = p_4 \text{ and } H = p_5, \text{so that } N = p_1 + p_2 + p_3 + p_4 + p_5$$

Further, by using vector notation $p = (p_1, p_2, p_3, p_4, p_5)^T$, the model can be written in the form $\frac{dp}{dt} = S(p)$, with $R = (s_1, s_2, s_3, s_4, s_5)^T$, as follows:

$\dot{p}_1 = s_1 = \sigma\pi + \tau p_5 + \omega p_2 + \Omega p_3 - \beta(p_3 + \eta_1 p_4 + \eta_2 p_5)p_1 - \mu p_1$ **(3.5.3.1)**,

$\dot{p}_2 = s_2 = (1 - \sigma)\pi - \kappa\beta(p_3 + \eta_1 p_4 + \eta_2 p_5)p_1 - (\omega + \mu)p_1$ **(3.5.3.2)**,

$\dot{p}_3 = s_3 = \beta(p_3 + \eta_1 p_4 + \eta_2 p_5)p_1 + \kappa\beta(p_3 + \eta_1 p_4 + \eta_2 p_5)p_2 - \forall_1 p_3$ **(3.5.3.3)**,

$\dot{p}_4 = s_4 = \theta p_3 + \alpha p_5 - \forall_2 p_4$ **(3.5.3.4)**,

$\dot{p}_5 = s_5 = \rho p_4 + \gamma p_3 - \forall_3 p_5$ **(3.5.3.5)**.

Considering the case when $R_C = 1$, we choose $\beta = \beta^*$ as a bifurcation parameter. Solving for β^* when $R_C = 1$ we get,

$$\beta^* = \frac{\forall_1(\alpha\rho - \forall_2\forall_3)}{(M^0 + \kappa N^0)(-(\alpha\gamma + \theta\forall_3)\eta_1 - \forall_2(\forall_3 + \gamma\eta_2) + \rho(\alpha - \theta\eta_2))}$$

Evaluating the Jacobian of the system $[(3.5.3.1) - (3.5.3.5)]$ at the radicalization free equilibrium point, denoted by $J(E_*^0)$. This gives:

$$J(E_*^0) = \begin{pmatrix} -\mu & \omega & \Omega - \beta^*M^o & -\eta_1\beta^*M^o & \tau - \eta_2\beta^*M^o \\ 0 & -(\omega + \mu) & -\kappa\beta^*N^o & -\kappa\eta_1\beta^*N^o & -\kappa\eta_2\beta^*N^o \\ 0 & 0 & \beta^*X - \forall_1 & \beta^*\eta_1 X & \beta^*\eta_2 X \\ 0 & 0 & \theta & -\forall_2 & \alpha \\ 0 & 0 & \gamma & \rho & -\forall_3 \end{pmatrix},$$

where $X = (\kappa M^o + N^o)$. It can be shown that the Jacobian of $\frac{dp}{dt} = R(p)$ at the DFE, with $\beta = \beta^*$, denoted by $J(E_*^0)$, has four eigenvalues with negative real parts and one simple zero eigenvalue. Hence, the centre manifold theory can be used to analyze the dynamics of the model using Castillo-Chavez and Song theorem [11, 12]. Eigenvectors of J_{β^*}: For the case when $R_C = 1$, it can be shown that the Jacobian of$[(3.5.3.1) - (3.5.3.5)]$ at $\beta = \beta^*$ (denoted by J_{β^*}) has a right eigenvector given by $u = [u_1, u_2, u_3, u_4, u_5]^T$, where,

$u_1 = \dfrac{-\beta^*M^o(u_3 + \eta_1 u_4 + \eta_2 u_5) + \omega u_2 + \Omega u_3 + \tau u_5}{\mu} < 0;$

$u_2 = \dfrac{-\kappa\beta^*N^o(u_3 + \eta_1 u_4 + \eta_2 u_5)}{(\omega + \mu)} < 0; \; u_3 = u_3 > 0; \; u_4 = \dfrac{\theta u_3 + \alpha u_5}{\forall_2} > 0;$

$u_5 = \dfrac{\gamma u_3 + \rho u_4}{\forall_3} > 0.$

Further, J_{β^*} has a left eigenvectors $v = [v_1, v_2, v_3, v_4, v_5]^T$, where,

$v_1 = 0; v_2 = 0; \; v_3 = v_3 > 0; v_4 = \dfrac{\beta^*\eta_1 Xv_2 + \rho v_5}{\forall_2} > 0; v_5 = \dfrac{\beta^*\eta_2 Xv_3 + \alpha v_4}{\forall_3} > 0.$

Since, $v_1 = 0$ and $v_2 = 0$, we only need to compute the partial derivatives of s_3, s_4 and s_5. For system the associated non-zero partial derivative of s_3 is given by

$\dfrac{\partial^2 s_3}{\partial p_1 \partial p_3} = \dfrac{\partial^2 s_3}{\partial p_3 \partial p_1} = \beta^*; \dfrac{\partial^2 s_3}{\partial p_1 \partial p_4} = \dfrac{\partial^2 s_3}{\partial p_4 \partial p_1} = \eta_1\beta^*; \dfrac{\partial^2 s_3}{\partial p_1 \partial p_5} = \dfrac{\partial^2 s_3}{\partial p_5 \partial p_1} = \eta_2\beta^*;$

$\dfrac{\partial^2 s_3}{\partial p_2 \partial p_3} = \dfrac{\partial^2 s_3}{\partial p_3 \partial p_2} = \kappa\beta^*; \dfrac{\partial^2 s_3}{\partial p_2 \partial p_4} = \dfrac{\partial^2 s_3}{\partial p_4 \partial p_2} = \kappa\eta_1\beta^*; \dfrac{\partial^2 s_3}{\partial p_2 \partial p_5} = \dfrac{\partial^2 s_3}{\partial p_5 \partial p_2} = \kappa\eta_2\beta^*$

It implies,

$a = v_3 \sum_{i,j=1}^{5} u_i u_j \dfrac{\partial^2 s_k}{\partial p_i \partial p_j}$

$\quad\quad = 2v_3\{u_1 u_3(\beta^*) + u_1 u_4(\eta_1\beta^*) + u_1 u_5(\eta_2\beta^*) + u_2 u_3(\kappa\beta^*) + u_2 u_4(\kappa\eta_1\beta^*) + u_2 u_5(\kappa\eta_2\beta^*)\}$

$\quad\quad > 0,$

To calculate b we compute the second order derivative so that,

$$\frac{\partial^2 s_3}{\partial p_3 \partial \beta^*} = \beta^*(M^0 + \kappa N^0); \frac{\partial^2 s_3}{\partial p_4 \partial \beta^*} = \eta_1 \beta^*(M^0 + \kappa N^0); \frac{\partial^2 s_3}{\partial p_5 \partial \beta^*} = \eta_2 \beta^*(M^0 + \kappa N^0)$$

$$b = v_3 \sum_{i,j=1}^{4} u_i \frac{\partial^2 s_k}{\partial p_i \partial \beta^*}$$

$$= v_3 \{ u_3(\beta^*(M^0 + \kappa N^0)) + u_4(\eta_1 \beta^*(M^0 + \kappa N^0)) + u_5(\eta_2 \beta^*(M^0 + \kappa N^0)) \} > 0.$$

Hence, it follows for a fact that $a > 0 \ and \ b > 0$, which means the system of equations $[(2.2.1) - (2.2.5)]$ exhibits a backward bifurcation $R_C = 1$ [12].

3.5.4. Global stability of the radicalization persistent stationary point(RPS).

Theorem

The radicalization persist equilibrium point is globally asymptotically stable in Lyapunov sense whenever $W < G$ and unstable otherwise, where,

$$\begin{aligned}
G = {}& \mu \frac{(M - M^{**})^2}{M} + (\omega + \mu) \frac{(N - N^0)^2}{N} + \frac{M^{**}}{M} \{ \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} + \tau H + \omega N + \Omega R \} \\
& + \frac{N^{**}}{N} \{ \kappa \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} \} + \frac{R^{**}}{R} \{ \beta(R + \eta_1 T + \eta_2 H)M + \kappa \beta(R + \eta_1 T + \eta_2 H)N \} \\
& + \frac{T^{**}}{T} \{ \theta R + \alpha H \} + \frac{H^{**}}{H} \{ \rho T + \gamma R \} + \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} \\
& + (\tau H^{**} + \omega N^{**} + \Omega R^{**} + \forall_2 T + \forall_1 R + \forall_3 H), \\
W = {}& \tau H + \omega N + \Omega R + \kappa \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} + \theta R + \alpha H + \rho T + \gamma R + \forall_3 H^{**} + \forall_2 T^{**} + \forall_1 R^{**} \\
& + \kappa \beta(R + \eta_1 T + \eta_2 H)N^{**} + \beta(R + \eta_1 T + \eta_2 H)M^{**} + \frac{M^{**}}{M}(\tau H^{**} + \omega N^{**} + \Omega R^{**}).
\end{aligned}$$

Proof

The RFS of the system of equations $[(2.2.1) - (2.2.5)]$ is denoted by $S^0 = [M^0, N^0, R^0, T^0, H^0]$. Let the RPS of the system of equations $[(2.2.1) - (2.2.5)]$ be denoted by $S^{**} = [M^{**}, N^{**}, R^{**}, T^{**}, H^{**}]$. The study propose the following Lyapunov function,

$$\begin{aligned}
L(M, N, R, T, H) = {}& M - M^{**} - M^{**} \text{Ln} \frac{M}{M^0} + N - N^{**} - N^{**} \text{Ln} \frac{N}{N^{**}} + R - R^{**} - R^{**} \text{Ln} \frac{R}{R^{**}} + T - T^{**} - T^{**} \text{Ln} \frac{T}{T^{**}} \\
& + H - H^{**} - H^{**} \text{Ln} \frac{H}{H^{**}},
\end{aligned}$$

The equation (i) satisfies the conditions; $L(M^{**}, N^{**}, R^{**}, T^{**}, H^{**}) = 0$ and $L(M, N, R, T, H) > 0$, therefore $L(M, N, R, T, H)$ is positive definite. For $\frac{dL(M,N,R,T,H)}{dt}$ to be negative definite, it must satisfies $\frac{dL(M^{**},N^{**},R^{**},T^{**},H^{**})}{dt} = 0$ and $\frac{dL(M,N,R,T,H)}{dt} < 0$. At RPS, $\lambda^{**} = \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})$, $\sigma \pi = \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} + \mu M^{**} - \tau H^{**} - \omega N^{**} - \Omega R^{**}$,

$(1 - \sigma)\pi = \kappa \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} + (\omega + \mu)N^{**}$,

$\beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} + \kappa \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} = \forall_1 R^{**}$,

$\rho T^{**} + \gamma R^{**} = \forall_3 H^{**}$.

Calculating the derivative of equation (i), we obtain,

$$\frac{dL(M, N, R, T, H)}{dt} = \left(1 - \frac{M^{**}}{M}\right)\frac{dM}{dt} + \left(1 - \frac{N^{**}}{N}\right)\frac{dN}{dt} + \left(1 - \frac{R^{**}}{R}\right)\frac{dR}{dt} + \left(1 - \frac{T^{**}}{T}\right)\frac{dT}{dt} + \left(1 - \frac{H^{**}}{H}\right)\frac{dH}{dt},$$

$$\begin{aligned}
= {}& \left(1 - \frac{M^{**}}{M}\right) \{ \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} + \mu M^{**} - \tau H^{**} - \omega N^{**} - \Omega R^{**} + \tau H + \omega N + \Omega R \\
& - \beta(R + \eta_1 T + \eta_2 H)M - \mu M \} \\
& + \left(1 - \frac{N^{**}}{N}\right) \{ \kappa \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} + (\omega + \mu)N^{**} - \kappa \beta(R + \eta_1 T + \eta_2 H)N - (\omega + \mu)N \} \\
& + \left(1 - \frac{R^{**}}{R}\right) \{ \beta(R + \eta_1 T + \eta_2 H)M + \kappa \beta(R + \eta_1 T + \eta_2 H)N - \forall_1 R \} \\
& + \left(1 - \frac{T^{**}}{T}\right) \{ \theta R + \alpha H - \forall_2 T \} + \left(1 - \frac{H^{**}}{H}\right) \{ \rho T + \gamma R - \forall_3 H \},
\end{aligned}$$

$$= \left(1 - \frac{M^{**}}{M}\right)\{\beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} + \mu M^{**} - \tau H^{**} - \omega N^{**} - \Omega R^{**} + \tau H + \omega N + \Omega R$$
$$- \beta(R + \eta_1 T + \eta_2 H)M - \mu M\}$$
$$+ \left(1 - \frac{N^{**}}{N}\right)\{\kappa\beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} + (\omega + \mu)N^{**} - \kappa\beta(R + \eta_1 T + \eta_2 H)N - (\omega + \mu)N\}$$
$$+ \left(1 - \frac{R^{**}}{R}\right)\{\beta(R + \eta_1 T + \eta_2 H)M + \kappa\beta(R + \eta_1 T + \eta_2 H)N - \not{V}_1 R\}$$
$$+ \left(1 - \frac{T^{**}}{T}\right)\{\theta R + \alpha H - \not{V}_2 T\} + \left(1 - \frac{H^{**}}{H}\right)\{\rho T + \gamma R - \not{V}_3 H\},$$

$$= -\left[\mu\frac{(M - M^{**})^2}{M} + (\omega + \mu)\frac{(N - N^0)^2}{N} + \frac{M^{**}}{M}\{\beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**} + \tau H + \omega N + \Omega R\}\right.$$
$$+ \frac{N^{**}}{N}\{\kappa\beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**}\} + \frac{R^{**}}{R}\{\beta(R + \eta_1 T + \eta_2 H)M + \kappa\beta(R + \eta_1 T + \eta_2 H)N\}$$
$$+ \frac{T^{**}}{T}\{\theta R + \alpha H\} + \frac{H^{**}}{H}\{\rho T + \gamma R\} + \beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})M^{**}$$
$$\left. + (\tau H^{**} + \omega N^{**} + \Omega R^{**} + \not{V}_2 T + \not{V}_1 R + \not{V}_3 H)\right]$$
$$+ \left[\tau H + \omega N + \Omega R + \kappa\beta(R^{**} + \eta_1 T^{**} + \eta_2 H^{**})N^{**} + \theta R + \alpha H + \rho T + \gamma R + \not{V}_3 H^{**} + \not{V}_2 T^{**}\right.$$
$$\left. + \not{V}_1 R^{**} + \kappa\beta(R + \eta_1 T + \eta_2 H)N^{**} + \beta(R + \eta_1 T + \eta_2 H)M^{**} + \frac{M^{**}}{M}(\tau H^{**} + \omega N^{**} + \Omega R^{**})\right],$$
$$= -G + W.$$

The condition necessary and sufficient for $\frac{dL(M,N,R,T,H)}{dt} < 0$, is $W < G$, which completes the proof.

4. Social Interpretation of Analytical Results
4.1. Local and Global Stability of Disease Free Equilibrium (DFE) Point and Endemic Equilibrium Point (EEP).

When the stationary point is locally stable then all the points near it tends to move towards it over time. This implies that if a radicalized individual is introduced into a vulnerable population, then the population is least likely to radicalize individuals with time. The radicalization free stationary point (RFS) was locally unstable and radicalization free stationary point (RPS) exhibited backward bifurcation at $R_C = 1$, this is a major social challenge because if a radicalized individual recruit less than one individual, that does not guarantee eradicating radicalization. The model analysis indicates that more effort should be focused in mitigating radicalization as opposed to eradication. An equilibrium point is globally stable if all initial conditions lead to it over time. This means that irrespective of how many radicalized individuals are introduced into susceptible population, the radicalization will die down over time.

4.2. Thresholds analysis

The rehabilitation thresholds are determined when R_C is equated to one and solving for γ^c *and* ρ^c (critical rehabilitation),

$$1 = \beta(\kappa M^0 + N^0)\left\{\frac{1}{\not{V}_1} + \eta_1\left(\frac{\alpha\gamma + \theta\not{V}_3}{\not{V}_1(-\alpha\rho + \not{V}_2\not{V}_3)}\right) + \eta_2\left(\frac{\gamma\not{V}_2 + \theta\rho}{\not{V}_1(-\alpha\rho + \not{V}_2\not{V}_3)}\right)\right\}$$

The critical rehabilitation γ^c for individuals in subclass R (t) when the rate of rehabilitation of individuals in subclass T(t) remains constants is obtained using Mathematica software as,

$$\gamma^c = ((-(\kappa M^0 + N^0)\beta + \theta + \Omega + \mu)\not{V}_2\not{V}_3 - (\kappa M^0 + N^0)\beta\theta\not{V}_3\eta_1$$
$$- \rho(\alpha(-(\kappa M^0 + N^0)\beta + \theta + \Omega + \mu) + (\kappa M^0 + N^0)\beta\theta\eta_2))$$
$$/(\alpha(\rho + (\kappa M^0 + N^0)\beta\eta_1 + \not{V}_2(-\not{V}_3 + (\kappa M^0 + N^0)\beta\eta_2)).$$

The critical rehabilitation ρ^c for individuals in subclass T (t) when the rate of rehabilitation of individuals in subclass R(t) remains constants is obtained using Mathematica software as,

$$\rho^c = (\not{V}_3\left((\delta_1 + \mu)((\kappa M^0 + N^0)\beta - \not{V}_2) + (\kappa M^0 + N^0)\beta\theta\eta_1\right)$$
$$+ (\kappa M^0 + N^0)\beta\gamma(\alpha\eta_1 + (\delta_1 + \mu)\eta_2))/(((\kappa M^0 + N^0)\beta - \not{V}_2)(\alpha - \not{V}_3)$$
$$- (\kappa M^0 + N^0)\beta(\gamma + \theta)\eta_2).$$

The policy makers should strive to make sure that actual rehabilitation rates ρ or γ is greater than critical rehabilitation ρ^c *or* γ^c to ensure mitigation of radicalization that is,
$$\rho > \rho^c \text{ } or \text{ } \gamma > \gamma^c.$$

Also, rehabilitation with sufficient coverage does not guarantee eradication of radicalization when R_c is below unity because the model exhibited backward bifurcation although it is likely to lower the impact. Following McLean and Blower [10], a measure of rehabilitation impact, based on the reproduction numbers can be defined as

$$(U) = 1 - \frac{R_C}{R_0},$$

$$= 1 + \frac{(-\gamma + \forall_1)(\rho - \forall_2)\left(-\alpha\rho + (\alpha\gamma + \theta\forall_3)\eta_1 + \theta\rho\eta_2 + \forall_2(\forall_3 + \gamma\eta_2)\right)}{\forall_1(\alpha\rho - \forall_2\forall_3)(\rho - \forall_2 - \theta\eta_1)} > 0.$$

Thus, population-level impact of rehabilitation is always positive provided $R_C < R_0$. This condition is likely to be satisfied for rehabilitation with effective counter radicalization measures.

4.3 Sensitivity of the control reproduction number($\boldsymbol{R_C}$).

The study investigated the sensitivity of R_C to changes in the intervention strategies currently undertaken by Kenya Government especially urging individuals in subclasses R(t) and T(t) to turn up in rehabilitation center under amnesty call. Determining partial derivatives of R_C with respect to;

    a) Rate at which individuals in subclass R(t) seek rehabilitation (γ).

$$\frac{dR_C}{d\gamma} = \beta(\kappa M^0 + N^0)\left\{-\frac{1}{\forall_1{}^2} + \eta_1\left(\frac{\alpha\forall_1 - (\alpha\gamma + \theta\forall_3)}{\forall_1{}^2(-\alpha\rho + \forall_2\forall_3)}\right) + \eta_2\left(\frac{\forall_1\forall_2 - (\gamma\forall_2 + \theta\rho)}{\forall_1{}^2(-\alpha\rho + \forall_2\forall_3)}\right)\right\} < 0.$$

    b) Rate at which individuals in subclass T(t) seek rehabilitation (ρ).

$$\frac{dR_C}{d\rho} = \beta(\kappa M^0 + N^0)\left\{\eta_1\left(-\frac{(\alpha\gamma + \theta\forall_3)(-\alpha + \forall_2)}{\forall_1(-\alpha\rho + \forall_2\forall_3)^2}\right) + \eta_2\left(\frac{(\gamma + \theta)(-\alpha\rho + \forall_2\forall_3) - (\gamma\forall_2 + \theta\rho)(-\alpha + \forall_3)}{\forall_1(-\alpha\rho + \forall_2\forall_3)^2}\right)\right\}$$
$$< 0.$$

Clearly R_C is inversely proportional to ρ and γ. Higher rates of efficient rehabilitations hold great promise in lowering radicalization burden.

5. Numerical Results
5.1. Parameter Estimation

According to the 2014 Kenya population estimates study [13], the total population is about 45010056. This study assumes: 39969 individuals are radicalized, 80% of the radicalized to be in R(t), 5% to be in T(t) and the rest in H(t). The birth rate (b) for the year 2014 according to study (13) was 2.11% of the total population (P). It is then assumed $\pi = b \times P$. According to study [13], the life expectancy in Kenya in 2014 was 63.52 years, the constant death rate(μ) is assumed to be the reciprocal of the life expectancy. The symbol A in the tables below represent assumed.

Table-1. Data of the initial conditions of the radicalization model

Classes/year 2014	(M)	(N)	Radicalized but peaceful(R)	Radicalized but Violent(T)	Radicalized in Rehabs	Total
Estimates	4946156	40013939	39969	2498	7494	45010056
Source	Assumed	Assumed	Assumed	Assumed	Assumed	[13]

Table-2. Estimated parameter values

Parameter	β	σ	τ	ω	Ω	η_1	η_2	μ	π	κ
Value	1.05×10^{-7}	0.111	0.00023	0.00013	0.0071	0.3	0.15	1/63.52	$b \times P$	1.2
Source	A	A	A	A	A	A	A	(13)		A

Table-3. Estimated parameter values continued

Parameter	θ	γ	ρ	δ_1	δ_2	α	b
Value	0.00452	0.00357	0.00113	0.0005184	0.0005159	0.00089	2.11%
Source	A	A	A	A	A	A	[13]

The estimated total numerical value of the reproduction numbers in section 3.3 is obtained by substituting the estimated parameters as $R_C = 233.69, R_\rho = 234.26, R_\gamma = 256.103$ and $R_0 = 256.723$. It is the Individuals in subclasses R(t), T(t) and H(t) who are involved in radicalization, the sum of $R(0) + T(0) + H(0) = 49961$ according to the table above. Expressing the reproductions as proportion of 49961 we obtain actual reproduction numbers as,

$$R_C = 0.00468, R_\rho = 0.00469, R_\gamma = 0.00513 \text{ and } R_0 = 0.00514$$

The research study [14] defined measure of rehabilitation impact based on the reproduction numbers can be defined as,

$$(U) = 1 - \frac{R_C}{R_0} = 1 - \frac{0.004685}{0.00514} = 0.0897, U_\rho = 1 - \frac{R_\rho}{R_0} = 1 - \frac{0.00469}{0.00514} = 0.0875 \text{ and}$$

$$U_\gamma = 1 - \frac{R_\gamma}{R_0} = 1 - \frac{1.99065}{0.00514} = 0.00243.$$

From above, the greatest impact of intervention is realized when individuals in both subclasses R(t) and T(t) are rehabilitated. The impact is least when only individuals in subclass R(t) are rehabilitated.

5.2. Numerical Sensitivity Analysis

According to the study [15], sensitive index is described as measure the relative change in a state variable when a parameter changes, and the normalized forward sensitivity index of a variable to a parameter as the ratio of the relative change in the variable to the relative change in the parameter.

Definition 5

The normalized forward sensitivity index of a variable, φ, that depends differentiability on a parameter, ϖ, is defined as[15]:

$$R_\varpi^\varphi = \frac{\partial \varphi}{\partial \varpi} \times \frac{\varpi}{\varphi}.$$

Sensitivity indices of R_C to parameters of the radicalization model, are obtained in the table below,

Table-4. Sensitivity index of the model parameters

Parameter	κ	η_1	η_2	σ	α	ω
Sensitivity index	0.139	0.073	0.03	0.0215	0.00148	0.00141

Table-5. Sensitivity indices of the model parameter continued

| Parameter	μ	Ω	γ	θ	ρ	δ_1	δ_2	τ
Sensitivity index	- 1.61	- 0.23	- 0.0848	- 0.0737	- 0.00229	- 0.00225	- 0.000986	- 0.000439

The parameters which have negative sensitivity index are inversely proportional to Control reproduction number(R_C) while those that have positive sensitivity index are directly proportional to Control reproduction number(R_C). Higher values of parameter with negative sensitivity index (Table 5) and lower values of parameters with positive sensitivity index (Table 4) hold great promise in lowering burden of radicalization. Effort should also be made to determine actual values of the parameters in Kenya in order to obtain realistic reproduction numbers. Since interventions in this study do not affect μ, efforts should concentrate on increasing Ω, γ and θ.

5.3. Numerical Simulation

The simulations below were obtained from Matlab inbuilt ode solver by using parameter values in tables 2 and 3, and using initial conditions of the table 1.
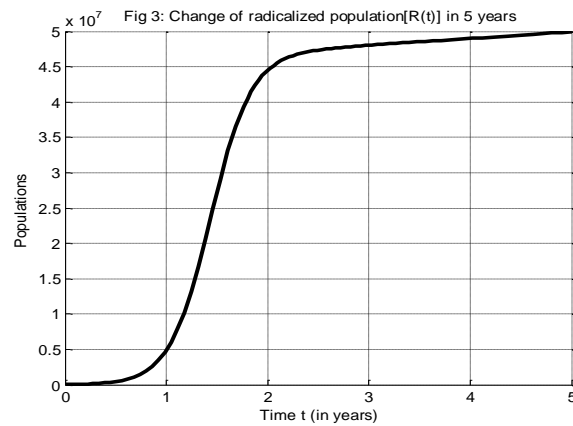
Fig-1. The simulated graph showing the change of the religious fanatics' population over a period of five years. It is not realistic for the abrupt decline indicated by the graph and therefore effort should be made to get actual parameter values.
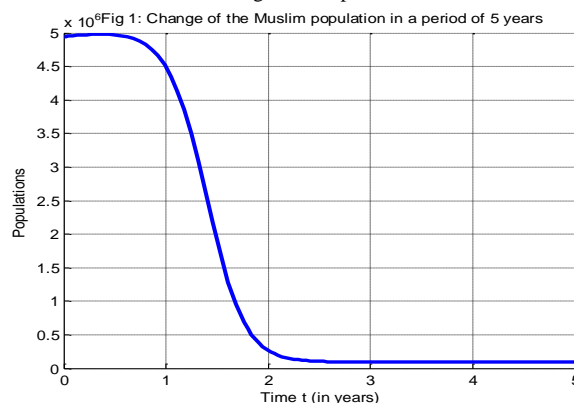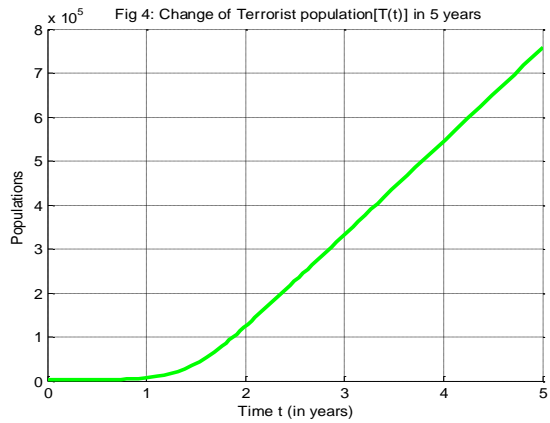
Fig-2. The simulated graph showing the change of the Non-religious fanatics (N) population over a period of five years. It is not realistic for the abrupt decline indicated by the graph and therefore effort should be made to get actual parameter values.
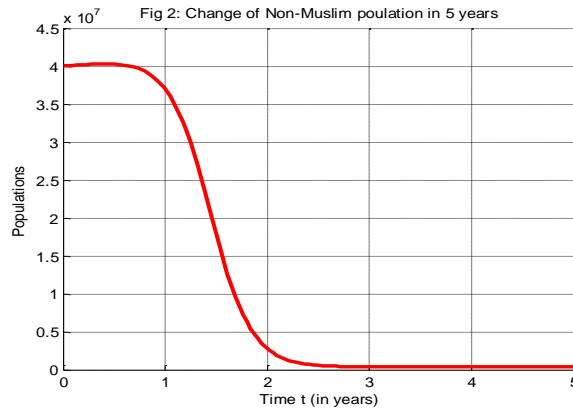

Fig-3. The simulated graph showing the change of the Radicalized but peaceful population over a period of five years. It is not realistic for the abrupt increase indicated by the graph and therefore effort should be made to get actual parameter values.

Fig-4. The simulated graph showing the change of the Radicalized but violent population (Terrorist) over a period of five years. It is not realistic for the abrupt increase indicated by the graph and therefore effort should be made to get actual parameter values.

Fig-5. The simulated graph showing the change of the Rehabilitated population over a period of five years. It is not realistic for the abrupt increase indicated by the graph and therefore effort should be made to get actual parameter values.
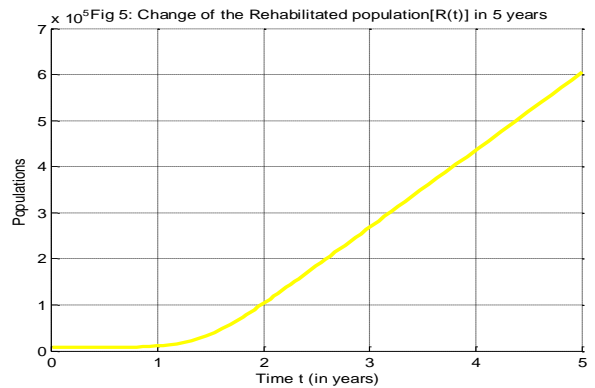
6. Results and Discussion

The research study concurred with the article [6], which point out the ability of mathematics modeling to give insight in underlying mechanism which influence radicalization. This study determined the reproduction numbers which can predict the number of secondary recruits one radicalized individual can influence. In concurrent with research study [8], this study determined equilibriums but formulated the model on religion perspective as opposed to economic. This research study concurred with dynamic model of study [9] however it differed on perspective and case study.

According to the extent of my research ability, for a long time there is no developed Mathematical deterministic model with Kenya specific attributes capable of giving insight into the dynamics of radicalizations in Kenya. This study will go a long way in introducing mathematical perspective in counter terrorism measures in Kenya.

7. Conclusion

Recovery of radicalized individual(R) from radicalized but peaceful subclass to religious fanatic population(M) subclass holds the great promise in lowering impact of radicalization since it has the highest negative sensitivity index.

References

[1] Odhiambo, E. O. S., Kennedy, O., Maito, T., and Leshan, 2015. "Domestic radicalisation in kenya." *Global Journal of Interdisciplinary Social Science,* vol. 4, pp. 48-56.

[2] Holmer, G., 2014. *Creating spaces for effective CVE approaches*. USA: UNITED STATES INSTITUTE OF PEACE. p. 178.

[3] Richard, S., Candace, S., and Alcorta. *Militants and Martyrs: Evolutionary perspectives on religion and terrorism*. Jerusalem : s.n.

[4] Victoroff, J., 2005. *The mind of the Terrorist. A review and critique of psychological approaches* vol. 49: Sage Publications.

[5] Otenyo and Eric, E., 2004. "New terrorism: Toward an explanation of cases in Kenya." *Illinois State University : African Security Review,*

[6] Alexander, G., 2009. "Terrorism as a Mathematical Problem." In *SIAM's conference on dynamical systems.*

[7] Botha, A., 2014. "Radicalisation in Kenya: Recruitment to al-Shabaab and the Mombasa Republican Council Kenya : Institute for security studies ISS PAPER 265.

[8] Brock, B. S., Gregory, D., Hess, Akila, W., and Wellesley, 2002. "Terrorism From Within: An economic model of Terrorism Brock Blomberg, Department of Economics, Wellesley College, ."

[9] Volintiru, C., 2010. *Towards a dynamic model of Terrorist radicalization Bucharest, sector 3, ISSN online: 1841 – 4281*. Romania : s.n.

[10] Braurer, F., 2004. "Backward bifurcation in simple vaccination models." *Journal of Mathematical Analysis and Application,* pp. 418-431.

[11] Junli, L., Tailei, Z., China, and ELSEVIER, 2011. "Global stability for a tuberculosis model." *Mathematical and Computer Modelling,* vol. 54, pp. 836-845.

[12] Okaka, C., Akinyi, Mugisha, J. Y. T., Manyonge, A., and Ouma, C., 2013. "Modelling the impact of misdiagnosis and treatment on the dynamics of malaria concurrent and co-infection with pneumonia." *Applied Mathematical Sciences,* vol. 7, pp. 6275-6296.

[13] Mundi, 2014. *Kenya demographic profile*. Kenya.

[14] Clean, H. I. V. A. M. and Blower, S., 1993. "Imperfect vaccines and herd Immunity to." *Pro Soc B,* pp. 9-13.

[15] Nakul, C., James, M., Hyman, Jim, M., and Cushing, 2008. "Determining important parameters in the spread of malaria through the sensitivity analysis of a Mathematical model." *Basel : Bulletin of Mathematical Biology.*