Academic Research Publishing Group

Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 2, No. 6, pp: 45-50, 2016
URL: http://arpgweb.com/?ic=journal&journal=17&info=aims

Hahn-Banach, Sandwich and Separation Theorems for Invariant Functionals with Values in Ordered Vector Spaces and Applications to Nonlinear Vector Programming

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Abstract: We give a direct proof of Hahn-Banach and sandwich-type theorems in the setting of convex subinvariant functionals, and a result of separation of convex sets by means of an invariant affine manifold. As consequences and applications, we give some conditions for an optimal solution of minimization problems, proving a Farkas and a Kuhn-Tucker-type theorem.

Keywords: Partially ordered vector space; Amenability; Hahn-Banach theorem; Sandwich theorem; Farkas theorem; Kuhn-Tucker theorem.

1. Introduction

The Hahn-Banach and sandwich-type theorems are widely studied and have several applications in different branches, among which, for instance, extension theorems for finitely additive measures ([1] and its bibliography, [2, 3]), convex optimization, Fenchel-type duality theorems for conjugate convex functions, which are deeply used in the study of the properties of the dual energy related with the problem of image reconstruction ([4-7]), representation theorems for (Riesz) MV-algebras ([8-10]), subdifferential calculus and variational analysis ([6, 11]) and their bibliographies), optimization theory and vector programming (see for instance [7, 12-15]), separation theorems for convex sets by means of affine manifolds (see also [16-19]).

Another field, related with these topics, is the study of the properties of measures and functionals invariant with respect to suitable (semi)groups of transformations, which are widely studied in several branches of Mathematics, for example in group theory, in which the concept of amenability is deeply investigated in various features (see for example [20]), and in Probability (see also [21, 22]). In particular, the theory of exchangeable processes is related to the group of all permutations of the set of the natural numbers, which keep fixed all but a finite number of integers. Note that this group is amenable but not abelian, while every abelian semigroup is amenable (see also [20]).

These subjects are widely studied in the context of functionals and measures with values in abstract spaces, and in particular in partially ordered vector spaces. A comprehensive survey can be found, for instance, in Buskes ([23]), Fuchssteiner and Lusky ([24]) and Lipecki ([3]). Moreover some extension, sandwich and Hahn-Banach-type theorems for invariant (sub)additive and (sub)linear partially order vector space-functionals were given, for instance, in [25-29].

In this paper we deal with *convex* subinvariant functionals, defined on suitable subsets of a given real vector space, which are not subspaces, and taking values in partially ordered vector spaces. We prove Hahn-Banach and sandwich theorems which allow to find invariant linear functionals. We give a direct proof, using a result proved in [26] on the existence of an invariant partially ordered vector space-valued mean. Moreover we prove the existence of an affine manifold "separating" two suitable convex sets. Furthermore, as consequences and applications, we prove a Farkas-type and a Kuhn-Tucker-type theorem, which are related with the problem of finding the minimum of nonlinear invariant functionals with values in order spaces, under suitable constraints.

2. Preliminaries

Let X be a real vector space. An *affine combination* of elements $x_1, x_2, \dots, x_n$ of X is any linear combination of the form $\sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ and $\sum_{i=1}^{n} \lambda_i = 1$. An *affine manifold* of X is a nonempty subset of X, closed under affine combinations.

If $\emptyset \neq Z \subset X$, then the *affine hull of* Z is the smallest affine manifold of X which contains Z, and we denote it by $\text{span}_{\text{aff}}(Z)$ (see also [30]).

A point $x_0 \in Z$ is said to be an *algebraic interior point of* Z iff for every $x \in X$ there is a positive real number r_0 with $(1-\lambda)x_0 + \lambda x \in Z$ for each $\lambda \in]-r_0, r_0[$. We say that $x_0 \in Z$ is an *algebraic relative interior point of* Z iff for each $x \in \text{span}_{\text{aff}}(Z)$ there is $\lambda_0 > 0$ such that $(1-\lambda)x_0 + \lambda x \in Z$ for each $\lambda \in]-\lambda_0, \lambda_0[$. We denote by $Z^{(0)}$ and $int(Z)$ the sets of all algebraic interior points of Z and of all algebraic relative interior points of Z, respectively.

A nonempty set $W \subset X$ is said to be *algebraically expanded* iff there is at least an element $a \in int(W)$ with $a + \lambda(b-a) \in int(W)$ for each $b \in W$ and $\lambda \in]0,1[$.

A nonempty subset D of any real vector space X is said to be *convex* iff $\lambda x_1 + (1-\lambda)x_2 \in D$ for every x_1, $x_2 \in D$ and $\lambda \in [0,1]$.

Given any two real vector spaces X, Y (where Y is equipped with a partial order compatible with the structure of real vector space) and a convex set $D \subset X$, we say that a function $f: D \to Y$ is *convex* (resp. *concave*) on D iff $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$ (resp. $f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda)f(x_2)$) for every x_1, $x_2 \in D$ and $\lambda \in [0,1]$. In this case, sometimes we will write $D(f)$ instead of D.

Let G be a semigroup, and $\mathcal{P}(G)$ be the family of all subsets of G. We say that G is *left* (resp. *right*) *amenable* iff there exists a finitely additive measure $\mu: \mathcal{P}(G) \to [0,1]$, with $\mu(G) = 1$ and $\mu(\{hg: g \in E\}$ (resp. $\mu(\{gh: g \in E\}) = \mu(E)$ for every $E \subset G$ and $g \in G$. We say that G is *amenable* iff it is both left and right amenable. Note that, in general, left and right amenability do not coincide, but are equivalent when G is a group (see also [31]).

Let $G \subset X^X$ be a semigroup of (linear) homomorphisms, with $(gh)(x) = g(hx)$ for any g, $h \in G$ and $x \in X$. Such a semigroup is said to be *acting* on X. Let R be a Dedekind complete partially ordered real vector space, $R^+ := \{y \in R: y \ge 0\}$, $l_b(G,R)$ be the space of all bounded R-valued functions defined on G. Given $f \in l_b(G,R)$ and $h \in G$, we call *left*(*right*)*h-translation of* f the function ${}_hf$ (resp. f_h) $\in l_b(G,R)$ defined by

$$ {}_hf(g) = f(hg) \text{ (resp. } f_h(g) = f(gh)), \; g \in G. $$

A linear positive function $m: l_b(G,R) \to R$ is called a *left* (resp. *right*)-*G-invariant R-mean* iff $m({}_hf) = m(f)$ (resp. $m(f_h) = m(f)$) for each $f \in l_b(G,R)$ and $h \in G$, and $m(\boldsymbol{y}) = y$ for each $y \in R$, where $\boldsymbol{y}$ is the constant function which associates the value y to every element $g \in G$.

A set $\emptyset \ne Z \subset X$ is said to be *G-invariant* iff $gz \in Z$ whenever $z \in Z$. A set $\emptyset \ne A \subset X \times R$ is *G-invariant* iff $(gx, y) \in A$ whenever $g \in G$ and $(x,y) \in A$.

A function $L: X \to R$ is said to be *G-subinvariant* (resp. *G-superinvariant*, *G-invariant*) iff $L(gx) \le L(x)$ (resp. $L(gx) \ge L(x)$, $L(gx) = L(x)$) for every $g \in G$ and $x \in X$.

By $\mathcal{L}(X,R)$ and $\mathcal{L}(R,R)$ we denote the sets of all linear functions from X to R and from R to R, respectively. We indicate with $\mathcal{L}_{\text{inv}}(X,R)$ (resp. $\mathcal{L}_{+,\text{inv}}(X,R)$) the set of all linear (resp. linear positive) G-invariant functions $L \in \mathcal{L}(X,R)$.

A nonempty set $A \subset X \times R$ is called a *cone* with vertex $x_0 \in X \times R$ iff $\lambda(A - x_0) \subset A - x_0$ for every non-negative real number λ.

Given $0 \not\equiv L \in \mathcal{L}(X,R)$, $0 \not\equiv L' \in \mathcal{L}(X,R)$ and u_0 in R, set

$$ H := \{(x,y) \in X \times R: L(x) + L'(y) = u_0\}. \qquad (1) $$

It is not difficult to check that the set H defined in (1) is empty or an affine manifold of $X \times R$ (see also [17]).

If A, B are two nonempty subsets of $X \times R$ and $H \ne \emptyset$ is as in (1), then we say that H *separates* A and B iff $A \subset H^-$ and $B \subset H^+$, where

$$ H^- := \{(x,y) \in X \times R: L(x) + L'(y) \le u_0\},\; H^- := \{(x,y) \in X \times R: L(x) + L'(y) \ge u_0\}. $$

The *projection* of $X \times R$ onto X is the function $P_X: X \times R \to X$ defined by $P_X(x,y) = x$ for every $(x,y) \in X \times R$. Moreover, for any nonempty set $A \subset X \times R$, put

$$ P_X(A) := \{x \in X: \text{ there exists } y \in R \text{ with } (x,y) \in A\}. $$

It is not difficult to see that $P_X(A+B) = P_X(A) + P_X(B)$ for any two nonempty subsets A, $B \subset X \times R$.

Given a set $\emptyset \ne A \subset X \times R$, we call *cone generated by* A the set $C(A) := \{a \in X \times R$: there exist $\lambda \in \mathbb{R}_0^+$ and $a' \in A$ with $a = \lambda a'\}$. It is not difficult to check that, if A is convex, then $C(A)$ is too, and that $A - B$ and $C(A-B)$ are G-invariant whenever A and B are G-invariant.

3. The Main Results

We begin with proving Theorem 3.1, which is a Hahn-Banach-type theorem, whose [32], is the particular case in which the involved functions are not necessarily G-invariant or G-subinvariant (or, equivalently, G is supposed to be a trivial group having one element). We always assume, when we do not say it explicitly, that X is a real vector space, G is a right amenable semigroup acting on X and R is a Dedekind complete partially ordered real vector space.

Theorem 3.1. *Let D be a convex and G-invariant subset of X, Z be a G-invariant subspace of X with $D^{(0)} \cap Z \ne \emptyset$, $p: D \to R$ be a convex and G-subinvariant function, and $L_0 \in \mathcal{L}_{inv}(Z,R)$ be such that $L_0(z) \le p(z)$ for every $z \in D \cap Z$. Then L_0 admits an extension $L \in \mathcal{L}_{inv}(X,R)$, with $L_0(x) \le p(x)$ for every $x \in D$.*

Proof. By Zowe [32] there is a function $L^* \in \mathcal{L}(X,R)$ (not necessarily G-invariant) with $L^*(x) \le p(x)$ for every $x \in D$ and $L^*(z) = L_0(z)$ for each $z \in Z$. Now, choose arbitrarily $x \in D$ and define $f_x \in l_b(G,R)$ by $f_x(g) = L^*(gx)$, $g \in G$. Since G is right amenable and R is Dedekind complete, by Chojnacki [26] there is a right G-invariant R-mean $m: l_b(G,R) \to R$. Set $L(x) := m(f_x)$, $x \in D$. Since $f_{hx}(g) = L^*(ghx) = f_x(gh) = (f_x)_h(g)$ for every $g \in G$, then $L(hx) = m(f_{hx}) = m((f_x)_h) = m(f_x) = L(x)$ for any $h \in G$, and hence L is G-invariant. Moreover, as $L^*(z) \le p(z)$ for every $z \in X$ and p is G-subinvariant, we get $f_x(g) = L^*(gx) \le p(gx) \le p(x)$ for every $g \in G$, and hence $L(x) = m(f_x) \le m(\boldsymbol{p(x)}) = p(x)$. Finally, if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in X$, then

$$ f_{\lambda_1 x_1 + \lambda_2 x_2}(g) = L^*\big(g(\lambda_1 x_1 + \lambda_2 x_2)\big) = L^*(\lambda_1 g x_1 + \lambda_2 g x_2) = $$

$$= \lambda_1 L^*(gx_1) + \lambda_2 L^*(gx_2) = \lambda_1 f_{x_1}(g) + \lambda_2 f_{x_2}(g)$$

for every $g \in G$, and hence

$$L(\lambda x_1 + (1-\lambda)(x_2)) = m(f_{\lambda_1 x_1 + \lambda_2 x_2}) =$$
$$= \lambda_1 m(f_{x_1}) + \lambda_2 m(f_{x_2}) = \lambda_1 L(x_1) + \lambda_2 L(x_2).$$

Thus, $L \in \mathcal{L}_{\text{inv}}(X,R)$. This ends the proof. □

By proceeding analogously as in Zowe [32], it is possible to give the following consequence of Theorem 3.1.

Theorem 3.2. *Let D be as in Theorem 3.1, and $p: D \to R$ be a convex and G-subinvariant function. If $0 \in int(D)$ and $p(0) \geq 0$, then there is $L \in \mathcal{L}_{inv}(X,R)$ with $L(x) \leq p(x)$ for every $x \in D$.*

Similarly as in the proof of Theorem 3.1, it is possible to prove the following sandwich-type theorem, which extends to invariance the analogous result given in [32], proved when the involved convex, concave and linear functions are not necessarily G-subinvariant, G-superinvariant and G-invariant, respectively.

Theorem 3.3. *Let D and E be two convex and G-invariant subsets of X with nonempty intersection, $p: D \to R$ be a convex function, $q: E \to R$ be a concave function. Assume that $0 \in int(D-E)$ and $q(x) \leq p(x)$ for every $x \in D \cap E$. Then there exist $L \in \mathcal{L}_{inv}(X,R)$ and $u_0 \in R$ with $L(x) - u_0 \leq p(x)$ for every $x \in D$ and $L(x) - u_0 \geq q(x)$ for each $x \in E$.*

Now, using Theorem 3.2, we prove the existence of G-invariant affine manifolds separating two convex sets in the context of partially ordered vector spaces, and extend to invariance earlier results proved in [16], [17] and [18].

Theorem 3.4. *Let A, B be two G-invariant subsets of $X \times R$ such that $C(A-B)$ is convex, $P_X(A-B)$ is algebraically expanded,*

$$0 \in int(P_X(A-B)) \tag{2}$$

and

$$y_1 \geq y_2 \text{ whenever } (x,y_1) \in A \text{ and } (x,y_2) \in B. \tag{3}$$

Then there exist $L \in \mathcal{L}_{inv}(X,R)$ and $u_0 \in R$ such that the affine manifold

$$H := \{(x,y) \in R \times X : L(x) - y = u_0\} \tag{4}$$

separates A and B.

Proof. First of all, observe that it is known that, if $X_1 := \text{span}_{\text{aff}}(int(P_X(A-B)))$, then X_1 is a subspace of X and

$$X_1 = \text{span}_{\text{aff}}(P_X(A-B)) = \text{span}_{\text{aff}}(int(P_{X_1}(A-B))) = \text{span}_{\text{aff}}(P_{X_1}(A-B))$$

(see also [17, 18]). From this and (2) it follows that

$$0 \in int(P_{X_1}(A-B)). \tag{5}$$

Moreover, since A and B are G-invariant, it is not difficult to deduce that X_1 is G-invariant too. By (5), for each $x \in X_1$ there is $\lambda_0 > 0$ such that for every $\lambda \in [0,\lambda_0[$ there is $y \in R$ with $(\lambda x, y) \in A - B$.

Set now $E_x := \{y \in R : (x,y) \in C(A-B)\}$, $x \in X_1$. Note that $\frac{1}{\lambda} y \in E_x$ for every $\lambda \in]0,\lambda_0[$, and hence $E_x \neq \emptyset$ for any $x \in X_1$.

We claim that $y \geq 0$ for each $y \in E_0$. Indeed, choose arbitrarily $y \in E_0 \setminus \{0\}$. Then there exist $\lambda > 0$, $(x_1,y_1) \in A$, $(x_2,y_2) \in B$, with $\lambda((x_1,y_1)-(x_2,y_2)) = (0,y)$, and hence $x_1 = x_2$. From this and (3) it follows that $y_1 \geq y_2$. Thus, $y = \lambda(y_1 - y_2) \geq 0$, getting the claim.

We now show that

$$E_{x_1} + E_{x_2} \subset E_{x_1+x_2} \quad \text{for each } x_1, x_2 \in X_1. \tag{6}$$

Fix arbitrarily x_1, $x_2 \in X_1$, and pick $y_i \in E_{x_i}$, $i = 1,2$. Then, (x_1,y_1), $(x_2,y_2) \in C(A-B)$. From this, since $C(A-B)$ is convex, it follows that $(x_1+x_2, y_1+y_2) \in C(A-B)$, and hence $y_1 + y_2 \in E_{x_1+x_2}$.

Now, we show that the set E_x is bounded from below for every $x \in X_1$. Choose arbitrarily $y \in E_x$. Since $E_{-x} \neq \emptyset$, there exists $y^* \in R$ with $-y^* \in E_{-x}$. From (6) we get $y - y^* \in E_x + E_{-x} \subset E_0$, and hence $y - y^* \geq 0$, namely $y \geq y^*$.

Thus, since R is Dedekind complete, it makes sense to define a function $p: X_1 \to R$ by

$$p(x) := \bigwedge \{y : y \in E_x\}.$$

We now claim that p is G-subinvariant. Pick arbitrarily $g \in G$ and $x \in X$. Since $C(A-B)$ is G-invariant, then $E_{gx} \supset E_x$, and hence

$$p(gx) = \bigwedge \{y : y \in E_{gx}\} \leq \bigwedge \{y : y \in E_x\} = p(x),$$

getting the claim.

Now, fix arbitrarily x_1, $x_2 \in X_1$. Taking into account (6), we have

$$p(x_1+x_2) = \bigwedge \{y : y \in E_{x_1+x_2}\} \leq \bigwedge \{y_1 + y_2 : y_1 \in E_{x_1}, y_2 \in E_{x_2}\} = \tag{7}$$
$$= \bigwedge \{y_1 : y_1 \in E_{x_1}\} + \bigwedge \{y_2 : y_2 \in E_{x_2}\} = p(x_1) + p(x_2).$$

Moreover for every $x \in X_1$ and $\lambda > 0$ we get

$$p(\lambda x) = \bigwedge \{y : y \in E_{\lambda x}\} = \bigwedge \{y : y \in \lambda E_x\} = \tag{8}$$
$$= \bigwedge \{\lambda z : z \in E_x\} = \lambda \bigwedge \{z : z \in E_x\} = \lambda\, p(x).$$

From (7) and (8) we deduce that p is sublinear, and also convex, on X_1. Moreover $p(0) \geq 0$, since $E_0 \subset R^+$. By Theorem 3.2 there exists $L' \in \mathcal{L}_{\text{inv}}(X_1, R)$ with $L'(x) \leq p(x)$ for every $x \in X_1$. From this it follows that for each $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ it is

$$L'(x_1) - L'(x_2) = L'(x_1 - x_2) \leq p(x_1 - x_2) \leq y_1 - y_2. \quad (9)$$

From (9) we get $L'(x_1) - y_1 \leq L'(x_2) - y_2$. Let $u_0 \in R$ be such that

$$\bigvee\{L'(x_1) - y_1 : (x_1, y_1) \in A\} \leq u_0 \leq \bigwedge\{L'(x_2) - y_2 : (x_2, y_2) \in B\}. \quad (10)$$

Note that such an element does exist in R, since R is Dedekind complete. Let X_2 be an *algebraic complement* of X_1, that is a subspace of X such that every element $x \in X$ can be expressed in a unique way as $x = x_1 + x_2$, where $x_i \in X_i$, $i = 1,2$. Such a space X_2 does exist (see also [33]). Let us define $L: X \to R$ by $L(x) = L'(x_1)$. From (10) it is not difficult to deduce that L and u_0 are such that the affine manifold H defined as in (4) separates A and B. This ends the proof.

4. Applications

In this section, as consequences and applications of the results proved in Section 3, we establish some conditions for an optimal solution of some minimization problems for invariant functionals with values in a partially ordered vector space.

Following [18], suppose that $X^+ \subset X$ is a G-invariant cone with vertex 0, which induces on X the natural order, defined by $x_1 \geq x_2$ if and only if $x_1 - x_2 \in X^+$, and suppose that $gx_1 \geq gx_2$ whenever $g \in G$ and $x_1 \geq x_2$. Assume that $U: D(U) \subset X \to X$ is a G-*equivariant* function (that is, $U(gx) = g(U(x))$ for every $g \in G$ and $x \in D(U)$, see also [34]), $V: D(V) \subset X \to R$ is a G-invariant function, $X_0 := D(U) \cap D(V) \neq \emptyset$, $D(U)$ and $D(V)$ are convex and G-invariant sets. Put

$$u := \bigwedge\{V(x) : x \in X_0, U(x) \leq 0\} \quad (11)$$

and

$$A := \{(U(x) + z, V(x) + w - u)\} : x \in X_0, z \in X^+, w \in R^+\}. \quad (12)$$

Suppose that $C(A)$ is convex, $U(X_0) + X^+$ is algebraically expanded, and

$$0 \in int(U(X_0) + X^+). \quad (13)$$

As a consequence of Theorem 3.4, we first prove the following Farkas-type theorem, extending earlier results proved in [12, 13].

Theorem 4.1. *Under the same hypotheses and notations as above, assume that*

$$V(x) \geq 0 \textit{ whenever } x \in X_0 \textit{ and } U(x) \leq 0. \quad (14)$$

Then there is $L \in \mathcal{L}_{+,inv}(X, R)$ with

$$V(x) + L(U(x)) \geq 0 \textit{ for each } x \in X_0. \quad (15)$$

Proof. First of all, observe that from (14) it follows that $u \geq 0$, where u is as in (11). If $x \in X_0$ and $z \in X^+$ are such that $U(x) + z = 0$, then $U(x) \leq 0$ and hence, by construction, $u \leq V(x)$. From this it follows that $0 \leq V(x) + w - u$ for any $w \in R^+$. Furthermore, since $gx \in X_0$, $U(gx) = g(U(x))$ and $V(gx) = V(x)$ for every $g \in G$ and $x \in X_0$, then $(g(U(x) + z), V(x) + w - u) = (U(gx) + gz, V(gx) + w - u)$ for each $g \in G$, $x \in X_0$, $z \in X^+$ and $w \in R^+$, and hence it is not difficult to deduce that the set A defined in (12) is G-invariant. Let $B := \{(0,0)\} \subset X \times R$. Then, the sets A and B satisfy the hypotheses of Theorem 3.4. Thus, there are $L \in \mathcal{L}_{\text{inv}}(X, R)$ and $y_0 \in R$, $y_0 \leq 0$, with $-L(x) - y \leq y_0$ for each $(x, y) \in A$ and

$$-L(U(x) + z) - V(x) - w + u \leq 0 \quad (16)$$

for each $w \in R^+$, $x \in X_0$, $z \in X^+$. From (16) used with $w = 0$, taking into account that $u \leq 0$, we obtain

$$L(U(x) + z) + V(x) \geq 0 \quad (17)$$

for each $x \in X_0$ and $z \in X^+$. In particular, choosing $z = 0$, we get $L(U(x)) + V(x) \geq 0$ for each $x \in X_0$, that is (15). We now claim that L is positive. Pick arbitrarily $z \in X^+$. For each $x \in X_0$, taking into account (17), we get

$$0 \leq L(U(x) + \lambda z) + V(x) = L(U(x)) + V(x) + \lambda L(z) \text{ for any } \lambda > 0,$$

and hence, passing to the infimum, we deduce

$$L(z) = \bigwedge\{L(z) + \frac{1}{\lambda}(L(U(x)) + V(x)) : \lambda > 0\} \geq 0.$$

This ends the proof. □

We now are in position to prove our version of the Kuhn-Tucker theorem for invariant functionals (see also [12, 14, 15]).

Theorem 4.2. *Under the same hypotheses and notations as in Theorem* 4.1, *let x_0 be a solution of the problem*

P1) *find $x_0 \in Z_0 := \{x \in X_0 : U(x) \leq 0\}$ such that $V(x_0) \leq V(x)$ for every $x \in Z_0$.*

Then there exists $L_0 \in \mathcal{L}_{+,inv}(X, R)$ such that (x_0, L_0) is a solution of the problem

P2) *find $x_0 \in X_0$ and $L_0 \in \mathcal{L}_{+,inv}(X, R)$ such that*

$$L(U(x_0)) + V(x_0) \leq L_0(U(x_0)) + V(x_0) \leq L_0(U(x)) + V(x) \quad (18)$$

for every $x \in X_0$ and $L \in \mathcal{L}_{+,inv}(X, R)$.

Proof. Let x_0 be a solution of the problem P1). Set $V'(x) := V(x) - V(x_0)$, $x \in D(V)$. It is not difficult to check that U and V' fulfil condition (14), and thus satisfy (15) too, thanks to Theorem 4.1. Hence there is $L_0 \in \mathcal{L}_{+,\mathrm{inv}}(X, R)$ with $V'(x) + L_0(U(x)) \geq 0$ for every $x \in X_0$, that is $L_0(U(x)) + V(x) \geq V(x_0)$ for each $x \in X_0$. Thus, $L_0(U(x_0)) \geq 0$. Since $U(x_0) \leq 0$ and L_0 is positive, then $L_0(U(x_0)) \leq 0$, and hence $L_0(U(x_0)) = 0$. Thus, it follows that

$$L_0(U(x_0)) + V(x_0) \leq L_0(U(x)) + V(x) \quad \text{for every } x \in X_0. \tag{19}$$

As $U(x_0) \leq 0$, we get $L(U(x_0)) \leq 0$ for every $L \in \mathcal{L}_{+,\mathrm{inv}}(X, R)$, and hence

$$L(U(x_0)) + V(x_0) \leq V(x_0) = L_0(U(x_0)) + V(x_0) \tag{20}$$

for each $L \in \mathcal{L}_{+,\mathrm{inv}}(X, R)$. From (19) and (20) it follows that (x_0, L_0) is a solution of Problem P2). □

Remark 4.3. Observe that both Dedekind completeness and (right) amenability of G are not only sufficient, but also necessary conditions in order that Hahn-Banach and sandwich-type theorems hold (see also [35] and [25], respectively).

5. Conclusions

We proved some versions of Hahn-Banach and sandwich-type theorems related to convex subinvariant functionals, taking values in a partially ordered vector space R. We used some similar classical results holding without invariance and a technique, by means of which it is possible to construct an invariant R-valued mean on all bounded R-valued functions defined on an amenable semigroup of homomorphisms and to get an invariant linear functional from a not necessarily invariant linear functional. We used the obtained Hahn-Banach theorem to prove a result on separation of convex sets by means of an invariant affine manifold. As consequences and applications, we got some conditions for an optimal solution of minimization problems related to nonlinear vector programming, extending to our context some Farkas and Kuhn-Tucker type theorems.

Acknowledgment

This work was supported by University of Perugia and the Italian National Group of Mathematical Analysis, Probability and Applications (G.N.A.M.P.A.).

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