Academic Research Publishing Group

Academic Journal of Applied Mathematical Sciences

ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 2, No. 8, pp: 93-97, 2016
URL: http://arpgweb.com/?ic=journal&journal=17&info=aims

Felicitous Labelings of Some Graphs

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Abstract: *A **felicitous** labeling of a graph G, with q edges is an injection $f : V(G) \to \{0, 1, 2, . . . , q\}$ so that the induced edge labels $f^*(xy) = (f(x) + f(y)) \pmod q$ are distinct.*

Keywords: Graph; Labeling; Felicitous labeling; Path; Cycle; Linear cactus; Bigraph; Splitting graph; Planar graph.

1. Introduction

Graham and Sloane [1] introduced the concept of a harmonious graph. A connected graph G = (V, E) with $|V| = p$ vertices and $|E| = q\ (\geq p)$ edges is said to be harmonious if it is possible to label the vertices $x \in V$ with distinct numbers f(x) of Z_q, the integers modulo q, in such a way that when each edge e = xy is labeled with f * (e) = (f(x) + f(y)) (mod q), the resulting edge labels are distinct. If the graph is a tree (with p vertices and q = p – 1 edges), it requires exactly one vertex label to be repeated.

To generalize the harmonious labeling and to keep the number of vertex labels that can be repeated to a minimum, Lakshmi, *et al.* [2] introduced the felicitous labeling. A graph which admits a felicitous labeling is said to be felicitous.

2. Definitions

Definition 2.1: A ***graph*** *G*(*V*,*E*) consists of a finite non-empty set *V* = *V* (*G*) of *p* ***points*** (called ***vertices***) together with a prescribed set *E* = *E*(*G*) of *q* unordered pair of distinct vertices of *V*. Each pair *e* = *{u, v}* of vertices in *E* is a ***line*** (called ***edge***) of *G*.

In an edge *e* = *uv*, *u* and *v* are ***adjacent*** *vertices*; vertex *u* and edge *e* are ***incident*** with each other. If two distinct edges e_1 and e_2 are incident with a common vertex, then they are *adjacent edges*. A graph with *p* vertices and *q* edges is called a (*p*, *q*) - *graph*. In this case, *p* is called the ***order*** of the graph and *q* is the ***size*** of the graph.

Definition 2.2: The ***corona*** $G_1 \odot G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1 (with *p* vertices) and *p* copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 2.3 [3]: A ***cycle with a chord*** is a graph having a cycle C_n, $n \geq 4$ and two of its non – adjacent vertices are joined by an edge (chord).

Definition 2.4: *Duplication of a vertex* v_k of graph G produces a new graph G_1 by adding a vertex v_k' with $N(v_k') = N(v_k)$. In other words, a vertex v_k' is said to be the duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v_k' also.

Definition 2.5: A *linear cactus* $P_m(K_n)$ is a connected graph in which all the blocks are isomorphic to a complete graph K_n and block - cutpoint is a path P_{2m-1}.

Definition 2.6: For a graph G, the *split graph* is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted by *spl* (G).

Definition 2.7 [3]: Let G be a graph. A 1 – 1 function $f : V(G) \to \{0, 1, 2, . . . , q\}$ is said to be an ***odd – edge labeling*** of G, if for every edge $e = uv \in E(G)$, f(u) + f(v) is odd and f * (E(G)) = {1, 3, 5, . . . , 2q – 1}.

Definition 2.8 [3]: Let G be a graph. A 1 – 1 function $f : V(G) \to \{0, 1, 2, . . . , q\}$ is said to be an ***even – edge labeling*** of G, if for every edge $e = uv \in E(G)$, f(u) + f(v) is even and f * (E(G)) = {2, 4, 6, . . . , 2q}.

Definition 2.9 [3]: A graph G with q edges is called ***harmonious*** if there is an injection $f : V(G) \to Z_q$, the additive group of integers modulo q such that when each edge xy of G is assigned the label (f(x) + f(y)) (mod q), the resulting edge labels are all distinct. If the graph is a tree (with p nodes and e = p – 1 edges), we require exactly one node label to be repeated.

Definition 2.10 [4]: For a given graph G, consider 2G. Let u be any vertex of one copy of G. Let u' be the corresponding vertex in the other copy of G. We define $G^*_{u,u'}$ = 2G + (uu'). Let G* denote any such graph $G^*_{u,u'}$ for some pair (u, u') of corresponding vertices. Further, for the sake of convenience, we define $(G^*)^0 = G$, $(G^*)^1 = G^*$ and $(G^*)^m = ((G^{*m-1})^*$.

Definition 2.11 [4]: A sub graph H of a graph G is said to be an ***even sub graph*** of G, if the degree of every vertex of H is even in H.

Definition 2.12 [5]: A ***felicitous*** labeling of a graph G, with q edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ so that the induced edge labels $f^*(xy) = (f(x) + f(y)) \pmod q$ are distinct.

3. Basic Results

Theorem 3.1 [5]: Every bipartite graph is a sub graph of a felicitous bipartite graph.
Theorem 3.2 [5]: Let G be a felicitous graph with an even number of edges. Then every cycle C of G contains an even number of odd edges.
Theorem 3.3 [5]: C_{4k+2} is not felicitous.
Theorem 3.4 [4]: If a graph G admits an odd – edge labeling, then G* is felicitous.
Theorem 3.5 [4]: Let G be a harmonious (or a felicitous) graph with an even number of edges. Then every even sub graph G' of G contains an even number of odd edges.
Theorem 3.6 [4]: No even graph with 4n + 2 edges is felicitous.
Theorem 3.7 [4]: Let G be a graph with an odd number of edges, and let $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ be an odd edge labeling of G. Then f is a felicitous labeling of G.
Remark 3.8: Let G be a (p, q) graph. Let f be a vertex labeling. We define f^* and f_1 on E(G) as follows : $f_1(uv) = f(u) + f(v)$ and $f^*(uv) = f_1(uv) \pmod q$ for every $uv \in E(G)$.
Remark 3.9: It is observed that as in 3.6, most of the even graphs are not felicitous. So, finding felicitousness for graphs with even number of edges are very difficult.

4. Main Results

Theorem 4.1: A linear cactus $P_m(K_4)$ is a felicitous graph.
Proof : Let $V(P_m(K_4)) = \{u_i : 1 \le i \le m\} \cup \{v_i, v_i' : 1 \le i \le m - 1\}$ and $E(P_m(K_4)) = \{(u_iu_{i+1}) : 1 \le i \le m - 1\} \cup \{(u_i v_i) : 1 \le i \le m - 1\} \cup \{(u_i v_i') : 1 \le i \le m - 1\} \cup \{(v_i v_i') : 1 \le i \le m - 1\} \cup \{(v_i u_{i+1}) : 1 \le i \le m - 1\} \cup \{(v_i' u_{i+1}) : 1 \le i \le m - 1\}$.
Define a function $f : V(P_m(K_4)) \rightarrow \{0,1,2, \ldots, q = 6(m - 1)\}$ by

$$f(u_i) = 3i - 2, \quad 1 \le i \le m$$
$$f(v_i) = 3(i - 1), \quad 1 \le i \le m - 1$$
$$f(v_i') = 3i - 1, \quad 1 \le i \le m - 1$$

The edge labels are as follows :

$$f^*(u_iu_{i+1}) = 6i - 1, \quad 1 \le i \le m - 1$$
$$f^*(u_i v_i) = 6i - 5, \quad 1 \le i \le m - 1$$
$$f^*(v_iv_i') = 6i - 4, \quad 1 \le i \le m - 1$$
$$f^*(u_iv_i') = 6i - 3, \quad 1 \le i \le m - 1$$
$$f^*(v_iv_{i+1}) = 6i - 2, \quad 1 \le i \le m - 1$$
$$f^*(v_i'u_{i+1}) = 6i, \quad 1 \le i \le m - 1$$

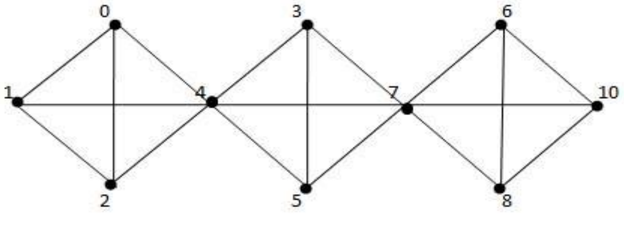
Then, $f^*(E(P_m(K_4))) = \{1, 7, \ldots, 6(m - 1) - 5\} \cup \{2, 8, \ldots, 6(m - 1) - 4\} \cup \{3, 9, \ldots, 6(m - 1) - 3\} \cup \{4, 8, \ldots, 6(m - 1) - 2\} \cup \{5, 11, \ldots, 6(m - 1) - 1\} \cup \{6, 12, \ldots, 6(m - 1)\}$

$= \{1, 7, \ldots, 6m - 11\} \cup \{2, 8, \ldots, 6m - 10\} \cup \{3, 9, \ldots, 6m - 9\} \cup \{4, 8, \ldots, 6m - 8\} \cup \{5, 11, \ldots, 6m - 7\} \cup \{6, 12, \ldots, 6m - 6\}$

$= \{1, 2, 3, \ldots, 6m - 7, 6m - 6\}$.

Hence, $P_m(K_4)$ is a felicitous graph.
For example, a felicitous labeling of $P_4(K_4)$ is shown in Figure 4.1.

Fig-4.1.



Theorem 4.2 : *Spl* $(B_{n,n})$ is a felicitous graph for all n.
Proof : Let $V(Spl\ (B_{n,n})) = \{u_i, u_i' : 1 \le i \le n\} \cup \{u, u', v, v'\} \cup \{v_i, v_i' : 1 \le i \le n\}$ and $E(Spl\ (B_{n,n})) = \{(u\, u_i) : 1 \le i \le n\} \cup \{(u\, v_i) : 1 \le i \le n\} \cup \{(v\, v_i) : 1 \le i \le n\} \cup \{(u'\, u_i) : 1 \le i \le n\} \cup \{(u'\, v_i) : 1 \le i \le n\} \cup \{(v'\, v_i) : 1 \le i \le n\} \cup \{(u\, u') \cup (u\, v') \cup (u'v)\}$.
Define a function $f : V(Spl\ (B_{n,n})) \rightarrow \{0,1,2, \ldots, q = 6n + 3\}$ by

$f(u) = 1, \quad f(u') = 6n$
$f(v) = 2n + 1, \quad f(v') = 6n + 2$
$f(u_i) = 2(i - 1), \quad 1 \le i \le n$

$$f(u_i') = \begin{cases} 2i + 3, & 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil \\ 2i + 5 & \left\lceil \frac{n-1}{2} \right\rceil + 1 \le i \le n \end{cases}$$

$f(v_i) = 2(n + i - 1), \quad 1 \le i \le n$
$f(v_i') = 2n + 4i + 3, \quad 1 \le i \le n$

The edge labels are as follows :

$f_1(u\, u_i) = 2i - 1, \quad 1 \le i \le n$
$f_1(u\, v_i) = 2(n + i) - 1, \quad 1 \le i \le n$
$f_1(v\, v_i) = 4n + 2i - 1, \quad 1 \le i \le n$

$$f_1(u'\, u_i') = \begin{cases} 6n + 2i + 3, & 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil \\ 6n + 2i + 5, & \left\lceil \frac{n-1}{2} \right\rceil + 1 \le i \le n \end{cases}$$

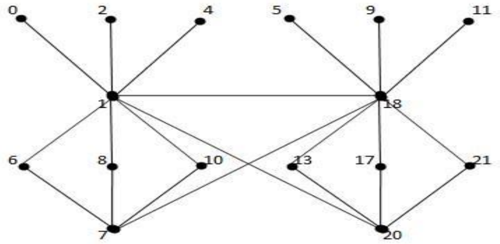
$f_1(u'\, v_i') = 8n + 4i + 3, \quad 1 \le i \le n$
$f_1(v'\, v_i') = 8n + 4i + 5, \quad 1 \le i \le n$
$f_1(u\, u') = 6n + 1,$
$f_1(u\, v') = 6n + 3,$
$f_1(u'\, v) = 8n + 1$

Clearly, $f_1(Spl\,(B_{n,n})) = \{1, 3, 5, \ldots, 2n - 1\} \cup \{2(n + 1) - 1, 2(n + 2) - 1, \ldots, 2(2n) - 1\} \cup \{4n + 1, 4n + 3, \ldots, 6n - 1\} \cup \{6n + 1\} \cup \{6n + 3\} \cup \{6n + 5, 6n + 7, \ldots, 6n + 2\left\lceil \frac{n-1}{2} \right\rceil + 3\} \cup \{8n + 1\} \cup \{6n + 2\left\lceil \frac{n-1}{2} \right\rceil + 1, \ldots, 8n + 5\} \cup \{8n + 7, \ldots, 12n + 3\} \cup \{8n + 9, \ldots, 12n + 5\}$

$= \{1, 3, 5, \ldots, 2n - 1, 2n + 1, \ldots 4n - 1, 4n + 1, \ldots, 6n - 1, 6n + 1, \ldots, 8n + 5, 8n + 7, \ldots, 12n + 3, 12 + 5\}.$

Hence, *Spl* $(B_{n,n})$ admits odd edge labeling and by 3.7, it is a felicitous graph for all n.
For example, a felicitous labeling of *Spl*$(B_{n,n})$ is shown in Figure 4.2.

Fig-4.2.



Theorem 4.3. Planar graph $Pl_{m,n}$ is a felicitous for all m and n.

Proof : Let $V(Pl_{m,n}) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\}$ and $E(Pl_{m,n}) = \{(u_1v_j) : 1 \le j \le n\} \cup \{(u_2v_j) : 1 \le j \le n\} \cup \{(u_iv_1) : 3 \le i \le m\} \cup \{(u_iv_n) : 3 \le i \le m\}$.

Define a function $f : V(Pl_{m,n}) \rightarrow \{0,1,2, \ldots, q = 2(m + n - 2)\}$ by

$f(u_i) = i - 1, \quad 1 \le i \le m$

$$f(v_j) = \begin{cases} m, & j = 1 \\ 2(m + j - 2), & 2 \le j \le n \end{cases}$$

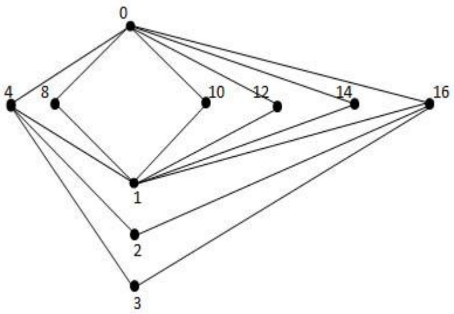
The edge labels are as follows :

$$f_1(u_1\, v_j) = \begin{matrix} m, & j = 1 \\ 2(m + j - 2), & 2 \le j \le n \\ m + 1, & j = 1 \\ 2(m + j - 2) + 1, & 2 \le j \le n \end{matrix}$$

$f_1(u_i\, v_1) = m + i - 1, \quad 3 \le i \le m$
$f_1(u_i\, v_n) = 2(m + n - 2) + (i - 1), \quad 3 \le i \le m$

Clearly, $f_1(Pl_{m,n})$ = {m, 2m, 2(m + 1), . . . , 2(m + n - 2)} ∪ {m + 1, 2m + 1, 2(m + 1) + 1, . . . , 2(m + n - 2) + 1)} ∪ {m + 2, m + 3, . . . , 2m - 1} ∪ {2(m + n - 2) + 2, 2(m + n - 2) + 3, . . . , 2(m + n - 2) + (m - 1)}.
= {m, m + 1, m + 2, . . . , 2m - 1, 2m, 2m + 1, . . . , 2(m + n - 2), 2(m + n - 2) + 1, . . . , 2(m + n - 2) + (m - 1)}.
Then, $f^*(E(Pl_{m,n}))$= {m, m + 1, m + 2, . . . , 2m - 1, 2m, 2m + 1, . . . , 2(m + n - 2), 1, 2, . . . , m - 1}
{1, 2, . . . , m - 1, m, m + 1, m + 2, . . . , 2m - 1, 2m, 2m + 1, . . . , 2(m + n - 2)}.
Hence, $Pl_{m,n}$ is a felicitous graph.
For example, a felicitous labeling of $Pl_{m,n}$ is shown in Figure 4.3.

Fig-4.3.



Theorem 4.4: $C_{2k+1} \odot S_m$ is a felicitous graph for all k ≥ 1 and m ≥ 1.
Proof : Let $V(C_{2k+1} \odot S_m) = \{u_i : 1 \le i \le 2k + 1\} \cup \{v_{i_j} : 1 \le i \le 2k + 1 \text{ and } 1 \le j \le m\}$ and $E(C_{2k+1} \odot S_m) = \{(u_i u_{i+1}) : 1 \le i \le 2k\} \cup \{(u_{2k+1} u_1)\} \cup \{(u_i v_{i_j}) : 1 \le i \le 2k + 1 \text{ and } 1 \le j \le m\}$.
Define a function $f : V(C_{2k+1} \odot S_m) \to \{0,1,2, . . . , q = (2k + 1)(m + 1)\}$ by

$f(u_i) = i - 1, \quad 1 \le i \le 2k + 1$

For $1 \le i \le 2k$,

$f(v_{i_j}) = j(2k + 1) + i, \quad 1 \le j \le m$

For i = 2k + 1,

$f(v_{i_j}) = j(2k + 1), \quad 1 \le j \le m$

The edge labels are as follows :

$f_1(u_i u_{i+1}) = 2i - 1, \quad 1 \le i \le 2k$
$f_1(u_{2k+1} u_1) = 2k$

For $1 \le i \le 2k$,

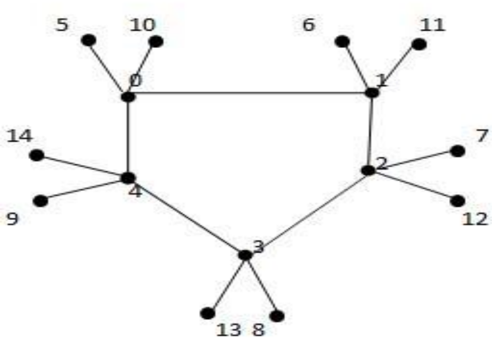
$f(u_i v_{i_j}) = j(2k + 1) + (2i - 1), \quad 1 \le j \le m$

For i = 2k + 1,

$f(u_i v_{i_j}) = j(2k + 1) + (i - 1), 1 \le j \le m$

Clearly, $f_1(C_{2k+1} \odot S_m)$ = {1, 3, 5, . . . , 2k - 1, . . . , 2(2k) - 1} ∪ {2k} ∪ {1(2k + 1) + 1, 2(2k + 1) + 2, . . . , m(2k + 1) + 1, 1(2k + 1) + 3, 2(2k + 1) + 3, . . . , m(2k + 1) + 3, . . . , 1(2k + 1) + 4k - 1, 2(2k + 1) + 4k - 1, . . . , m(2k + 1) + 4k - 1} ∪ {1(2k + 1) + 2k, 2(2k + 1) + 2k, . . . , m(2k + 1) + 2k}.
= {1, 3, 5, . . . , 2k - 1, . . . , 4k - 1} ∪ {2k} ∪ {2k + 2, 4k + 3, . . . , m(2k + 1) + 1, 2k + 4, 4k + 5, . . . , m(2k + 1) + 3, . . . , 6k, 8k + 1, . . . , m(2k+1) + 4k - 1} ∪ {4k + 1, 6k + 2, . . . , m(2k + 1) + 2k}.
= {1, 3, 5, . . . , 2k - 1, 2k, 2k + 2, 2k + 4, . . . , 4k - 1, 4k + 3, 4k + 5, . . . , 6k, 6k + 2, . . . , 8k + 1, . . . , m(2k + 1) + 1, m(2k + 1) + 3, . . . , m(2k + 1) + 2k, . . . , m(2k + 1) + 4k - 1}
= {1, 3, 5, . . . , 2k - 1, 2k, . . . , 8k + 1, . . . , m(2k + 1) + 1, m(2k + 1) + 3, . . . , m(2k + 1) + 2k, m(2k + 1) +(2k + 1), . . . , m(2k + 1) + (2k + 1) + (2k - 2)}.
Then, $f^*(E(C_{2k+1} \odot S_m))$ = $f_1(E(C_{2k+1} \odot S_m))$ (mod (m + 1) (2k + 1))
= {1, 2, 3, . . . , (m + 1) (2k + 1)}.
Hence, $C_{2k+1} \odot S_m$ is a felicitous graph.
For example, a felicitous labeling of $C_5 \odot S_2$ is shown in Figure 4.4.

Fig. 4.4.



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