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Graphs with Appended End Vertices: Laplacian Spectra, Laplacian Energy, and Laplacian Eigen-Bi-Balance

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Abstract: In this paper, we determine the Laplacian spectra of graphs obtained by appending h end vertex to all vertices of a defined class of graphs called the *base graph*. The end vertices allow for a quick solution to the eigen-vector equations of the Laplacian matrix satisfying the characteristic equation, and the solutions to the eigenvalues of the Laplacian matrix of the base graph arise. We determine the relationship between the eigenvalues of the Laplacian matrix of the base graph and the eigenvalues of the Laplacian matrix of the new graph as constructed above, and determine that if α is an eigenvalue of the Laplacian matrix of the base graph, then $\lambda = \dfrac{(\alpha+h+1)\pm\sqrt{(\alpha+h+1)^2-4\alpha}}{2}$ is an eigenvalue of the Laplacian matrix of the constructed graph.

We then determine the Laplacian spectra for such graphs where the base graph is one of the well-known classes of graphs, namely the complete, complete split-bipartite, cycle, path, wheel and star graphs. We then use the Laplacian spectra to determine the Laplacian energy of the graph, constructed from the base graphs, for each of the above classes of graphs. We then analyse the case where only one end vertex is appended to each vertex in the base graph, and determine the Laplacian energy for large values of n, the total number of vertices in the constructed graph. In the last section, we investigate the eigen-bi-balance of the graphs using the eigenvalues of the Laplacian matrix for graphs with appended end vertices, and consider the example of the star sun graph.

Keywords: Laplacian spectra of graphs; Graphs with many end vertices; Laplacian energy of graphs; Laplacian eigen-bi-balance.
JEL Classification: 05C50

1. Introduction

In this paper, we use the concept of the energy of a graph using the eigenvalues of a graph Gutman [1] and use it to define the energy of a graph using the eigenvalues of the Laplacian matrix of a graph Fath-Tabar, *et al.* [2] and Gutman [1]. There are some similarities between these definitions, and some differences – see Radenkovic and Gutman [3].

We determine the Laplacian spectra of graphs obtained by appending h end vertex to all vertices of a defined class of graphs called the *base graph*. The end vertices allow for a quick solution to the eigen-vector equations satisfying the characteristic equation for the Laplacian matrix, and the solutions to the eigenvalues of the Laplacian matrix of the base graph arise. We determine the relationship between the eigenvalues of the Laplacian matrix of the base graph and the eigenvalues of the Laplacian matrix of the new graph as constructed above, and determine that if α is an eigenvalue of the Laplacian matrix of the base graph, then $\lambda = \dfrac{(\alpha+h+1)\pm\sqrt{(\alpha+h+1)^2-4\alpha}}{2}$ is an eigenvalue of the Laplacian matrix of the constructed graph.

We then determine the spectra for such graphs where the base graph is one of the well-known classes of graphs, namely the complete, complete split-bipartite, cycle, path, wheel and star graphs. We also determine the energy of the constructed graph for each of these classes of graphs.

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In the last section, we apply the definitions of eigen-bi-balance of graphs using the eigenvalues of the Laplacian matrix of a graph Winter and Jessop [4], and then apply these definitions to the graphs with appended end vertices, and calculate the ratios for the star sun graph.

2. Laplacian Energy

The energy of a graph was introduced almost 30 years ago, and has a clear connection to chemical problems.

Definition 2.1

The *energy of a graph* is $E(G)=\sum_{i=1}^{n}|\lambda_i|$, where λ_i, $1\le i\le n$ are the eigenvalues of the adjacency matrix of the graph G.

It is easily verified that the eigenvalues obey the following well-known relations $\sum_{i=1}^{n}\lambda_i=0$ and $\sum_{i=1}^{n}\lambda_i^2=2m$

See Gutman and Zhou [5].

This quantity has in recent times attracted much attention of mathematicians and mathematical chemists. There has also been much learnt about a graph by creating an adjacency matrix for the graph, and then computing the eigenvalues of the Laplacian of the adjacency matrix. We now define the *Laplacian energy* of a graph as follows.

Definition 2.2

The *Laplacian eigenvalues* $\mu_1,\mu_2,\mu_3,...,\mu_n$ of a graph G, are the eigenvalues of the Laplacian matrix $L(G)$ of G, where $L(G)=D(G)-A(G)$, $D(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. Then we have

$$\sum_{i=-1}^{n}\mu_i=2m \text{ and } \sum_{i=-1}^{n}\mu_i^2=2m+\sum_{i=1}^{n}d_i^2 .$$

See Gutman and Zhou [5].

The *Laplacian energy of the graph* G, was defined to get a graph energy concept that is defined in terms of the eigenvalues of the Laplacian matrix instead of the eigenvalues of the adjacency matrix of the graph, and that would preserve the main features of the original graph energy definition.

The *Laplacian energy of the graph* G, denoted by $L_iE(G)$, $i=1,2$ has been defined in the following two ways:

1) The first definition is a direct adaptation of the definition of the Energy of a graph, using the eigenvalues of the Laplacian matrix, ie.

$$L_1E(G)=\sum_{i=1}^{n}|\mu_i| .$$

However, since the eigenvalues of the Laplacian matrix is always non-negative,

$$\sum_{i=-1}^{n}|\mu_i|=\sum_{i=-1}^{n}\mu_i=2m=tr(L(G)),$$

2) The definition for the Laplacian energy of a graph was then adapted, in a natural way, to become

$$L_2E(G)=\sum_{i=1}^{n}|\mu_i-\bar{\mu}|, \text{ where } \bar{\mu}=\frac{\sum_{i=1}^{n}\mu_i}{n} .$$

ie. The sum of the absolute value of the deviation of the Laplacian eigenvalues from the mean of the Laplacian eigenvalues. This gives a measure of the variance of the Laplacian eigenvalues about their mean. This definition preserves the main features of the original graph energy $E(G)$ definition as follows:

$$\bar{\mu}=\frac{\sum_{i=1}^{n}\mu_i}{n}=\frac{2m}{n}, \text{ and therefore } L_2E(G)=\sum_{i=1}^{n}\left|\mu_i-\frac{2m}{n}\right| .$$

We note that (similar to the original graph energy $E(G)$ definition):

$\sum_{i=1}^{n}\left(\mu_i - \frac{2m}{n}\right) = 0$ and $\sum_{i=1}^{n}\left(\mu_i - \frac{2m}{n}\right)^2 = 2M$ where $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$ and that if G is a regular graph, then $M = m$. See Gutman and Zhou [5].

There are a great deal of similarities between the properties of the energy of a graph $E(G)$ and the two definitions of the Laplacian energy of a graph $L_iE(G), i = 1,2$, but there are also some significant differences. See Gutman and Zhou [5], Radenkovic and Gutman [3].

For regular graphs, we have the following:

Lemma 6.1.1

If the graph G is regular of degree r, then $L_2E(G) = E(G)$.

Proof

If a $(n,m)-$ graph is regular of degree r, then $\frac{2m}{n} = r$.

Now $A(G)\underline{x_i} = \lambda_i x_i$ for eigenvalue λ_i and eigenvector $\underline{x_i}$, $for\ i = 1,\ldots,n$, and

$L(G)\underline{y_j} = (rI - A(G))\underline{y_j} = rI\underline{y_j} - A(G)\underline{y_j} = \mu_j \underline{y_j}$ for eigenvalue μ_j and eigenvector $\underline{y_j}$ of $L(G)$, $for\ j = 1,\ldots,n$.

Therefore $A(G)\underline{y} = rI\underline{y} - \mu_j\underline{y}$

$\Rightarrow A(G)\underline{y} = (rI - \mu_j)\underline{y}$.

$\Rightarrow (r - \mu_i) = \lambda_i$.

Therefore, $\mu_i - \frac{2m}{n} = \mu_i - r = -\lambda_i,\quad for\ i = 1,2,\ldots,n.$

Then $L_2E(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| = \sum_{i=1}^{n}\left|-\lambda_i\right| = E(G)$. □

The following theorem is useful in determining the Laplacian eigenvalues for a number of classes of graphs.

Theorem 2.1

Let K and L be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_l$, respectively. Then the Laplacian eigenvalues of $K \otimes L$ are

$\mu_1 = k + l$;

$\mu_{1+i} = l + \lambda_i,\ 1 \leq i \leq k-1$;

$\mu_{k+j} = k + \omega_j,\ 1 \leq j \leq l-1$; and

$\mu_{k+l} = 0$.

See Mohannadian and Tayfeh-Rezaie [6].

3. End Vertices Appended to Each Vertex of a Base Graph

Let the generalized sun graph $Gsun(h,p)$ be a graph which consists of the base graph G on p vertices, with h end vertices appended to each of the p vertices in the graph G. Then the graph $Gsun(h,p)$ has $n = p(h+1)$ vertices, and the (nxn) Laplacian matrix of $GSun(h,p)$ is:

$$L(GSun(h,p)) = \begin{bmatrix} L(G) + hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{p,p} & \cdots & I_{p,p} \end{bmatrix}$$

For example, the Laplacian matrix of $CompSun(2,3)$ is

$$L(CompSun(2,3)) = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 3.1

If α_j are the eigenvalues of $L(G)$, $1 \le j \le p$, then $\{\lambda_{2j-1}, \lambda_{2j}\} = \dfrac{(\alpha_j + h + 1) \pm \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2}$ are two eigenvalues of $L(GSun(h,p))$, for $1 \le j \le p$. The remaining eigenvalues of $L(GSun(h,p))$ are $\lambda_{2p+j} = 1$, $1 \le j \le p(h-1)$.

Proof

Let $\underline{x} = (x_1, x_2 x_3, \ldots, x_n)$ be an eigenvector of $L(GSun(h,p))$, with eigenvalue λ . Then we have:

$L(GSun(h,p))\underline{x} = \lambda \underline{x}$

$$\Rightarrow \begin{bmatrix} L(G) + hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{p,p} & \cdots & I_{p,p} \end{bmatrix} \underline{x} = \lambda \underline{x}$$

$$\Rightarrow \begin{bmatrix} L(G)_1 (x_1, \ldots, x_p)^T + hx_1 - \sum_{i=1}^{h} x_{ip+1} \\ L(G)_2 (x_1, \ldots, x_p)^T + hx_2 - \sum_{i=1}^{h} x_{ip+2} \\ \vdots \\ L(G)_k (x_1, \ldots, x_p)^T + hx_k - \sum_{i=1}^{h} x_{ip+k} \\ \vdots \\ L(G)_p (x_1, \ldots, x_p)^T + hx_p - \sum_{i=1}^{h} x_{ip+p} \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_1 \\ \vdots \\ \lambda x_k \\ \vdots \\ \lambda x_p \end{bmatrix}$$ where $L(G)_k$ is the k the row of $L(G)$,

and

$-x_k + x_{ip+k} = \lambda x_{p+k}$, $1 \le i \le h$ and $1 \le k \le p$.

$\Rightarrow -x_k = (\lambda - 1) x_{ip+k}$

$\Rightarrow x_{ip+k} = \dfrac{1}{1-\lambda} x_k$.

Substituting the values for $x_{ip+k} = \dfrac{1}{1-\lambda} x_k$ into equation k in the matrix equation above, we get

$$L(G)_k (x_1, \ldots, x_p)^T + hx_k - \sum_{i=1}^{h} x_{ip+k} = \lambda x_k$$

$$\Rightarrow L(G)_k (x_1, \ldots, x_p)^T - \left(\frac{1}{1-\lambda} x_k + \frac{1}{1-\lambda} x_k + \ldots + \frac{1}{1-\lambda} x_k \right) = (\lambda - h) x_k$$

$$\Rightarrow L(G)_k (x_1,\ldots,x_p)^T - \frac{h}{1-\lambda} x_k = (\lambda - h) x_k$$

$$\Rightarrow L(G)_k (x_1,\ldots,x_p)^T = \left(\lambda - h + \frac{h}{1-\lambda}\right) x_k$$

Therefore $\left(\lambda - h + \frac{h}{1-\lambda}\right)$ is an eigenvalue of $L(G)$ corresponding to eigenvector $(x_1,\ldots,x_p)$, and $\left(\lambda - h + \frac{h}{1-\lambda}\right) = \alpha_j$, for some j, $1 \le j \le p$, where α_j is an eigenvalue of $L(G)$. Therefore,

$$\left(\lambda - h + \frac{h}{1-\lambda}\right) = \alpha_j$$

$$\Rightarrow (\lambda - h)(1-\lambda) + h - \alpha_j(1-\lambda) = 0$$

$$\Rightarrow \lambda - h - \lambda^2 + h\lambda + h - \alpha_j + \alpha_j \lambda = 0$$

$$\Rightarrow \lambda^2 - (\alpha_j + h + 1)\lambda + (\alpha_j) = 0$$

$$\Rightarrow \{\lambda_{2j-1}, \lambda_{2j}\} = \frac{(\alpha_j + h + 1) \pm \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2}.$$

Setting $x_j = 0$ in equation $-x_j + x_{ip+j} = \lambda x_{ip+j}$, we get $\lambda = 1$ for the remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $L(GSun(h, p))$, i.e. $\lambda_{2p+j} = 1$, $1 \le j \le p(h-1)$. □

Theorem 3.2

The Laplacian energy L_1E of the generalized sun graph $GSun(h, p)$ is

$$L_1E(GSun(h, p)) = \sum_{j=1}^{p} (\alpha_j + h + 1)$$

where α_j are the eigenvalues of $L(G)$.

Proof

$$L_1E(GSun(h, p))$$

$$= \sum_{j=1}^{p(h+1)} |\lambda_j| \text{ where } \lambda_j \text{ are the eigenvalues of } L(GSun(h, p))$$

$$= \sum_{j=1}^{p} \left(\left| \frac{(\alpha_j + h + 1) + \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_k}}{2} \right| + \left| \frac{(\alpha_j + h + 1) - \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_k}}{2} \right| \right)$$

$$+ p(h-1)$$

Now all the Laplacian eigenvalues of a graph are non-negative, therefore $\alpha_j \ge 0$, for $1 \le j \le p$, and so $\frac{(\alpha_j + h + 1) + \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2} \ge 0$ and $\frac{(\alpha_j + h + 1) - \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2} \ge 0$. Therefore,

$$L_1E(GSun(h, p))$$

$$= \sum_{j=1}^{p} \left(\frac{(\alpha_j + h + 1) - \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2} + \frac{(\alpha_j + h + 1) - \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2} \right) + p(h-1)$$

$$= \sum_{j=1}^{p} (\alpha_j + h + 1) + p(h-1)$$

□

Theorem 3.3

The Laplacian energy L_2E of the generalized sun graph $GSun(h, p)$ is

$$L_2E(GSun(h, p)(h, p)) = \sum_{j=1}^{p(h+1)} \left| \lambda_j - \frac{L_1E(GSun(h, p))}{p(h+1)} \right| \text{ and}$$

$$L_2E\left(GSun\left(1, \frac{n}{2}\right)\right) = \sum_{j=1}^{n} \left| \lambda_j - \frac{L_1E\left(GSun\left(1, \frac{n}{2}\right)\right)}{n} \right|.$$

Proof

$$\bar{\lambda} = \frac{\sum_{j=1}^{p(h+1)} \lambda_j}{p(h+1)} = \frac{2m}{p(h+1)} = \frac{L_1E(GSun(h, p))}{p(h+1)} \text{ and}$$

$$L_2E(GSun(h, p)(h, p)) = \sum_{j=1}^{p(h+1)} \left| \lambda_j - \bar{\lambda} \right| = \sum_{j=1}^{p(h+1)} \left| \lambda_j - \frac{L_1E(GSun(h, p))}{p(h+1)} \right|.$$

Setting $h = 1$, and having $n = 2p$,

$$L_2E\left(GSun\left(1, \frac{n}{2}\right)\right) = \sum_{j=1}^{n} \left| \lambda j - \frac{L_1E\left(GSun\left(1, \frac{n}{2}\right)\right)}{n} \right|. \qquad \square$$

4. Examples

4.1. The Complete Sun Graph

Let G be the complete sun graph $CompSun(h, p)$ on $n = (h+1)p$ vertices, constructed by taking a complete graph K_p and appending h pendant vertices to each of the vertices of K_p. The complete sun graph $CompSun(h, p)$ has $p\left(\frac{p-1}{2}\right) + hp = p\left(\frac{p-1}{2} + h\right)$ edges. Then the (nxn) Laplacian matrix of the $CompSun(h, p)$ graph is:

$$L(CompSun(h, p)) = \begin{bmatrix} A(K_p) + hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{n,n} & \cdots & I_{p,p} \end{bmatrix}.$$

Theorem 4.1.1

The eigenvalues of $L(CompSun(h, p))$ are

$$\{\lambda_{2j-1}, \lambda_{2j}\} = \frac{(p+h+1) \pm \sqrt{(p+h+1)^2 - 4p}}{2}, \; 1 \le j \le p-1,$$

$$\{\lambda_{2p-1}, \lambda_{2p}\} = \{h+1, 0\}, \text{ and}$$

$\lambda_{2p+j} = 1,\ 1 \le j \le p(h-1)$.

Proof

The eigenvalues of $L(K_p)$ are $\alpha_k = p$, $1 \le k \le p-1$ and $\alpha_p = 0$. See Brouwer and Haemers [7].

From Theorem 3.1, the eigenvalues of $L(CompSun(h, p))$ are

$\{\lambda_{2j-1},\lambda_{2j}\}=\dfrac{(\alpha_j+h+1)\pm\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}$, where α_j are the eigenvalues of $L(K_p)$, $1\le j\le p$, and the remaining eigenvalues are equal to 1.
Therefore,

$$
\begin{aligned}
\{\lambda_{2j-1},\lambda_{2j}\} &= \frac{(\alpha_j+h+1)\pm\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}\\
&= \frac{(p+h+1)\pm\sqrt{(p+h+1)^2-4p}}{2},\ 1\le j\le p-1, \text{ and}\\
\{\lambda_{2p-1},\lambda_{2p}\} &= \frac{(\alpha_p+h+1)\pm\sqrt{(\alpha_p+h+1)^2-4\alpha_p}}{2}\\
&= \frac{(h+1)\pm\sqrt{(h+1)^2}}{2}\\
&= \{h+1,0\}.
\end{aligned}
$$

The remaining $p(h+1)-2p=p(h-1)$ eigenvalues of $L(CompSun(h,p))$ are $\lambda_{2p+j}=1$, $1\le j\le p(h-1)$. □

Theorem 4.1.2

The Laplacian energy L_1E of the complete sun graph is

$$
\begin{aligned}
L_1E(CompSun(h,p)) &= p(p+2h-1),\\
L_1E\left(CompSun\left(1,\frac{n}{2}\right)\right) &= \frac{n}{2}\left(\frac{n}{2}+1\right), \text{ and for large } n,\\
L_1E\left(CompSun\left(1,\frac{n}{2}\right)\right) &\approx \frac{n^2}{4}.
\end{aligned}
$$

Proof

From Theorem 3.2, where α_k are the eigenvalues of $L(K_p)$,

$$
\begin{aligned}
L_1E(CompSun(h,p)) &= \sum_{j=1}^{p}(\alpha_j+h+1)+p(h-1)\\
&=(p-1)(p+h+1)+(0+h+1)+p(h-1)\\
&=p^2-p+hp-h+p-1+h+1+ph-p\\
&=p^2+2hp-p\\
&=p(p+2h-1)
\end{aligned}
$$
□

Setting $h=1$, and having $n=2p$,

$$L_1E\left(CompSun\left(1,\frac{n}{2}\right)\right)=\left(\frac{n}{2}\right)^2+2\left(\frac{n}{2}\right)-\left(\frac{n}{2}\right)=\frac{n^2}{4}+\frac{n}{2}=\frac{n}{2}\left(\frac{n}{2}+1\right).$$

So, for large n, we have $L_1E\left(CompSun\left(1,\frac{n}{2}\right)\right)\approx\dfrac{n^2}{4}$.

Theorem 4.1.3

The Laplacian energy L_2E of the complete sun graph is

$$
\begin{aligned}
L_2E(CompSun(h,p)) &= \sum_{j=1}^{p(h+1)}\left|\lambda_j-\frac{(p+2h-1)}{(h+1)}\right|,\\
L_2E\left(CompSun\left(1,\frac{n}{2}\right)\right) &= \sum_{j=1}^{2p}\left|\lambda_j-\frac{(n+2)}{4}\right|, \text{ and}
\end{aligned}
$$

$$L_2E\left(CompSun\left(1,\frac{n}{2}\right)\right)\geq\frac{n^2-4}{4}.$$

Proof

$$\overline{\lambda}=\frac{\sum_{j=1}^{p(h+1)}\lambda_j}{p(h+1)}=\frac{2m}{p(h+1)}=\frac{(p^2+2hp-p)}{p(h+1)}=\frac{(p+2h-1)}{(h+1)}\text{ and}$$

$$L_2E(CompSun(h,p))=\sum_{j=1}^{p(h+1)}\left|\lambda_j-\overline{\lambda}\right|=\sum_{j=1}^{p(h+1)}\left|\lambda_j-\frac{(p+2h-1)}{(h+1)}\right|$$

Setting $h=1$, and having $n=2p$,

$$L_2E\left(CompSun\left(1,\frac{n}{2}\right)\right)=\sum_{j=1}^{2p}\left|\lambda_j-\frac{\left(\frac{n}{2}+2-1\right)}{(1+1)}\right|$$

$$=\sum_{j=1}^{2p}\left|\lambda_j-\frac{(n+2)}{4}\right|$$

So, for large n, we have

$$L_2E\left(CompSun\left(1,\frac{n}{2}\right)\right)$$

$$=\sum_{j=1}^{n}\left|\lambda_j-\frac{(n+2)}{4}\right|$$

$$=\left(\frac{\left(\frac{n}{2}+2\right)+\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}-\frac{(n+2)}{4}\right)\left(\frac{n}{2}-1\right)$$

$$+\left(\frac{\left(\frac{n}{2}+2\right)-\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}-\frac{(n+2)}{4}\right)\left(\frac{n}{2}-1\right)+\left|2-\frac{(n+2)}{4}\right|+\left|0-\frac{(n+2)}{4}\right|$$

$$\geq\left(\frac{\left(\frac{n}{2}+2\right)+\frac{n}{2}}{2}-\frac{(n+2)}{4}\right)\left(\frac{n}{2}-1\right)$$

$$+\left(\frac{\left(\frac{n}{2}+2\right)-\frac{n}{2}}{2}-\frac{(n+2)}{4}\right)\left(\frac{n}{2}-1\right)+\left|\frac{(6-n)}{4}\right|+\left|-\frac{(n+2)}{4}\right|$$

$$=\left(\frac{n+2}{4}\right)\left(\frac{n}{2}-1\right)+\left(\frac{n-2}{4}\right)\left(\frac{n}{2}-1\right)+\left(\frac{n-6}{4}\right)+\left(\frac{n+2}{4}\right)$$

$$=\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)+\left(\frac{2n-4}{4}\right)$$

$$= \frac{n^2 - 2n}{4} + \frac{2n-4}{4}$$

$$= \frac{n^2 - 4}{4}$$ □

4.2. The Complete Split-Bipartite Sun Graph

Let G be the complete split-bipartite sun graph $BipSun(h, p)$ on $n = (h+1)p$ vertices, constructed by taking a complete split-bipartite graph $K_{\frac{p}{2},\frac{p}{2}}$ and appending h pendant vertices to each of the vertices of $K_{\frac{p}{2},\frac{p}{2}}$.

The complete split-bipartite sun graph $BipSun(h, p)$ has $\frac{p^2}{4} + ph$ edges. Then the (nxn) Laplacian matrix of the $BipSun(h, p)$ graph is:

$$L(BipSun(h,p)) = \begin{bmatrix} L\left(K_{\frac{p}{2},\frac{p}{2}}\right) + hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{p,p} & \cdots & I_{p,p} \end{bmatrix}.$$

Theorem 4.2.1

The eigenvalues of $L(BipSun(h, p))$ are

$$\{\lambda_{2j-1}, \lambda_{2j}\} = \frac{\left(\frac{p}{2}+h+1\right) \pm \sqrt{\left(\frac{p}{2}+h+1\right)^2 - 2p}}{2},\ 1 \le j \le p-2,$$

$$\{\lambda_{2p-3}, \lambda_{2p-2}\} = \frac{(p+h+1) \pm \sqrt{(p+h+1)^2 - 4p}}{2},$$

$\{\lambda_{2p-1}, \lambda_{2p}\} = \{h+1, 0\}$, and

$\lambda_{2p+j} = 1,\ 1 \le j \le p(h-1).$

Proof

Let G be the complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Then $K_{\frac{n}{2},\frac{n}{2}} = L_{\frac{n}{2}} \otimes L_{\frac{n}{2}}$ where $L_{\frac{n}{2}}$ is the graph consisting of $\frac{n}{2}$ isolated vertices. Then, by Theorem 2.1, the eigenvalues of $L\left(K_{\frac{n}{2},\frac{n}{2}}\right)$ are $\alpha_j = \frac{p}{2}$, $1 \le j \le p-2$, $\alpha_{p-1} = p$, and $\alpha_p = 0$.

From Theorem 3.1, the eigenvalues of $L(BipSun(h, p))$ are

$\{\lambda_{2j-1}, \lambda_{2j}\} = \frac{(\alpha_j + h + 1) \pm \sqrt{(\alpha j + h + 1)^2 - 4\alpha_j}}{2}$, where α_j are the eigenvalues of $L\left(K_{\frac{p}{2},\frac{p}{2}}\right)$, $1 \le j \le p$,

and the remaining eigenvalues are equal to 1.

Therefore,

$$\{\lambda_{2j-1}, \lambda_{2j}\} = \frac{\left(\frac{p}{2}+h+1\right) \pm \sqrt{\left(\frac{p}{2}+h+1\right)^2 - 4\left(\frac{p}{2}\right)}}{2},\ 1 \le j \le p-2,$$

$$\{\lambda_{2p-3}, \lambda_{2p-2}\} = \frac{(p+h+1) \pm \sqrt{(p+h+1)^2 - 4(p)}}{2},$$

$$\{\lambda_{2p-1}, \lambda_{2p}\} = \frac{(h+1) \pm \sqrt{(h+1)^2}}{2} = \{h+1, 0\}.$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $L(BipSun(h,p))$ are
$\lambda_{2p+j} = 1$, $1 \le j \le p(h-1)$. □

Theorem 4.2.2

The Laplacian energy L_1E of the complete split-bipartite sun graph is

$$L_1E(BipSun(h,p)) = p\left(\frac{p}{2} + 2h\right),$$

$$L_1E\left(BipSun\left(1, \frac{n}{2}\right)\right) = n\left(\frac{n}{8} + 1\right),$$ and for large n,

$$L_1E\left(BipSun(1, \frac{n}{2})\right) \approx \frac{n^2}{8}.$$

Proof

From Theorem 3.2, where α_k are the eigenvalues of $L\left(K_{\frac{p}{2},\frac{p}{2}}\right)$,

$$\begin{aligned}
L_1E(BipSun(h,p)) &= \sum_{j=1}^{p} (\alpha_j + h + 1) + p(h-1) \\
&= (p-2)\left(\frac{p}{2} + h + 1\right) + (p + h + 1) + (h+1) + ph - p \\
&= \frac{p}{2}(p-2) + hp - 2h + p - 2 + p + 2h + 2 + ph - p \\
&= \frac{p^2}{2} + 2hp \\
&= p\left(\frac{p}{2} + 2h\right)
\end{aligned}$$ □

Setting $h = 1$, and having $n = 2p$, we get

$$L_1E\left(BipSun\left(1, \frac{n}{2}\right)\right) = n\left(\frac{n}{8} + 1\right).$$

So, for large n, we have

$$L_1E\left(BipSun\left(1, \frac{n}{2}\right)\right) \approx \frac{n^2}{8}.$$

Theorem 4.2.3

The Laplacian energy L_2E of the complete sun graph is

$$L_2E(BipSun(h,p)) = \sum_{j=1}^{p(h+1)} \left|\lambda_j - \frac{(p+4h)}{2(h+1)}\right|,$$

$$L_2E\left(BipSun\left(1, \frac{n}{2}\right)\right) = \sum_{j=1}^{n} \left|\lambda_j - \frac{(n+8)}{8}\right|,$$ and for large n,

$$L_2E\left(BipSun\left(1, \frac{n}{2}\right)\right) \approx \frac{n^2}{8} + \frac{n}{4}.$$

Proof

$$\bar{\lambda} = \frac{\sum_{j=1}^{p(h+1)} \lambda_j}{p(h+1)} = \frac{2m}{p(h+1)} = \frac{\frac{p^2}{2} + 2hp}{p(h+1)} = \frac{(p+4h)}{2(h+1)} \text{ and}$$

$$L_2E(BipSun(h,p)) = \sum_{j=1}^{p(h+1)} \left|\lambda_j - \bar{\lambda}\right| = \sum_{j=1}^{p(h+1)} \left|\lambda_j - \frac{(p+4h)}{2(h+1)}\right|$$

Setting $h=1$, and having $n=2p$,

$$L_2E\left(BipSun\left(1,\frac{n}{2}\right)\right) = \sum_{j=1}^{n} \left|\lambda_j - \frac{\left(\frac{n}{2}+4\right)}{2(1+1)}\right|$$

$$= \sum_{j=1}^{n} \left|\lambda_j - \frac{(n+8)}{8}\right|$$

So, for large n, we have

$$L_2E\left(BipSun\left(1,\frac{n}{2}\right)\right)$$

$$= \sum_{j=1}^{n} \left|\lambda j - \frac{(n+8)}{8}\right|$$

$$= \left|\frac{\left(\frac{n}{4}+2\right) \pm \sqrt{\left(\frac{n}{4}+2\right)^2 - n}}{2} - \frac{(n+8)}{8}\right| \left(\frac{n}{2}-1\right)$$

$$+ \left|\frac{\left(\frac{n}{2}+2\right) \pm \sqrt{\left(\frac{n}{2}+2\right)^2 - 2n}}{2} - \frac{(n+8)}{8}\right| .1 + \left|2 - \frac{(n+8)}{8}\right| + \left|0 - \frac{(n+8)}{8}\right|$$

$$\approx \left|\frac{\left(\frac{n}{4}+2+\frac{n}{4}\right)}{2} - \frac{(n+8)}{8}\right| \left(\frac{n}{2}-1\right) + \left|\frac{\left(\frac{n}{4}+2-\frac{n}{4}\right)}{2} - \frac{(n+8)}{8}\right| \left(\frac{n}{2}-1\right)$$

$$+ \left|\frac{\left(\frac{n}{2}+2+\frac{n}{2}\right)}{2} - \frac{(n+8)}{8}\right| + \left|\frac{\left(\frac{n}{2}+2-\frac{n}{2}\right)}{2} - \frac{(n+8)}{8}\right| + \left|\frac{8-n}{8}\right| + \left|\frac{-(n+8)}{8}\right|$$

$$= \left|\frac{(n+4)}{4} - \frac{(n+8)}{8}\right| \left(\frac{n}{2}-1\right) + \left|\frac{(2)}{2} - \frac{(n+8)}{8}\right| \left(\frac{n}{2}-1\right)$$

$$+ \left|\frac{(n+2)}{2} - \frac{(n+8)}{8}\right| + \left|\frac{(2)}{2} - \frac{(n+8)}{8}\right| + \frac{n-8}{8} + \frac{(n+8)}{8}$$

$$= \left|\frac{n}{8}\right| \left(\frac{n}{2}-1\right) + \left|\frac{-n}{8}\right| \left(\frac{n}{2}-1\right) + \left|\frac{3n}{8}\right| + \left|\frac{-n}{8}\right| + \frac{2n}{8}$$

$$=\frac{2n}{8}\left(\frac{n}{2}-1\right)+\frac{6n}{8}$$
$$=\frac{n^2}{8}-\frac{2n}{8}+\frac{6n}{8}$$
$$=\frac{n^2}{8}+\frac{n}{4}$$ □

4.3. The Caterpillar Sun Graph

Let G be the path sun graph $Caterpillar(h,p)$ on $n=(h+1)p$ vertices. $n=(h+1)p$ vertices, constructed by taking a path graph P_p and appending h pendant vertices to each of the vertices of P_p. The caterpillar graph $Caterpillar(h,p)$ has $(p-1)+hp$ edges. Then the (nxn) Laplacian matrix of the $Caterpillar(h,p)$ graph is:

$$L(Caterpillar(h,p))=\begin{bmatrix} L(P_p)+hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{p,p} & \cdots & I_{p,p} \end{bmatrix}.$$

Theorem 4.3.1

The eigenvalues of $L(Caterpillar(h,p))$ are

$$\{\lambda_{2j+1,2j+2}\}=\frac{1}{2}\left(3-\sigma_k+h\pm\sqrt{\sigma_k{}^2-2(h+1)\sigma_k+h^2+6h+1}\right),$$

where $\sigma_j=2\cos\left(\frac{\pi j}{p}\right)$, $0\le j\le p-1$, and

$\lambda_{2p+j}=1$, $1\le j\le p(h-1)$.

Proof

The Laplacian eigenvalues of the path graph P_n are $\alpha_{j+1}=2-2\cos\left(\frac{\pi j}{p}\right)$; $0\le j\le p-1$, for $p\ge 3$, - see Brouwer and Haemers [7]. From Theorem 3.1, the eigenvalues of $L(Caterpillar(h,p))$ are

$$\{\lambda_{2j+1},\lambda_{2j+2}\}=\frac{(\alpha_j+h+1)\pm\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}$$, where α_j are the eigenvalues of $L(P_p)$, $1\le j\le p$, and remaining eigenvalues are equal to 1.

Therefore,

$$\{\lambda_{2j+1},\lambda_{2j+2}\}=\frac{\left(2-2\cos\left(\frac{\pi j}{p}\right)+h+1\right)\pm\sqrt{\left(2-2\cos\left(\frac{\pi j}{p}\right)+h+1\right)^2-4\left(2-2\cos\left(\frac{\pi j}{p}\right)\right)}}{2}$$

$$=\frac{\left(3-2\cos\left(\frac{\pi j}{p}\right)+h\right)\pm\sqrt{\left(3-2\cos\left(\frac{\pi j}{p}\right)+h\right)^2-4\left(2-2\cos\left(\frac{\pi j}{p}\right)\right)}}{2},$$

$0\le j\le p-1$.

Set $\sigma_j=2\cos\left(\frac{\pi j}{p}\right)$, $0\le j\le p-1$, then

$$\{\lambda_{2j+1},\lambda_{2j+2}\}=\frac{1}{2}\left(3-\sigma_j+h\pm\sqrt{(3-\sigma_j+h)^2-4(2-\sigma_j)}\right)$$

$$=\frac{1}{2}\left(3-\sigma_j+h\pm\sqrt{(3+h)^2-2(3+h)\sigma_j+\sigma_j{}^2-8+4\sigma_j}\right)$$

$$=\frac{1}{2}\left(3-\sigma_j+h\pm\sqrt{\sigma_j{}^2-2(h+1)\sigma_j+h^2+6h+1}\right).$$

The remaining $p(h+1)-2p=p(h-1)$ eigenvalues of $L(Caterpillar(h,p))$ are $\lambda_{2p+k}\ j=1$, $1\le j\le p(h-1)$ □

Theorem 4.3.2

The Laplacian energy of the caterpillar graph is

$$L_1E(Caterpillar(h,p)) = 2(p+hp-1).$$

$$L_1E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) = 2(n-1)$$, and for large n,

$$L_1E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) \approx 2n.$$

Proof

From Theorem 3.2, where α_k are the eigenvalues of $L(P_p)$,

$$\begin{aligned}L_1E(Caterpillar(h,p)) &= \sum_{j=1}^{p}(\alpha_j+h+1)+p(h-1)\\ &=\sum_{j=0}^{p-1}(2-\sigma_j+h+1)+p(h-1)\\ &=3p+hp-\sum_{j=0}^{p-1}\sigma_j+ph-p\\ &=2p+2hp-2\\ &=2(p+hp-1)\end{aligned}$$

Setting $h=1$, and having $n=2p$, we get

$$L_1E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) = 2\left(\frac{n}{2}+\frac{n}{2}-1\right)=2(n-1)$$, and for large n,

$$L_1E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) \approx 2n.$$

Theorem 4.3.3

The Laplacian energy L_2E of the caterpillar graph is

$$L_2E(Caterpillar(h,p)) = \sum_{j=1}^{p(h+1)}\left|\lambda_j-\frac{2(p+hp-1)}{p(h+1)}\right|,$$

$$L_2E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) = \sum_{j=1}^{n}\left|\lambda_j-\frac{2(n-1)}{n}\right|$$, and for large n,

$$L_2E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) > 0.618n.$$

Proof

$$\bar{\lambda}=\frac{\sum_{j=1}^{p(h+1)}\lambda_j}{p(h+1)}=\frac{2m}{p(h+1)}=\frac{2(p+hp-1)}{p(h+1)}$$ and

$$L_2E(Caterpillar(h,p)) = \sum_{j=1}^{p(h+1)} \left|\lambda_j - \bar{\lambda}\right| = \sum_{j=1}^{p(h+1)} \left|\lambda_j - \frac{2(p+hp-1)}{p(h+1)}\right|$$

Setting $h=1$, and having $n=2p$,

$$L_2E\left(Caterpillar\left(1,\frac{n}{2}\right)\right) = \sum_{j=1}^{n} \left|\lambda_j - \frac{2(2p-1)}{2p}\right|$$

$$= \sum_{j=1}^{n} \left|\lambda_j - \frac{2(n-1)}{n}\right|$$

So, for large n, we have

$$L_2E\left(Caterpillar\left(1,\frac{n}{2}\right)\right)$$

$$= \sum_{j=1}^{n} \left|\lambda_j - \frac{2(n-1)}{n}\right|$$

$$= \sum_{j=0}^{\frac{n}{2}-1} \left|\left(\frac{1}{2}\left(3-\sigma_j + h \pm \sqrt{\sigma_j^{\,2} - 2(h+1)\sigma_j + h^2 + 6h + 1}\right) - \frac{2(n-1)}{n}\right)\right|$$

$$> \sum_{j=0}^{\frac{n}{2}-1} \left|\left(\frac{1}{2}\left(4-(2)+\sqrt{(2)^2 - 4(2)+8}\right) - \frac{2(n-1)}{n}\right)\right|$$

$$+ \sum_{j=0}^{\frac{n}{2}-1} \left|\left(\frac{1}{2}\left(4-(-2)-\sqrt{(-2)^2 - 4(-2)+8}\right) - \frac{2(n-1)}{n}\right)\right|$$

$$= \sum_{j=0}^{\frac{n}{2}-1} \left|\left(\frac{1}{2}\left(2+\sqrt{4}\right) - \frac{2(n-1)}{n}\right)\right| + \sum_{j=0}^{\frac{n}{2}-1} \left|\left(\frac{1}{2}\left(6-\sqrt{20}\right) - \frac{2(n-1)}{n}\right)\right|$$

$$= \frac{n}{2}\left(\frac{1}{2}\left(2+\sqrt{4}\right) - \frac{2(n-1)}{n}\right) + \frac{n}{2}\left(\frac{2(n-1)}{n} - \frac{1}{2}\left(6-\sqrt{20}\right)\right)$$

$$= \frac{n}{4}\left(2+\sqrt{4} - 6 + \sqrt{20}\right)$$

$$= 0.618n$$

□

4.4. The Cycle Sun Graph

Let G be the cycle sun graph $CycleSun(h,p)$ on $n=(h+1)p$ vertices, constructed by taking a cycle graph C_p and appending h pendant vertices to each of the vertices of C_p. The cycle sun graph $CycleSun(h,p)$ has $(h+1)p$ edges. Then the (nxn) Laplacian matrix of $CycleSun(h,p)$ is:

$$L(CycleSun(h,p)) = \begin{bmatrix} L(C_p)+hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{p,p} & \cdots & I_{p,p} \end{bmatrix}.$$

Theorem 4.4.1

The eigenvalues of $L(CycleSun(h,p))$ are

$$\{\lambda_{2j+1,2j+2}\} = \frac{1}{2}\left(3-\varsigma_j + h \pm \sqrt{\varsigma_j^{\,2} - 2(h+1)\varsigma_j + h^2 + 6h + 1}\right),$$

where $\varsigma_j = 2\cos\left(\frac{2\pi j}{p}\right)$, $0 \le j \le p-1$, and

$\lambda_{2p+j} = 1$, $1 \le j \le p(h-1)$.

Proof

The Laplacian eigenvalues of the cycle graph C_n are $\alpha_{j+1} = 2 - 2\cos\left(\frac{2\pi j}{p}\right)$; $0 \le j \le p-1$, for $p \ge 3$, - Brouwer and Haemers [7]. From Theorem 3.1, the eigenvalues of $L(CycleSun(h, p))$ are

$\{\lambda_{2j+1}, \lambda_{2j+2}\} = \frac{(\alpha_j + h + 1) \pm \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2}$ where α_j are the eigenvalues of $L(C_p)$, $1 \le j \le p$, and the remaining eigenvalues are equal to 1.

Therefore,

$$\{\lambda_{2j+1}, \lambda_{2j+2}\}$$

$$= \frac{\left(2 - 2\cos\left(\frac{2\pi j}{p}\right) + h + 1\right) \pm \sqrt{\left(2 - 2\cos\left(\frac{2\pi j}{p}\right) + h + 1\right)^2 - 4\left(2 - 2\cos\left(\frac{2\pi j}{p}\right)\right)}}{2}$$

$$= \frac{\left(3 - 2\cos\left(\frac{2\pi j}{p}\right) + h\right) \pm \sqrt{\left(3 - 2\cos\left(\frac{2\pi j}{p}\right) + h\right)^2 - 4\left(2 - 2\cos\left(\frac{2\pi j}{p}\right)\right)}}{2}, \ 0 \le j \le p-1.$$

Set $\varsigma_j = 2\cos\left(\frac{2\pi j}{p}\right)$, $0 \le j \le p-1$, then

$$\{\lambda_{2j+1}, \lambda_{2j+2}\} = \frac{1}{2}\left(3 - \varsigma_j + h \pm \sqrt{(3 + h - \varsigma_j)^2 - 4(2 - \varsigma_j)}\right)$$

$$= \frac{1}{2}\left(3 - \varsigma_j + h \pm \sqrt{\varsigma_j^{\,2} - 2(h+1)\varsigma_j + h^2 + 6h + 1}\right).$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $L(CycleSun(h, p))$ are $\lambda_{2p+j} = 1$, $1 \le j \le p(h-1)$. □

Theorem 4.4.2

The energy of the cycle sun graph is

$$L_1E(CycleSun(h, p)) = 2p(1+h).$$

$$L_1E\left(CycleSun\left(1, \frac{n}{2}\right)\right) = 2n, \text{ and for large } n,$$

$$L_1E\left(CycleSun\left(1, \frac{n}{2}\right)\right) = 2n.$$

Proof

From Theorem 3.2, where α_k are the eigenvalues of $L(C_p)$

$$\begin{aligned} L_1E(CycleSun(h, p)) &= \sum_{j=1}^{p}(\alpha_j + h + 1) + p(h-1) \\ &= \sum_{j=0}^{p-1}(2 - \varsigma_j + h + 1) + p(h-1) \\ &= 3p + hp - \sum_{j=0}^{p-1}\varsigma_j + ph - p \\ &= 2p + 2hp \\ &= 2p(1+h) \end{aligned}$$

Setting $h = 1$, and having $n = 2p$, we get

$$L_1E\left(CycleSun\left(1,\frac{n}{2}\right)\right) = 2\left(\frac{n}{2}+\frac{n}{2}\right) = 2n$$

Theorem 4.4.3

The Laplacian energy L_2E of the cycle sun graph is

$$L_2E(CycleSun(h,p)) = \sum_{j=1}^{p(h+1)}\left|\lambda_j - 2\right|,$$

$$L_2E\left(CycleSun\left(1,\frac{n}{2}\right)\right) = \sum_{j=1}^{n}\left|\lambda_j-2\right|,$$ and for large n,

$$L_2E\left(CycleSun\left(1,\frac{n}{2}\right)\right) > 0.618n.$$

Proof

$$\bar{\lambda} = \frac{\sum_{j=1}^{p(h+1)}\lambda_j}{p(h+1)} = \frac{2m}{p(h+1)} = \frac{2p(1+h)}{p(h+1)} = 2$$ and

$$L_2E(CycleSun(h,p)) = \sum_{j=1}^{p(h+1)}\left|\lambda_j-\bar{\lambda}\right| = \sum_{j=1}^{p(h+1)}\left|\lambda_j-2\right|$$

Setting $h=1$, and having $n=2p$,

$$L_2E\left(CycleSun\left(1,\frac{n}{2}\right)\right) = \sum_{j=1}^{n}\left|\lambda_j-2\right|$$

So, for large n, we have

$$\begin{aligned}
L_2E\left(CycleSun\left(1,\frac{n}{2}\right)\right)
&= \sum_{j=1}^{n}\left|\lambda_j-2\right| \\
&= \sum_{j=0}^{\frac{n}{2}-1}\left|\frac{1}{2}\left(4-\varsigma_j \pm \sqrt{\varsigma_j^{\,2}-4\varsigma_j+8}\right)-2\right| \\
&= \sum_{j=0}^{\frac{n}{2}-1}\left|\frac{1}{2}\left(4-\varsigma_k+\sqrt{\varsigma_k^{\,2}-4\varsigma_k+8}\right)-2\right| + \sum_{j=0}^{\frac{n}{2}-1}\left|\frac{1}{2}\left(4-\varsigma_k-\sqrt{\varsigma_k^{\,2}-4\varsigma_k+8}\right)-2\right| \\
&> \frac{n}{2}\left|\frac{1}{2}\left(4-2+\sqrt{2^2-4.2+8}\right)-2\right| + \frac{n}{2}\left|\frac{1}{2}\left(4-(-2)-\sqrt{(-2)^2-4(-2)+8}\right)-2\right| \\
&= \frac{n}{2}\left|2-2\right| + \frac{n}{2}\left|\frac{1}{2}\left(6-\sqrt{20}\right)-2\right| \\
&= 0.618n
\end{aligned}$$

4.5. The Wheel Sun Graph

Let G be the wheel sun graph $WheelSun(h,p)$ on $n=(h+1)p$ vertices, constructed by taking a wheel graph W_p and appending h pendant vertices to each of the vertices of W_p. The wheel sun graph $WheelSun(h,p)$ has $(h+2)(p-1)+h$ edges. Then the (nxn) Laplacian matrix of $WheelSun(h,p)$ is:

$$L(WheelSun(h,p)) = \begin{bmatrix} A(W_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Theorem 4.5.1

The eigenvalues of $L(WheelSun(h,p))$ are

$$\{\lambda_1,\lambda_2\}=\frac{(p+h+1)\pm\sqrt{(p+h+1)^2-4(p)}}{2},$$

$$\{\lambda_{2j+1,2j+2}\}=\frac{1}{2}\left(4-\tau_j+h\pm\sqrt{\tau_j^{\,2}-(2h+4)\tau_j+h^2+8h+4}\right),$$

where $\tau_j=2\cos\left(\dfrac{2\pi j}{p-1}\right)$, $1\le j\le p-2$,

$\{\lambda_{2p-1},\lambda_{2p}\}=\{h+1,0\}$, and

$\lambda_{2p+j}=1$, $1\le j\le p(h-1)$.

Proof

Let G be the wheel graph W_n with $n-1$ spokes, with $n\ge 4$. Then $W_n=C_{n-1}\otimes L_1$ where L_1 is the graph consisting of 1 isolated vertex. Then, by Theorem 3.2, the Laplacian eigenvalues of the wheel graph W_n are $\alpha_1=p$, $\alpha_{j+1}=3-2\cos\left(\dfrac{2\pi j}{p-1}\right)$ $1\le j\le p-2$, and $\alpha_p=0$. From Theorem 3.1, the eigenvalues of $L(WheelSun(h,p))$ are

$\{\lambda_{2j-1},\lambda_{2j}\}=\dfrac{(\alpha_j+h+1)\pm\sqrt{(\alpha_j+h+1)^2-4\alpha j}}{2}$, where α_j are the eigenvalues of $L(W_p)$, $1\le j\le p$, and the remaining eigenvalues are equal to 1.

Therefore,

$$\{\lambda_1,\lambda_2\}=\frac{(p+h+1)\pm\sqrt{(p+h+1)^2-4(p)}}{2},$$

$$\{\lambda_{2j+1},\lambda_{2j+2}\}=\frac{\left(3-2\cos\left(\frac{2\pi j}{p-1}\right)+h+1\right)\pm\sqrt{\left(3-2\cos\left(\frac{2\pi j}{p-1}\right)+h+1\right)^2-4\left(3-2\cos\left(\frac{2\pi j}{p-1}\right)\right)}}{2}$$

$$=\frac{\left(4-2\cos\left(\frac{2\pi j}{p-1}\right)+h\right)\pm\sqrt{\left(4-2\cos\left(\frac{2\pi j}{p-1}\right)+h\right)^2-4\left(3-2\cos\left(\frac{2\pi j}{p-1}\right)\right)}}{2},\ 1\le j\le p-2.$$

Set $\tau_j=2\cos\left(\dfrac{2\pi j}{p-1}\right)$, $1\le j\le p-2$, then

$$\{\lambda_{2j+1},\lambda_{2j+2}\}=\frac{1}{2}\left(4-\tau_j+h\pm\sqrt{(4-\tau_j+h)^2-4(3-\tau_j)}\right)$$

$$=\frac{1}{2}\left(4-\tau_j+h\pm\sqrt{\tau_j^{\,2}-(2h+4)\tau_j+h^2+8h+4}\right)$$

and

$$\{\lambda_{2p-1},\lambda_{2p}\}=\frac{(h+1)\pm\sqrt{(h+1)^2-4(0)}}{2}=\{h+1,0\}.$$

The remaining $p(h+1)-2p=p(h-1)$ eigenvalues of $L(WheelSun(h,p))$ are $\lambda_{2p+j}=1$, $1\le j\le p(h-1)$. □

Theorem 4.5.2

The energy of the wheel sun graph is

$L_1E(WheelSun(h,p)) = 4p-4+2hp$.

$L_1E\left(WheelSun\left(1,\frac{n}{2}\right)\right) = 3n-4$, and for large n,

$L_1E\left(WheelSun\left(1,\frac{n}{2}\right)\right) \approx 3n$.

Proof

From Theorem 3.2, where α_k are the eigenvalues of $L(W_p)$

$$
\begin{aligned}
&L_1E(WheelSun(h,p))\\
&=\sum_{j=1}^{p}(\alpha_j+h+1)+p(h-1)\\
&=(p+h+1)+\sum_{j=1}^{p-2}(3-\tau_j+h+1)+(h+1)+p(h-1)\\
&=p+h+1+4(p-2)+h(p-2)-\sum_{j=1}^{p-2}\tau_j+h+1+ph-p\\
&=p+h+1+4(p-2)+h(p-2)-2(-1)+h+1+ph-p\\
&=4p-4+2hp
\end{aligned}
$$

Setting $h=1$, and having $n=2p$, we get

$$L_1E\left(WheelSun\left(1,\frac{n}{2}\right)\right) = \left(4.\frac{n}{2}\right)-4+\left(2.\frac{n}{2}\right)=3n-4$$

Theorem 4.5.3

The Laplacian energy L_2E of the wheel sun graph is

$L_2E(WheelSun(h,p)) = \sum_{j=1}^{p(h+1)}\left|\lambda_j-\frac{4p-4+2hp}{p(h+1)}\right|$,

$L_2E\left(WheelSun\left(1,\frac{n}{2}\right)\right) = n\sum_{j=1}^{2p}\left|\lambda_j-\frac{3n-4}{n}\right|$, and for large n,

$$L_2E\left(WheelSun\left(1,\frac{n}{2}\right)\right) = \sum_{j=1}^{\frac{n}{2}-2}\left|\frac{1}{2}\left(5-\tau_j\pm\sqrt{\tau_j^{\,2}-6\tau_j+13}\right)-\frac{3n-4}{n}\right|+\sqrt{\left(\frac{n}{2}+2\right)^2-2n+\frac{4n-8}{n}}$$

Proof

$$\bar{\lambda}=\frac{\sum_{j=1}^{p(h+1)}\lambda_j}{p(h+1)}=\frac{2m}{p(h+1)}=\frac{4p-4+2hp}{p(h+1)} \text{ and}$$

$$L_2E(WheelSun(h,p)) = \sum_{j=1}^{p(h+1)}\left|\lambda_j-\bar{\lambda}\right| = \sum_{j=1}^{p(h+1)}\left|\lambda_j-\frac{4p-4+2hp}{p(h+1)}\right|$$

Setting $h=1$, and having $n=2p$,

$$
\begin{aligned}
&L_2E\left(WheelSun\left(1,\frac{n}{2}\right)\right)\\
&=\sum_{j=1}^{n}\left|\lambda_j-\frac{3n-4}{n}\right|
\end{aligned}
$$

$$=\sum_{j=1}^{\frac{n}{2}-2}\left|\frac{1}{2}\left(4-\tau_j+1\pm\sqrt{\tau_j^{\,2}-6\tau_j+13}\right)-\frac{3n-4}{n}\right|$$

$$+\left|\frac{\left(\frac{n}{2}+1+1\right)\pm\sqrt{\left(\frac{n}{2}+1+1\right)^2-4\frac{n}{2}}}{2}-\frac{3n-4}{n}\right|+\left|\frac{(1+1)\pm\sqrt{(1+1)^2}}{2}-\frac{3n-4}{n}\right|$$

$$=\sum_{j=1}^{\frac{n}{2}-2}\left|\frac{1}{2}\left(5-\tau_j\pm\sqrt{\tau_j^{\,2}-6\tau_j+13}\right)-\frac{3n-4}{n}\right|$$

$$+\left|\frac{\left(\frac{n}{2}+2\right)+\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}-\frac{3n-4}{n}\right|+\left|\frac{\left(\frac{n}{2}+2\right)-\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}-\frac{3n-4}{n}\right|+\frac{n-4}{n}+\frac{3n-4}{n}$$

$$=\sum_{j=1}^{\frac{n}{2}-2}\left|\frac{1}{2}\left(5-\tau_j\pm\sqrt{\tau_j^{\,2}-6\tau_j+13}\right)-\frac{3n-4}{n}\right|+\frac{2\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}+\frac{4n-8}{n}$$

$$=\sum_{j=1}^{\frac{n}{2}-2}\left|\frac{1}{2}\left(5-\tau_j\pm\sqrt{\tau_j^{\,2}-6\tau_j+13}\right)-\frac{3n-4}{n}\right|+\sqrt{\left(\frac{n}{2}+2\right)^2-2n}+\frac{4n-8}{n}\qquad\square$$

4.6. The Star Sun Graph

Let G be the star sun graph $StarSun(h,p)$ on $n=(h+1)p$ vertices, constructed by taking a star graph $S_{p-1,1}$, with $p-1$ rays of length 1, and appending m pendant vertices to each of the vertices of $S_{p-1,1}$. The star sun graph $StarSun(h,p)$ has $(h+1)(p-1)+h$ edges. Then the (nxn) Laplacian matrix of $StarSun(h,p)$ is:

$$L(StarSun(h,p))=\begin{bmatrix} L(S_{p-1,1})+hI_{p,p} & -I_{p,p} & \cdots & -I_{p,p} \\ -I_{p,p} & I_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ -I_{p,p} & 0_{p,p} & \cdots & I_{p,p} \end{bmatrix}.$$

Theorem 4.6.1

The eigenvalues of $L(StarSun(h,p))$ are

$\{\lambda_1,\lambda_2\}=\dfrac{(p+h+1)\pm\sqrt{(p+h+1)^2-4p}}{2}$,

$\{\lambda_{2j+1},\lambda_{2j+2}\}=\dfrac{(2+h)\pm\sqrt{(2+h)^2-4}}{2}$, $1\le j\le p-2$

$\{\lambda_{2p-1},\lambda_{2p}\}=\{h+1,0\}$, and

$\lambda_{2p+j}=1,\ 1\le j\le p(h-1)$.

Proof

Let G be the star graph $S_{n-1,1}$ with $n-1$ rays of length 1. Then $S_{n-1,1} = L_{n-1} \otimes L_1$ where L_i is the graph consisting of i isolated vertices. Then, by Theorem 3.2, the Laplcaian eigenvalues of the star graph $S_{n-1,1}$ are $\alpha_1 = p$, $\alpha_j = 1$, $2 \le j \le p-1$ and $\alpha_p = 0$. From Theorem 3.1, the eigenvalues of $L(StarSun(h,p))$ are $\lambda_j = \dfrac{(\alpha_j + h + 1) \pm \sqrt{(\alpha_j + h + 1)^2 - 4\alpha_j}}{2}$, where α_j are the eigenvalues of $L(StarSun(h,p))$, $1 \le j \le p$, and the remaining eigenvalues are equal to 1.

Therefore,

$$\{\lambda_1, \lambda_2\} = \frac{(p+h+1) \pm \sqrt{(p+h+1)^2 - 4(p)}}{2},$$

$$\{\lambda_{2j-1}, \lambda_{2j}\} = \frac{(1+h+1) \pm \sqrt{(1+h+1)^2 - 4(1)}}{2} = \frac{(2+h) \pm \sqrt{(2+h)^2 - 4}}{2},\ 2 \le j \le p-1,$$

$$\{\lambda_{2p-1}, \lambda_{2p}\} = \frac{(0+h+1) \pm \sqrt{(0+h+1)^2 - 4(0)}}{2} = \{h+1, 0\}.$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $L(StarSun(h,p))$ are $\lambda_{2p+j} = 1$, $1 \le j \le p(h-1)$.

Theorem 4.6.2

The Laplacian energy L_1E of the star sun graph is

$$L_1E(StarSun(h,p)) = 2(p + ph - 1).$$

$$L_1E\left(StarSun\left(1, \frac{n}{2}\right)\right) = 2n - 2, \text{ and, for large } n,$$

$$L_1E\left(StarSun\left(1, \frac{n}{2}\right)\right) \approx 2n.$$

Proof

From Theorem 3.2, where α_k are the eigenvalues of $L(S_{p-1,1})$,

$$\begin{aligned} L_1E(StarSun(h,p)) &= \sum_{j=1}^{p} (\alpha_j + h + 1) + p(h-1) \\ &= (p+h+1) + (p-2)(1+h+1) + (h+1) + p(h-1) \\ &= p + h + 1 + 2p - 4 + ph - 2h + h + 1 + ph - p \\ &= 2(p + ph - 1) \end{aligned}$$

Setting $h = 1$, and having $n = 2p$, we get

$$L_1E\left(StarSun\left(1, \frac{n}{2}\right)\right) = 2\left(\frac{n}{2} + \frac{n}{2} - 1\right) = 2n - 2,$$

and for large n,

$$L_1E\left(StarSun\left(1, \frac{n}{2}\right)\right) = 2n - 2$$

Theorem 4.6.3

The Laplacian energy L_2E of the star sun graph is

$$L_2E(StarSun(h,p)) = \sum_{j=1}^{p(h+1)} \left| \lambda_j - \frac{2(p + ph - 1)}{p(h+1)} \right|,$$

$$L_2E\left(StarSun\left(1, \frac{n}{2}\right)\right) = \sum_{j=1}^{n} \left| \lambda_j - \frac{2(n-1)}{n} \right|, \text{ and, for large } n,$$

$$L_2E\left(StarSun\left(1,\frac{n}{2}\right)\right)\approx 1.823n.$$

Proof

$$\overline{\lambda}=\frac{\sum_{j=1}^{p(h+1)}\lambda_j}{p(h+1)}=\frac{2m}{p(h+1)}=\frac{2(p+ph-1)}{p(h+1)}$$ and

$$L_2E(StarSun(h,p)) \qquad =\sum_{j=1}^{p(h+1)}\left|\lambda_j-\overline{\lambda}\right|=\sum_{j=1}^{p(h+1)}\left|\lambda_j-\frac{2(p+ph-1)}{p(h+1)}\right|$$

Setting $h=1$, and having $n=2p$,

$$L_2E\left(StarSun\left(1,\frac{n}{2}\right)\right) \qquad =\sum_{j=1}^{n}\left|\lambda_j-\frac{2(n-1)}{n}\right|$$

So, for large n, we have

$$L_2E\left(StarSun\left(1,\frac{n}{2}\right)\right)$$

$$=\sum_{j=1}^{n}\left|\lambda_j-\frac{2(n-1)}{n}\right|$$

$$=\left|\frac{\left(\frac{n}{2}+2\right)\pm\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}-\frac{2(n-1)}{n}\right|+\left|\frac{(2+1)\pm\sqrt{(2+1)^2-2}}{2}-\frac{2(n-1)}{n}\right|\left(\frac{n}{2}-2\right)$$

$$+\left|2-\frac{2(n-1)}{n}\right|+\left|0-\frac{2(n-1)}{n}\right|$$

$$=\left(\frac{\left(\frac{n}{2}+2\right)+\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}-\frac{2(n-1)}{n}\right)+\left(\frac{2(n-1)}{n}-\frac{\left(\frac{n}{2}+2\right)-\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}\right)$$

$$+\left|\frac{3\pm\sqrt{7}}{2}-\frac{2(n-1)}{n}\right|\left(\frac{n}{2}-2\right)+\frac{2}{n}+\frac{2(n-1)}{n}$$

$$=\left(\frac{2\sqrt{\left(\frac{n}{2}+2\right)^2-2n}}{2}\right)$$

$$+\left(\frac{3+\sqrt{7}}{2}-\frac{2(n-1)}{n}\right)\left(\frac{n}{2}-2\right)+\left(\frac{2(n-1)}{n}-\frac{3-\sqrt{7}}{2}-\right)\left(\frac{n}{2}-2\right)+\frac{2}{n}+\frac{2(n-1)}{n}$$

$$=\left(\frac{\sqrt{n^2+16}}{2}\right)+\sqrt{7}\left(\frac{n}{2}-2\right)+2$$

$$\approx \frac{\left(1+\sqrt{7}\right)}{2}n$$
$$\approx 1.823n \qquad \square$$

5. Summary of Results

Theorem 5.1.

The Laplacian eigenvalues and the Laplacian energy for the following classes $\Im$ of graphs on $n=(h+1)p$ vertices are:

Section	Class of graph	Eigenvalues of Laplacian matrix		Laplacian Energy	for $h=1$, large n
4.1	Complete sun graph $CompSun(h, p)$	$\left(\frac{(p+h+1)\pm\sqrt{(p+h+1)^2-4p}}{2}\right)^{p-1}$ $(h+1)^1$ $(1)^{p(h-1)}$ $(0)^1$	L_1E	$p(p+2h-1)$	$\frac{n}{2}\left(\frac{n}{2}+1\right)$ $\approx\frac{n^2}{4}$
			L_2E	$\sum_{j=1}^{p(h+1)}\left\lvert\lambda_j-\frac{(p+2h-1)}{(h+1)}\right\rvert$	$\sum_{j=1}^{n}\left\lvert\lambda_j-\frac{(n+2)}{4}\right\rvert$ $\approx\frac{n^2-4}{4}$
4.2	Complete spit-bipartite sun graph $BipSun(h, p)$	$\left(\frac{\left(\frac{p}{2}+h+1\right)\pm\sqrt{\left(\frac{p}{2}+h+1\right)^2-2p}}{2}\right)^{p-2}$ $\left(\frac{(p+h+1)\pm\sqrt{(p+h+1)^2-4p}}{2}\right)^{1}$ $(h+1)^1$ $(1)^{p(h-1)}$ $(0)^1$	L_1E	$p\left(\frac{p}{2}+2h\right)$	$n\left(\frac{n}{8}+1\right)$ $\approx\frac{n^2}{8}$
			L_2E	$\sum_{j=1}^{p(h+1)}\left\lvert\lambda_j-\frac{(p+4h)}{2(h+1)}\right\rvert$	$\sum_{j=1}^{n}\left\lvert\lambda_j-\frac{(n+8)}{8}\right\rvert$ $\approx\frac{n^2}{8}+\frac{n}{4}$
4.3	Caterpillar graph $Caterpillar(h, p$	$\left(\frac{1}{2}\left(3-\sigma_j+h\pm\sqrt{\sigma_j^2-2(h+1)\sigma_j+h^2+6h+1}\right)\right.$	L_1E	$2(p+hp-1)$	$2(n-1)$ $\approx 2n$

	$(1)^{p(h-1)}$	$\sigma_j = 2\cos\left(\frac{\pi j}{p}\right),\ 0 \le j \le p-1,$			
			L_2E	$\sum_{j=1}^{p(h+1)} \left\lvert \lambda_j - \frac{2(p+hp-1)}{p(h+1)} \right\rvert$	$\sum_{j=1}^{n} \left\lvert \lambda_j - \frac{2(n-1)}{n} \right\rvert$ $> 0.618n$
4.4	Cycle sun graph $CycleSun(h, p)$	$\left(\frac{1}{2}\left(3-\varsigma_j + h \pm \sqrt{\varsigma_j^{\,2} - 2(h+1)\varsigma_j + h^2 + 6h + 1}\right)\right)$, $\varsigma_j = 2\cos\left(\frac{2\pi j}{p}\right),\ 0 \le j \le p-1,$ $(1)^{p(h-1)}$	L_1E	$2p(1+h)$	$2n$
			L_2E	$\sum_{j=1}^{p(h+1)} \lvert \lambda_j - 2 \rvert$	$\sum_{j=1}^{n} \lvert \lambda_j - 2 \rvert$ $> 0.618n$
4.5	Wheel sun graph $WheelSun(h, p)$	$\left(\frac{1}{2}\left(4-\tau_j + h \pm \sqrt{\tau_j^{\,2} - (2h+4)\tau_j + h^2 + 8h + 4}\right)\right)$ $\tau_j = 2\cos\left(\frac{2\pi j}{p}\right),\ 1 \le j \le p-2$ $\left(\frac{(p+h+1) \pm \sqrt{(p+h+1)^2 - 4p}}{2}\right)^1$ $(h+1)^1$ $(1)^{p(h-1)}$ $(0)^1$	L_1E	$4p-4+2hp$	$3n-4$ $\approx 3n$
			L_2E	$\sum_{j=1}^{p(h+1)} \left\lvert \lambda_j - \frac{4p-4+2hp}{p(h+1)} \right\rvert$	$\sum_{j=1}^{n} \left\lvert \lambda_j - \frac{3n-4}{n} \right\rvert$
4.6	Star sun graph $StarSun(h, p)$	$\left(\frac{(p+h+1) \pm \sqrt{(p+h+1)^2 - 4p}}{2}\right)^1$	L_1E	$2(p+ph-1)$	$2n-2$ $\approx 2n$

		$\left(\frac{(2+h)\pm\sqrt{(2+h)^2-4}}{2}\right)^{p-2}$ $(h+1)^1$ $(1)^{p(h-1)}$ $(0)^1$			
			L_2E	$\sum_{j=1}^{p(h+1)}\left\lvert\lambda_j-\frac{2(p+ph-1)}{p(h+1)}\right\rvert$	$\sum_{j=1}^{n}\left\lvert\lambda_j-\frac{2(n-1)}{n}\right\rvert$ $\left(\frac{\sqrt{n^2+16}}{2}\right)+\sqrt{7}\left(\frac{n}{2}-2\right)+2$ $\approx 1.823n$

6. Laplacian Eigen-Bi-Balance of Classes of Graphs

The concept of eigen-bi-balance was introduced for the eigenvalues of the adjacency matrix of a graph. See Winter and Jessop [4]. These definitions can also be applied to the eigenvalues of the Laplacian matrix of a graph as follows.

Definition 6.1

A class $\Im$ of graphs is said to be *sum*(s)*Laplacian eigen-pair (integral) balanced* if there exists a pair (a,b) of *distinct non-zero* eigenvalues (eigenvalues considered once so we ignore multiplicities) of the Laplacian matrix associated with each member of the class, called a *Laplacian eigen-pair*, such that $a+b=s$ where s is the **same integer** as a fixed constant for each member in the class, or s is the **same integer** as a **function of each member** in the class. The sum balance is *exact*, if *s* is the same integer as a fixed constant for each member in the class, or *non-exact*.

Definition 6.2

A classes $\Im$ of graphs is said to be *product*(t)*Laplacian eigen-pair (integral) balanced* if there exists a Laplacian eigen-pair (a,b), such that $a.b=t$ where t is the **same integer** as a fixed constant for each member in the class, or t is the **same integer** as a **function of each member** in the class. The sum balance is *exact*, if t is the same integer as a fixed constant for each member in the class, or *non-exact*.

Definition 6.3

Classes of graphs, which are both sum and product Laplacian eigen-pair balanced are said to be *Laplacian eign-bi-balanced* with respect to the Laplacian eigen-pair a,b.

Definition 6.4

The reciprocals of eigenvalues are connected to the idea of *robustness* or *tightness* of graphs Brouwer and Haemers [7]. Since *a* and *b* are non-zero, the sum of their reciprocals is defined, and we define the *Laplacian eigen-balanced ratio* of the structure (with respect to the Laplacian eigen-pairs) as:

$Lr(a\Im b)=\frac{1}{b}+\frac{1}{a}=\frac{a+b}{ab}$, where $ab\neq 0$.

Definition 6.5

If the Laplacian eigen-balanced ratio is a function $f(n)$ of the size *n* of the graph, and has a horizontal asymptote, we call this asymptote the *Laplacian eigen-balanced ratio asymptote* with respect to the Laplacian eigen-pair *a,b* and denoted by $Lr(a\Im b)^{\infty}$ or $asymp(Lr(a\Im b))$.

This asymptote can be seen as the behavior of the ratio as the size of the graph becomes increasingly large.

Theorem 6.1

Let $\Im$ be a class of graphs consisting of the graphs $GSun(h,p)$, for $p=1,2,...$ and some $h=1,2,....$ Then, if α_j, $1\le j\le p$, is an integral eigenvalue of $L(G)$, then:

1. $\Im$ is sum* (α_j+h+1)*Laplacian eigen-pair (integral) balanced with respect to $(\lambda_{2k-1}.\lambda_{2k})$, where $\{\lambda_{2j-1},\lambda_{2j}\}$ are two eigenvalues of $L(GSun(h,p))$;
2. $\Im$ is product* $(-\alpha_j)$* Laplacian eigen-pair (integral) balanced with respect to $(\lambda_{2k-1}.\lambda_{2k})$;
3. The Laplacian eigen-bi-balanced ratio of the class of graphs $\Im$ is

$$Lr(\lambda_{2j-1}\Im\lambda_{2j})=\frac{-(\alpha_j+h+1)}{\alpha_j}$$, and

There is no Laplacian eigen-bi-balanced ratio asymptote of $Lr(\lambda_{2j-1}\Im\lambda_{2j})$ for the class of graphs $\Im$.

Proof

As per Theorem 3.1, if α_j is an eigenvalues of $L(G)$, $1\le j\le p$, then $\{\lambda_{2j-1},\lambda_{2j}\}=\dfrac{(\alpha_j+h+1)\pm\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}$ are two eigenvalues of $L(GSun(h,p))$, for $1\le j\le p$.

Now if α_j is integral, then

$$\lambda_{2j-1}+\lambda_{2j}=\frac{(\alpha_j+h+1)+\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}+\frac{(\alpha_j+h+1)-\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}$$

$=\alpha_j+h+1$ is integral,

and $\Im$ is sum* (α_j+h+1)*Laplacian eigen-pair (integral) balanced with respect to $(\lambda_{2j-1}.\lambda_{2j})$.

$$\lambda_{2j-1}.\lambda_{2j}=\left(\frac{(\alpha_j+h+1)+\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}\right).\left(\frac{(\alpha_j+h+1)-\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}\right)$$

$$=\frac{(\alpha_j+h+1)^2-\left(\sqrt{(\alpha_j+h+1)^2-4\alpha_j}\right)^2}{4}$$

$=\alpha_j$ is integral,

and $\Im$ is product* (α_j)*Laplacian eigen-pair (integral) balanced with respect to $(\lambda_{2j-1}.\lambda_{2j})$.

The Laplacian eigen-bi-balanced ratio of the class of graphs $\Im$ is

$r(\lambda_{2j-1}\Im\lambda_{2j})=\dfrac{\lambda_{2j-1}+\lambda_{2j}}{\lambda_{2j-1}.\lambda_{2j}}=\dfrac{\alpha_j+h+1}{\alpha_j}$, and there is no asymptote of this ratio for large h. □

For example, let $\Im$ be the class of graphs consisting of the star sun graphs $StarSun(h,p)$, for $p=1,2,...$ and some $h=1,2,....$ Then $\alpha_j=1$, $2\le j\le p-1$, is an integral eigenvalue of $L(StarSun(h,p))$, and $\lambda_{2j-1}+\lambda_{2j}=\alpha_j+h+1=h+2$ is integral, and $\lambda_{2j-1}.\lambda_{2j}=\alpha_j=1$ is integral. Therefore,

- $\Im$ is sum* $(h+2)$*Laplacian eigen-pair (integral) balanced with respect to $(\lambda_{2k-1}.\lambda_{2k})$, where $\{\lambda_{2k-1},\lambda_{2k}\}$ are the two eigenvalues of $L(StarSun(h,p))$ as per Theorem 3.1, i.e.

$$\{\lambda_{2j-1},\lambda_{2j}\}=\frac{(\alpha_j+h+1)\pm\sqrt{(\alpha_j+h+1)^2-4\alpha_j}}{2}=\frac{(2+h)\pm\sqrt{(2+h)^2-4}}{2};$$

4. $\Im$ is product* (1)*Laplacian eigen-pair (integral) balanced with respect to $(\lambda_{2k-1}.\lambda_{2k})$;
5. The Laplacian eigen-bi-balanced ratio of the class of graphs $\Im$ is $Lr(\lambda_{2j-1}\Im\lambda_{2j})=\dfrac{\alpha_j+h+1}{\alpha_j}=\dfrac{h+2}{1}=h+2$; and
6. There is no asymptote of the Laplacian eigen-bi-balanced ratio $Lr(\lambda_{2j-1}\Im\lambda_{2j})$ for large h.

7. Conclusion

In this paper, we determined the Laplacian spectra of graphs obtained by appending h end vertices to all the vertices in the well-known classes of graphs, namely the complete, complete split-bipartite, cycle, path, wheel and star (with rays of length 1) graphs. The end vertices allowed for a quick solution to the eigen-vector equations of the Laplacian matrix satisfying the characteristic equation. We determined the Laplacian energy (based on both definitions in section 6.1) for each of these classes of graphs, and anaylsed this energy for $h = 1$ and $n = 2p$. We then determined the Laplacian energy for each class of graph, for large values of n.

We finally defined the Laplacian eigen-bi-balanced characteristics of a graph and showed that if a graph G has a non-zero, integral eigenvalue of its Laplacian matrix, then the graph constructed by appending end vertices to each of the vertices in G has a pair of Laplacianeigenvalues, such that the new graph is sum and product Laplacian eigen-bi-balanced with respect to this pair of Laplacian eigenvalues of the constructed graph. We also determined the Laplacian eigen-bi-balanced characteristics for the star sun graph.

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