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More on Possibility Intuitionistic Fuzzy Soft Set

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Abstract: In this paper, we extend the possibility intuitionistic fuzzy soft sets defining new operations in it and a few properties of these operations are studied. An application of the new operations have been shown using example. Moreover the possibility intuitionistic fuzzy soft relations on PIFSS have been defined and their properties are discussed. As an end point, the notation of mapping on possibility intuitionistic fuzzy soft classes is defined and some properties of possibility intuitionistic fuzzy soft images and inverse images have been investigated. Examples and counterexamples for these concepts have been illustrated.
Keywords: Intuitionistic fuzzy soft set; Possibility intuitionistic fuzzy soft set; Mapping on intuitionistic fuzzy soft classes.

1. Introduction

In reality, the limitation of precise research is increasingly being recognized in many fields, such as economics, social science, and management science, etc. It is well known that the real world is full of uncertainty, imprecision and vagueness, so researches on these areas are of great importance. In recent years, uncertain theories such as probability theory, fuzzy set theory [1], intuitionistic fuzzy set theory [2]. However, these theories have their inherent difficulties, which are pointed in order to overcome these disadvantages, Molodtsov [3] proposed a new mathematical tool called soft set theory to deal with uncertainty and imprecision and it has been demonstrated that this new theory brings about a rich potential for applications in decision making, measurement theory, game theory, etc. After Molodtsov's work, some different operations and application of soft sets were studied by Maji, *et al.* [4]; Alkhazaleh, *et al.* [5] and Maji, *et al.* [6]. Also Alkhazaleh, *et al.* [5] introduced soft multiset as a generalization of Molodtsov's soft set. Furthermore Roy and Maji [7] presented the definition of fuzzy soft set as a generalization of Molodtsov's soft set, Maji, *et al.* [8] presented an application of fuzzy soft sets in a decision making problem. Moreover, Maji, *et al.* [8] defined the concept of intuitionistic fuzzy soft set and Maji also in Maji [9] defined some new operations on intuitionistic fuzzy soft sets and studied some results relating to the properties of these operations. Majumdar and Samanta [10] defined and studied the generalised fuzzy soft sets where the degree is attached with the parameterization of fuzzy sets while defining a fuzzy soft set and they defined the generalised fuzzy soft relation they also introduced in Majumdar and Samanta [11] several similarity measures of fuzzy soft sets and made a comparative study of these measures. Alkhazaleh, *et al.* [12] defined the concepts of possibility fuzzy soft set and gave their application in decision making and medical diagnosis. After that Bashir and Salleh [13] defined the concept of possibility intuitionistic fuzzy soft set and gave their applications in decision making and medical diagnosis. In Kharal and Ahmad [14], introduced the notions of a mapping on the classes of soft and fuzzy soft classes and studied their properties of images and inverse images. Bashir and Salleh [15] defined the notion of mappings on intuitionistic fuzzy soft classes and studied some properties of intuitionistic fuzzy soft images and inverse images. In this paper new operations on possibility intuitionistic fuzzy soft sets are introduced. Some results related to the properties of these operations are studied. An example is presented as an application of these operations. Finally, the notion of a mapping on classes of possibility intuitionistic fuzzy soft sets is introduced and the properties of possibility intuitionistic fuzzy soft images and inverse images are established.

2. Preliminaries

In this section we recall some definitions and properties regarding intuitionistic fuzzy soft set, possibility intuitionistic fuzzy soft set, and mappings on intuitionistic fuzzy soft classes.

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Definition 2.1.

Maji, *et al.* [8]. Consider U and E as a universe set and a set of parameters respectively. Let $P(U)$ denote the set of all intuitionistic fuzzy sets of U. Let $A \subseteq E$. A pair (F,E) is an intuitionistic fuzzy soft sets over U, where F is mapping given by $F: A \to P(U)$.

Let U universal set of elements, E be a set of parameters and A be a subset of E. Let the intuitionistic fuzzy soft set $(F,A) = \{\langle m, \mu_{F(e)}(m), \nu_{F(e)}(m)\rangle : m \in U, e \in A\}$ where $\mu_{F(e)}(m)$, $\nu_{F(e)}(m)$ are the membership and non-membership function respectively.

Definition 2.2.

Maji, *et al.* [8]. The necessity operation $\square$ on intuitionistic fuzzy soft set (F,A) is denoted by $\square(F,A)$ and is defined as $\square(F,A) = \{\langle m, \mu_{F(e)}(x), 1-\mu_{F(e)}(m)\rangle : m \in U, e \in A\}$ where $\mu_{F(e)}(m)$ is the membership function of m and F is a mapping $F: A \to P(U)$, $P(U)$ is the set of all intuitionistic fuzzy sets of U.

Definition 2.3.

Maji, *et al.* [8]. The possibility operation $\Diamond$ on intuitionistic fuzzy soft set (F,A) is denoted by $\Diamond(F,A)$ and is defined as $\Diamond(F,A) = \{\langle m, 1-\nu_{F(e)}(m), \nu_{F(e)}(m)\rangle : m \in U, e \in A\}$.

Definition 2.4.

Alkhazaleh, *et al.* [12]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \ldots, e_m\}$ be the universal set of parameters. The pair (U,E) will be called a soft universe. Let $F: E \to (I \times I)^U \times I^U$ and p be a fuzzy subset of E, i.e. $p: E \to I^U$, where I^U is the collection of all fuzzy subsets of U. Let $F_p: E \to (I \times I)^U \times I^U$ be a function defined as follows:

$F_p(e) = (F(e)(x),\ p(e)(x))$, where $F(e)(x) = (p(x), \nu(x))\ \forall x \in U$.

Then F_p is called a possibility intuitionistic fuzzy soft set (PIFSS in short) over the soft universe (U,E). For each parameter e_i, $F_p(e_i) = (F(e_i)(x), p(e_i)(x))$ indicates not only the degree of belongingness of the elements of U in $F(e_i)$, but also the degree of possibility of belongingness of the elements of U in $F(e_i)$, which is represented by $p(e_i)$. So we can write $F_p(e_i)$ as follows:

$$F_p(e_i) = \left\{\left(\frac{x_1}{F(e_i)(x_1)},\ p(e_i)(x_1)\right), \left(\frac{x_2}{F(e_i)(x_2)},\ p(e_i)(x_2)\right), \ldots, \left(\frac{x_n}{F(e_i)(x_n)},\ p(e_i)(x_n)\right)\right\}.$$

Sometime we write F_p as (F_p, E). If $A \subseteq E$ we can also have a PIFSS (F_p, A).

Definition 2.5.

Alkhazaleh, *et al.* [12]. Similarity between two PIFSSs F_μ and G_δ, denoted by $S(F_\mu, G_\delta)$, is defined as follows:

$S(F_\mu, G_\delta) = M(F(e), G(e)) \cdot M(\mu(e), \delta(e))$ such that

$M(F(e), G(e)) = \max_i M_i(F(e), G(e))$ and $M(\mu(e), \delta(e)) = \max_i M_i(\mu(e), \delta(e))$ where

$M_i(F(e), G(e)) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} \left(\phi_{F(e)}(i) - \phi_{G(e)}(i)\right)^p}$ $\quad 1 \le p \le \infty$,such that $\phi_{F(e)}(i) = \frac{\left(\mu_{F(e)} + \nu_{F(e)}\right)}{2}$ and

$\phi_{G(e)}(i) = \frac{\left(\mu_{G(e)} + \nu_{G(e)}\right)}{2}$ and

$$M_i\left(\mu(e),\delta(e)\right)=1-\frac{\sum_{j=1}^{n}\left|\mu_{ij}(e)-\delta_{ij}(e)\right|}{\sum_{j=1}^{n}\left|\mu_{ij}(e)+\delta_{ij}(e)\right|}.$$

Definition 2.6.

Alkhazaleh, *et al.* [12]. Union of two PIFSS F_p and G_q, denoted by $F_p\tilde{\bigcup}G_q$, is a PIFSS $H_r:E\to(I\times I)^U\times I^U$ defined by

$H_r(e)=\left(H(e)(x),\, r(e)(x)\right),\forall e\in E$

such that $H(e)=S\left(F(e),G(e)\right)$ and $r(e)=s\left(p(e),q(e)\right)$ where S is S-norm and s is an s- norm.

Definition 2.7.

Alkhazaleh, *et al.* [12]. Intersection of two PIFSS F_p and G_q, denoted by $F_p\tilde{\bigcap}G_q$, is a PIFSS $H_r:E\to(I\times I)^U\times I^U$ defined by

$H_r(e)=\left(H(e)(x), r(e)(x)\right)$,$\forall\ e\in E$

such that $H(e)=T\left(F(e),G(e)\right)$ and $r(e)=t\left(p(e),q(e)\right)$ Where T is an intuitionistic fuzzy intersection and t is an fuzzy t-norm.

Definition 2.8.

Majumdar and Samanta [10]. Let F_μ and G_σ GFSS over the parameterized universe (U,E) and $C\in E^2$. Then a fuzzy soft relation R from F_μ to G_σ is a function $R:C\to I^U\times I$, defined as follows:

$R(e,f)=F_\mu(e)\tilde{\bigcap}G_\sigma(f)\quad \forall(e,f)\in C.$

Definition 2.9.

Kharal and Ahmad [14]. Let X be a universe and E be a set of parameters. Then the collection of all intuitionistic fuzzy soft sets over X, with parameters from E is called an intuitionistic fuzzy soft class and is denoted by (X,E).

Definition 2.10.

Kharal and Ahmad [14]. Let (X,E) and $(Y,E^{'})$ be intuitionistic fuzzy soft classes. Let $r:X\to Y$ and $s:E\to E^{'}$ be mappings. Then a mapping $f:(X,E)\to(Y,E^{'})$ is defined as follows: for an intuitionistic fuzzy soft set (F,A) in (X,E), $f(F,A)$ is an intuitionistic fuzzy soft set in $(Y,E^{'})$ obtained as follows for $\beta\in s(E)\subseteq E^{'}$, $y\in Y$

and for all $\alpha\in s^{-1}(\beta)\cap A$

$$f(F,A)(\beta)(y)=\begin{cases}\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}F(\alpha))(x)\right), & \text{if } r^{-1}(y)\neq\phi,\ s^{-1}(\beta)\cap A\neq\varnothing\\ (0,0) & ,\quad \text{otherwise}\end{cases}$$

$f(F,A)$ is called an intuitionistic fuzzy soft image of an intuitionistic fuzzy soft set (F,A).

Definition 2.11.

Kharal and Ahmad [14]. Let (X,E) and $(Y,E^{'})$ be intuitionistic fuzzy soft classes. Let $r:X\to Y$ and $s:E\to E^{'}$ be mappings. Then a mapping $f^{-1}:(Y,E^{'})\to(X,E)$ is defined as follows: for an intuitionistic fuzzy soft set (G,B) in (Y,E'), $f^{-1}(G,B)$ is an intuitionistic fuzzy soft set in (X,E) obtained as follows for $\alpha \in s^{-1}(B) \subseteq E$, and $x \in X$

$$f^{-1}(G,B)(\alpha)(x) = \begin{cases} G(s(\alpha))(r(x)), & \text{for } s(\alpha) \in B \\ (0,0) & , \text{ otherwise} \end{cases}$$

$f^{-1}(G,B)$ is called an intuitionistic fuzzy soft inverse image of an intuitionistic fuzzy soft set (G,B).

3. More on Possibility Intuitionistic Fuzzy Soft Sets

In this section we extend the concept of possibility intuitionistic fuzzy soft sets as introduced by Alkhazaleh, *et al.* [12]. Let U universal set of elements, E be a set of parameters. Let $F: E \to (I \times I)^U \times I^U$ and p be a fuzzy subset of E, i.e. $p: E \to I^U$, where I^U is the collection of all fuzzy subsets of U. Let $F_p: E \to (I \times I)^U \times I^U$ be a function defined as follows:

$F_p(e) = (F(e)(x),\ p(e)(x))$, where $F(e)(x) = (p(x), \nu(x))\ \forall x \in U$.

Then F_p is possibility intuitionistic fuzzy soft set $(F_p) = \left\{ \left(\frac{x}{\mu_{F(e)}(x), \nu_{F(e)}(x)}, p(e)(x) \right) : x \in U, e \in E \right\}$,

where $\mu_{F(e)}(x)$, $\nu_{F(e)}(x)$ be the membership and non-membership function of $F(e)$ respectively.

Definition 3.1.

The necessity operation on possibility intuitionistic fuzzy soft set (F_p, E) is denoted $\square(F_p, E)$ in short $\square(F_p)$ is defined as $\square(F_p) = \left\{ \left(\frac{x}{\left(\mu_{F(e)}(x), 1-\mu_{F(e)}(x)\right)}, p(e)(x) \right) : x \in U, e \in E \right\}$ where $\mu_{F(e)}(x)$ is the membership function of $F(e)$ and F is mapping $F: E \to (I \times I)^U \times I^U$, $(I \times I)^U$ is the collection of all intuitionistic fuzzy subset of U.

Example 3.2.

Let $U = \{x_1, x_2, x_3\}$ be a set of universe, $E = \{e_1, e_2, e_3\}$ be a set of parameters and let $p: E \to I^U$. We define a function $F_p: E \to (I \times I)^U \times I^U$ as follows:

$$F_p(e_1) = \left\{ \left(\frac{x_1}{(0.4, 0.3)}, 0.7 \right), \left(\frac{x_2}{(0.7, 0.1)}, 0.5 \right), \left(\frac{x_3}{(0.5, 0.2)}, 0.6 \right) \right\},$$

$$F_p(e_2) = \left\{ \left(\frac{x_1}{(0.5, 0.1)}, 0.6 \right), \left(\frac{x_2}{(0.6, 0)}, 0.5 \right), \left(\frac{x_3}{(0.6, 0.3)}, 0.5 \right) \right\},$$

$$F_p(e_3) = \left\{ \left(\frac{x_1}{(0.7, 0)}, 0.5 \right), \left(\frac{x_2}{(0.6, 0.2)}, 0.5 \right), \left(\frac{x_3}{(0.5, 0.1)}, 0.7 \right) \right\}.$$

Then the necessity operation on possibility intuitionistic fuzzy soft set is

$$\square\left(F_p(e_1)\right)=\left\{\left(\frac{x_1}{(0.4,0.6)},0.7\right),\left(\frac{x_2}{(0.7,0.3)},0.5\right),\left(\frac{x_3}{(0.5,0.5)},0.6\right)\right\},$$

$$\square\left(F_p(e_2)\right)=\left\{\left(\frac{x_1}{(0.5,0.5)},0.6\right),\left(\frac{x_2}{(0.6,0.4)},0.5\right),\left(\frac{x_3}{(0.6,0.4)},0.5\right)\right\},$$

$$\square\left(F_p(e_3)\right)=\left\{\left(\frac{x_1}{(0.7,0.3)},0.5\right),\left(\frac{x_2}{(0.6,0.4)},0.5\right),\left(\frac{x_3}{(0.5,0.5)},0.7\right)\right\}.$$

Theorem 3.3.

Let F_p, G_q be any two PIFSES then over (U,E). Then the following results hold:

i. $\square\left[(F_p)\cup(G_q)\right]=\square(F_p)\cup\square(G_q)$,

ii. $\square\left[(F_p)\cap(G_q)\right]=\square(F_p)\cap\square(G_q)$,

iii. $\square\square(F_p)=\square(F_p)$,

iv. $\square\left[F_p\right]^n=\left[\square(F_p)\right]^n$, for any finite positive integer n,

v. $\square\left[(F_p)\cup(G_q)\right]^n=\left[\square\left[(F_p)\cup(G_q)\right]\right]^n$,

vi. $\square\left[(F_p)\cap(G_q)\right]^n=\left[\square(F_p)\cap\square(G_q)\right]^n$.

Proof: **i.**

$\square\left[(F_p)\cup(G_q)\right]$ For all $e\in E$,

$$=\square\left\{\left(\left\langle x,\max\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right),\min\left(\nu_{F(e)}(x),\nu_{G(e)}(x)\right)\right\rangle,\max\left(p(e)(x),q(e)(x)\right):x\in U\right)\right\}$$

$$=\left\{\left(\left\langle x,\max\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right),1-\max\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right)\right\rangle,\max\left(p(e)(x),q(e)(x)\right):x\in U\right)\right\}$$

$$=\left\{\left(\left\langle x,\max\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right),\min\left(1-\mu_{F(e)}(x),1-\mu_{G(e)}(x)\right)\right\rangle,\max\left(p(e)(x),q(e)(x)\right):x\in U\right)\right\}$$

$$=\left\{\left(\left\langle x,\mu_{F(e)}(x),1-\mu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U\right\}\cup\left\{\left(\left\langle x,\mu_{G(e)}(x),1-\mu_{G(e)}(x)\right\rangle,q(e)(x)\right):x\in U\right\}=$$

$$\square(F_p)\cup\square(G_q)$$

Hence the result is proved.

ii.

$\square\left[(F_p)\cap(G_q)\right]$

$$=\square\left\{\left(\left\langle x,\min\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right),\max\left(\nu_{F(e)}(x),\nu_{G(e)}(x)\right)\right\rangle,\min\left(p(e)(x),q(e)(x)\right):x\in U\right)\right\}$$

$$=\left\{\left(\left\langle x,\min\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right),1-\min\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right)\right\rangle,\min\left(p(e)(x),q(e)(x)\right):x\in U\right)\right\}$$

$$=\left\{\left(\left\langle x,\min\left(\mu_{F(e)}(x),\mu_{G(e)}(x)\right),\max\left(1-\mu_{F(e)}(x),1-\mu_{G(e)}(x)\right)\right\rangle,\min\left(p(e)(x),q(e)(x)\right):x\in U\right)\right\}$$

$$=\left\{\left(\left\langle x,\mu_{F(e)}(x),1-\mu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U\right\}\cap\left\{\left(\left\langle x,\mu_{G(e)}(x),1-\mu_{G(e)}(x)\right\rangle,q(e)(x)\right):x\in U\right\}=$$

$$\square(F_p)\cap\square(G_q)$$

iii.

Let $F_p = \left\{\left(\left\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x)\right\rangle, p(e)(x)\right) : x \in U\right\}$.

Then $\Box(F_p) = \left\{\left(\left\langle x, \mu_{F(e)}(x), 1-\mu_{F(e)}(x)\right\rangle, p(e)(x)\right) : x \in U\right\}$.

So $\Box\Box(F_p) = \left\{\left(\left\langle x, \mu_{F(e)}(x), 1-\mu_{F(e)}(x)\right\rangle, p(e)(x)\right) : x \in U\right\}$.

Hence the result is proved.

iv. Consider a possibility intuitionistic fuzzy soft set

$(F_p) = \left\{\left(\left\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x)\right\rangle, p(e)(x)\right) : x \in U, e \in E\right\}$.

Then for any finite positive integer n,

$(F_p)^n = \left\{\left(\left\langle x, \left[\mu_{F(e)}(x)\right]^n, 1-\left[1-\nu_{F(e)}(x)\right]^n\right\rangle, \left[p(e)(x)\right]^n\right) : x \in U, e \in E\right\}$.

Then $\Box\left[(F_p)\right]^n = \left\{\left(\left\langle x, \left[\mu_{F(e)}(x)\right]^n, 1-\left[\mu_{F(e)}(x)\right]^n\right\rangle, \left[p(e)(x)\right]^n\right) : x \in U, e \in E\right\}$.

Since $\Box(F_p) = \left\{\left(\left\langle x, \mu_{F(e)}(x), 1-\mu_{F(e)}(x)\right\rangle, p(e)(x)\right) : x \in U, e \in E\right\}$,

Then $\left[\Box(F_p)\right]^n = \left\{\left(\left\langle x, \left[\mu_{F(e)}(x)\right]^n, 1-\left[\mu_{F(e)}(x)\right]^n\right\rangle, \left[p(e)(x)\right]^n\right) : x \in U, e \in E\right\}$.

Hence the result.

Proof: **v.** From the fact that $\left[(F_p) \cup (G_q)\right]^n = (F_p)^n \cup (G_q)^n$ we have

$\Box\left[(F_p) \cup (G_q)\right]^n = \left[\Box\left[(F_p) \cup (G_q)\right]\right]^n$, by the therom 3.1i.iv.

$= \left[\Box(F_p) \cup \Box(G_q)\right]^n$, by the therom 3.2i.iv.ii.

Proof: **vi.** From the fact $(F_p)^n \cap (G_q)^n = \left[(F_p) \cap (G_q)\right]^n$ we have

$\Box\left[(F_p) \cap (G_q)\right]^n = \left[\Box\left[(F_p) \cap (G_q)\right]\right]^n$, by the therom 3.1.iv.

$= \left[\Box(F_p) \cap \Box(G_q)\right]^n$, by the therom3.1.ii.

Definition 3.4.

The possibility operation on possibility intuitionistic fuzzy soft set (F_p, E), denoted $\Diamond(F_p, E)$ in short $\Diamond(F_p)$, is defined as $\Diamond(F_p) = \left\{\left(\left(\dfrac{x}{\left(1-\nu_{F(e)}(x), \nu_{F(e)}(x)\right)}\right), p(e)(x)\right) : x \in U, e \in E\right\}$.

Example 3.5.

Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2, e_3\}$. Let F_p be a PIFSS defined as follows:

$$F_p(e_1) = \left\{\left(\frac{x_1}{(0.3, 0.4)}, 0.1\right), \left(\frac{x_2}{(0.7, 0.1)}, 0.4\right), \left(\frac{x_3}{(0.2, 0.6)}, 0.6\right)\right\},$$

$$F_p(e_2) = \left\{\left(\frac{x_1}{(0.2, 0.6)}, 0.3\right), \left(\frac{x_2}{(0.2, 0.5)}, 0.2\right), \left(\frac{x_3}{(0.1, 0.3)}, 0.4\right)\right\},$$

$$F_p(e_3)=\left\{\left(\frac{x_1}{(0.7,0.1)},\,0.1\right),\left(\frac{x_2}{(0.2,0,5)},\,0\right),\left(\frac{x_3}{(0.5,0.3)},\,0.6\right)\right\},$$

Then the possibility operation on possibility intuitionistic fuzzy soft set is

$$\lozenge F_p(e_1)=\left\{\left(\frac{x_1}{(0.6,0.4)},\,0.1\right),\left(\frac{x_2}{(0.9,0.1)},\,0.4\right),\left(\frac{x_3}{(0.4,0.6)},\,0.6\right)\right\},$$

$$\lozenge F_p(e_2)=\left\{\left(\frac{x_1}{(0.4,0.6)},\,0.3\right),\left(\frac{x_2}{(0.5,0.5)},\,0.2\right),\left(\frac{x_3}{(0.7,0.3)},\,0.4\right)\right\},$$

$$\lozenge F_p(e_3)=\left\{\left(\frac{x_1}{(0.9,0.1)},\,0.1\right),\left(\frac{x_2}{(0.5,0,5)},\,0\right),\left(\frac{x_3}{(0.7,0.3)},\,0.6\right)\right\},$$

Theorem 3.6.

Let F_p, G_q be any two PIFSS over (U,E). Then the following results hold:

i. $\lozenge\left[(F_p)\cup(G_q)\right]=\lozenge(F_p)\cup\lozenge(G_q)$,

ii. $\lozenge\left[(F_p)\cap(G_q)\right]=\lozenge(F_p)\cap\lozenge(G_q)$,

iii. $\lozenge\lozenge(F_p)=\lozenge(F_p)$,

iv. $\lozenge\left[F_p\right]^n=\left[\lozenge(F_p)\right]^n$, for any finite positive integer n ,

v. $\lozenge\left[(F_p)\cup(G_q)\right]^n=\left[\lozenge\left[(F_p)\cup(G_q)\right]\right]^n$,

vi. $\lozenge\left[(F_p)\cap(G_q)\right]^n=\left[\lozenge(F_p)\cap\lozenge(G_q)\right]^n$.

Proof: The proof is straightforward by using Definition 3.4. and similar to the proof of theorem 3.3.

Theorem 3.7.

Let F_p be a PIFSS over (U,E). Then the following results hold:

i. $\Box(F_p)\subset(F_p)\subset\lozenge(F_p)$,

ii. $\lozenge\Box(F_p)=\Box(F_p)$,

iii. $\Box\lozenge(F_p)=\lozenge(F_p)$.

Proof:

i. Let (F_p) possibility intuitionstic fuzzy soft set over the universe U.

Then $(F_p)=\left\{\left(\left\langle x,\mu_{F(e)}(x),\nu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U,e\in E\right\}$.

So, $\Box(F_p)=\left\{\left(\left\langle x,\mu_{F(e)}(x),1-\mu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U,e\in E\right\}$, and

$\lozenge(F_p)=\left\{\left(\left\langle x,1-\nu_{F(e)}(x),\nu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U,e\in E\right\}$.

$1-\mu_{F(e)}(x)\geq\nu_{F(e)}(x)$ then $\Box(F_p)\subseteq(F_p)$. $\left(\text{since } \mu_{F(e)}(x)+\nu_{F(e)}\leq 1\right)$.

Since we have $1-\nu_{F(e)}(x)\geq\mu_{F(e)}(x)$ then $(F_p)\subseteq\lozenge(F_p)$.

Hence the result follows.

ii. For possibility intuitionistic fuzzy soft set

$$(F_p)=\left\{\left(\left\langle x,\mu_{F(e)}(x),\nu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U,e\in E\right\},\text{ we have}$$

$$\Box(F_p)=\left\{\left(\left\langle x,\mu_{F(e)}(x),1-\mu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U\right\}\text{so}$$

$$\Diamond\Box(F_p)=\left\{\left(\left\langle x,1-\left(1-\mu_{F(e)}(x)\right),1-\mu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U\right\}.$$

$$=\left\{\left(\left\langle x,\mu_{F(e)}(x),1-\mu_{F(e)}(x)\right\rangle,p(e)(x)\right):x\in U\right\}.$$

$$=\Box(F_p).$$

iii. The proof is similar to the above proof.

Theorem 3.8.

Let F_p, G_q and be any two PIFSS over (U,E). then the following results hold:

i. $\Box\left[(F_p,A)\wedge(G_q,B)\right]=\Box(F_p,A)\wedge\Box(G_q,B)$,

ii. $\Box\left[(F_p,A)\vee(G_q,B)\right]=\Box(F_p,A)\vee\Box(G_q,B)$,

iii. $\Diamond\left[(F_p,A)\wedge(G_q,B)\right]=\Diamond(F_p,A)\wedge\Diamond(G_q,B)$,

iv. $\Diamond\left[(F_p,A)\vee(G_q,B)\right]=\Diamond(F_p,A)\vee\Diamond(G_q,B)$.

Proof:

i. Let $(H_\lambda,A\times B)=(F_p,A)\wedge(G_q,B)$, then for $\forall(\alpha,\beta)\in A\times B$

$$(H_\lambda,A\times B)=\left\{\left(\left\langle x,\mu_{H_\lambda(\alpha,\beta)}(x),\nu_{H_\lambda(\alpha,\beta)}(x)\right\rangle,\mu_{\lambda(\alpha,\beta)}(x)\right):x\in U\right\},$$

where $\mu_{H_\lambda(\alpha,\beta)}(x)=\min\left\{\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right\}$,

$\nu_{H_\lambda(\alpha,\beta)}(x)=\max\left\{\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right\}$ and $\mu_{\lambda(\alpha,\beta)}(x)=\min\left\{\mu_{p(\alpha)}(x),\nu_{q(\beta)}(x)\right\}$.So

$$\Box(H_\lambda,A\times B)=$$

$$=\left\{\left(\left\langle x,\mu_{H_\lambda(\alpha,\beta)}(x),1-\mu_{H_\lambda(\alpha,\beta)}(x)\right\rangle,\mu_{\lambda(\alpha,\beta)}(x)\right):x\in U\right\},$$

$$=\left\{\left(\left\langle x,\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),1-\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right)\right\rangle,\mu_{\lambda(\alpha,\beta)}(x)\right):x\in U\right\},$$

$$=\left\{\left(\left\langle x,\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),\max\left(1-\mu_{F(\alpha)}(x),1-\mu_{G(\beta)}(x)\right)\right\rangle,\min\left\{\mu_{p(\alpha)}(x),\nu_{q(\beta)}(x)\right\}\right):x\in U\right\},$$

$$=\left\{\left(\left\langle x,\mu_{F(\alpha)}(x),1-\mu_{F(\alpha)}(x)\right\rangle,\mu_{p(\alpha)}(x)\right):x\in U\right\}\text{AND}\left\{\left(\left\langle x,\mu_{G(\beta)}(x),1-\mu_{G(\beta)}(x)\right\rangle,\mu_{q(\beta)}(x)\right):x\in U\right\},$$

$$=\Box(F_p,A)\wedge\Box(G_q,B).$$

Hence the result is proved.

ii. similar to the proof of (i).

iii. Let $(H_\lambda,A\times B)=(F_p,A)\wedge(G_q,B)$.Then for $\forall(\alpha,\beta)\in A\times B$

$$(H_\lambda,A\times B)=\left\{\left(\left\langle x,\mu_{H_\lambda(\alpha,\beta)}(x),\nu_{H_\lambda(\alpha,\beta)}(x)\right\rangle,\mu_{\lambda(\alpha,\beta)}(x)\right):x\in U\right\},$$

where $\mu_{H_\lambda(\alpha,\beta)}(x) = \min\{\mu_{F(\alpha)}(x), \mu_{G(\beta)}(x)\}$,

$\nu_{H_\lambda(\alpha,\beta)}(x) = \max\{\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)\}$ and $\mu_{\lambda(\alpha,\beta)}(x) = \min\{\mu_{p(\alpha)}(x), \nu_{q(\beta)}(x)\}$.So

$$\lozenge(H_\lambda, A\times B) =$$
$$= \left\{\left(\left\langle x, 1-\nu_{H_\lambda(\alpha,\beta)}(x), \nu_{H_\lambda(\alpha,\beta)}(x)\right\rangle, \mu_{\lambda(\alpha,\beta)}(x)\right) : x\in U\right\},$$
$$= \left\{\left(\left\langle x, 1-\max\left(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)\right), \max\left(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)\right)\right\rangle, \mu_{\lambda(\alpha,\beta)}(x)\right) : x\in U\right\},$$
$$= \left\{\left(\left\langle x, \min\left(1-\nu_{F(\alpha)}(x), 1-\nu_{G(\beta)}(x)\right), \max\left(\nu_{F(\alpha)}(x), \nu_{G(\beta)}(x)\right)\right\rangle, \min\left\{\mu_{p(\alpha)}(x), \nu_{q(\beta)}(x)\right\}\right) : x\in U\right\},$$
$$= \left\{\left(\left\langle x, 1-\nu_{F(\alpha)}(x), \nu_{F(\beta)}(x)\right\rangle, \mu_{p(\alpha)}(x)\right) : x\in U\right\} \text{AND} \left\{\left(\left\langle x, 1-\nu_{G(\beta)}(x), \nu_{G(\beta)}(x)\right\rangle, \mu_{q(\beta)}(x)\right) : x\in U\right\},$$
$$= \lozenge(F_p, A) \wedge \lozenge(G_q, B).$$

Hence the result is proved.

iv. The proof is similar to that of (iii).

4. An Application of Newly Defined Operation on PIFSS

Alkhazaleh, *et al.* [12] defined some new operations on intuitionistic fuzzy soft sets and studied their application. In this section, we present an application of our newly defined operation on PIFSS by generalizing Maji's Algorithm using our similarity measurement method [11] of PIFSS to be compatible with our work. We consider the following problem:

Here we contemplate the problem of picking the most suitable object out of n options based on e parameters where information available are possibility intuitionistic fuzzy soft in nature. Suppose a hospital administrator wants to select an imaging machine such as MRI with certain characteristics. If all the said imaging machines are of similar quality, then it is very difficult to choose the appropriate machine. For the sake of taking the best decision in this case, let the characteristics in terms of parameter be E= {big, very big, small, costly, very costly, moderate}. Suppose that due to budgetary constraint and limited space, the administrator wishes to select an imaging machine from the view point of its possible 'size' and 'cost'. Suppose there are four machines made by different manufacturers with almost the same quality as the universe, $U = \{x_1, x_2, x_3, x_4\}$. These four machines are chosen for the parameter big, very big and moderate. The 'cost' of these four machines are considered for the parameters costly and moderate. Since the data provided are not crisp but possibility intuitionistic fuzzy soft, it is problematic to pick the appropriate machine as usual. The decision to pick the best choice will be made by employing the 'similarity measurement method'.

For this particular problem let the possibility intuitionistic fuzzy soft set (F_p, A) which represents the size of the machines be given as

$$F_p(\text{big}) = \left\{\left(\frac{x_1}{(0,0.1)}, 0.1\right), \left(\frac{x_2}{(0.2,0.6)}, 0.4\right), \left(\frac{x_3}{(0.3,0)}, 0.2\right)\right\},$$
$$F_p(\text{very big}) = \left\{\left(\frac{x_1}{(0,0.1)}, 0.2\right), \left(\frac{x_2}{(0.1,0.3)}, 0.6\right), \left(\frac{x_3}{(0.8,0)}, 0.1\right)\right\},$$
$$F_p(\text{moderate}) = \left\{\left(\frac{x_1}{(0.3,0.1)}, 0.4\right), \left(\frac{x_2}{(0,0.5)}, 0.1\right), \left(\frac{x_3}{(0.5,0)}, 0.2\right)\right\}.$$

Also let the cost of the machines represented by the possibility intuitionistic fuzzy soft set (G_q, B) be given

$$G_q(\text{moderate}) = \left\{\left(\frac{x_1}{(0.1,0.6)}, 0.1\right), \left(\frac{x_2}{(0.3,0.2)}, 0.3\right), \left(\frac{x_3}{(0.8,0)}, 0.4\right)\right\},$$

$$G_q(\text{costly}) = \left\{ \left(\frac{x_1}{(0,0)}, 0.3 \right), \left(\frac{x_2}{(0.2,0.4)}, 0.2 \right), \left(\frac{x_3}{(0.6,0)}, 0.1 \right) \right\}.$$

The following algorithm may be followed by a hospital administrator to pick the best choice of the machines.

Algorithm

a. Input the possibility intuitionistic fuzzy soft set $(F_p, A), (G_q, B)$.

b. Compute $c_1 = \left[(F_p, A) \wedge (G_q, B) \right]$.

c. Compute $c_2 = \Diamond\left[(F_p, A) \wedge (G_q, B) \right]$.

d. Compute the similarity $S(S, F_i)$ between the two PIFSSs c_1 and c_2.

e. The optimal decision is to select $S(S, F_k)$ if $S(S, F_k) \geq S(S, F_i), \forall i$, where F_k is the criteria value x_k.

f. If $S(S, F_k)$ has more than one value then any one of $S(S, F_k)$ may be chosen.

Table-1. Tabular representation of the possibility intuitionistic fuzzy soft sets c_1.

U	(big, moderate)	(big, costely)	(very big, moderate)	(very big, costely)	(moderate, moderate)	(moderate, costely)
x_1	$((0,0.6),1)$	$((0,0.1),0.1)$	$((0,0.6)0.1)$	$((0,0.1),0.2)$	$((0.1,0.6),0.1)$	$((0,0.1),0.3)$
x_2	$((0.2,0.6),0.3)$	$((0.2,0.6),0.2)$	$((0.1,0.3),0.3)$	$((0.1,0.4),0.2$	$((0,0.5),0.1)$	$((0.5,0),0.1)$
x_3	$((0.3,0),0.2)$	$((0.3,0),0.1)$	$((0.6,0),0.1)$	$((0.6,0)0.1)$	$((0.5,0)0.2)$	$((0.5,0),0.1)$

Table-2. Tabular representation of the a possibility intuitionistic fuzzy soft sets c_2.

U	(big, moderate)	(big, costely)	(very big, moderate)	(very big, costely)	(moderate, moderate)	(moderate, costely)
x_1	$((0.4,0.6),0.1)$	$((0.9,0.1),0.1)$	$((0.4,0.6),0.1$	$((0.9,0.1),0.2)$	$((0.4,0.6),0.1)$	$((0.9,0.1),0.3)$
x_2	$((0.4,0.6),0.3)$	$((0.4,0.6),0.2)$	$((0.7,0.3),0.3)$	$((0.6,0.4),0.2)$	$((0.5,0.5),0.1$	$((1,0),0.1)$
x_3	$((1,0),0.2)$	$((1,0),0.1)$	$((1,0),0.1)$	$((1,0),0.1)$	$((1,0),0.2)$	$((1,0),0.1)$

By similarity measurement of PIFSS we have

$S(S, F_1) = 1.32,$

$S(S, F_2) = 1.21,$

$S(S, F_3) = 1.27.$

Then the a hospital administrator will select the machine with the maximum value. Hence, they will buy machine x_1.

5. Relation on Possibility Intuitionistic Fuzzy Soft Sets

In this section we introduce the definitions of union and intersection of PIFSS, by using the s-norm ($\circ$) operator which is the standard union operator (max) and t-norm (*) operator which is the standard intersection operator (min), to define the relation on PIFSS.

Definition 5.1.

Let F_p and G_q two PIFSS over soft universe (U,E). Union of F_p and G_q, denoted by $F_p \tilde{\bigcup} G_q$, is a PIFSS $H_r : E \to (I \times I)^U \times I^U$ defined by

$H_r(e) = (H(e)(x), r(e)(x))$, where $H(e) \in (I \times I)^U$, $r(e) \in I^U$.

such that $H(e) = (F(e) \circ G(e))$ and $r(e) = (p(e) \circ q(e))$ where $\circ$ is the standard union operator for the intuitionistic fuzzy set and fuzzy soft set respectively.

Definition 5.2.

Let F_p and G_q two PIFSS over soft universe (U,E). *Intersection* of F_p and G_q, denoted by $F_p \tilde{\bigcap} G_q$, is a PIFSS $H_r : E \to (I \times I)^U \times I^U$ defined by

$H_r(e) = (H(e)(x), r(e)(x))$, where $H(e) \in (I \times I)^U$, $r(e) \in I^U$.

such that $H(e) = (F(e) * G(e))$ and $r(e) = (p(e) * q(e))$ Where * is the standard intersection operator for the intuitionistic fuzzy set and fuzzy soft set respectively.

Proposition 5.3.

Let F_p, G_q and H_r be any PIFSSs over (U,E). Then the following results hold:

i. $F_p \tilde{\bigcup} G_q = G_p \tilde{\bigcup} F_q$,

ii. $F_p \tilde{\bigcap} G_q = G_p \tilde{\bigcap} F_q$,

iii. $F_p \tilde{\bigcup} (G_q \tilde{\bigcup} H_r) = (F_p \tilde{\bigcup} G_q) \tilde{\bigcup} H_r$,

iv. $F_p \tilde{\bigcap} (G_q \tilde{\bigcap} H_r) = (F_p \tilde{\bigcap} G_q) \tilde{\bigcap} H_r$.

Proof: The properties of commutatively and associatively of $\tilde{\bigcup}$ and $\tilde{\bigcap}$ trivially follows from the definitions.

Proposition 5.4.

Let F_p be a PIFSS over (U,E). Then the following results hold:

i. $F_p \tilde{\bigcup} F_p = F_p$,

ii. $F_p \tilde{\bigcap} F_p = F_p$,

iii. $F_p \tilde{\bigcup} A_1 = A_1$,

iv. $F_p \tilde{\bigcap} A_1 = F_p$,

i. $F_\mu \underset{\sim}{\bigcup} \varnothing_0 = F_\mu$,

ii. $F_\mu \tilde{\bigcap} \varnothing_0 = \varnothing_0$.

Proof: The proof is straightforward by using the definitions of union and intersection.

Proposition 5.5. Let F_p, G_q and H_r be PIFSSs over (U,E). Then the following results hold:

i. $F_p \tilde{\bigcup} (G_q \tilde{\bigcap} H_r) = (F_p \tilde{\bigcup} G_q) \tilde{\bigcap} (F_p \tilde{\bigcup} H_r)$,

ii. $F_p \tilde{\bigcap} (G_q \tilde{\bigcup} H_r) = (F_p \tilde{\bigcap} G_q) \tilde{\bigcup} (F_p \tilde{\bigcap} H_q)$,

Proof: Since the standard $\tilde{\bigcup}$ (max/union) and $\tilde{\bigcap}$ (min/intersection) operators as defined in 5.1and 5.2 are commutative and associative, the above properties hold good, when these standard operators are used for s-norm and t-norm.

Note: - If ($\circ$) and (*) are different from the standard union and intersection operators respectively, the above rules may not hold good.

Definition 5.6.

A possibility intuitionistic fuzzy soft relation (PIFSR) $\tilde{R}_t$ between two PIFSS F_p and G_q over soft universes (U,E) and (U,F) respectively, is defined as $R_t(e,f) = F_p(e) \tilde{\bigcap} G_q(f) \quad \forall e \in E,\ f \in F,$ where $R_t : K \to (I \times I)^U \times I^U$ is a PIFSS over a soft universe (U,K) , $K \subseteq E \times F.$

Example 5.7.

Let $U = \{x_1, x_2, x_3\}$ be a set of universe Let $E = \{e_1, e_2, e_3\}$ be a set of parameters and let $p : E \to I^U$. Define a function $F_p : E \to (I \times I)^U \times I^U$ as follows:

$$F_p(e_1) = \left\{ \left(\frac{x_1}{(0.2, 0.1)}, 0.4 \right), \left(\frac{x_2}{(0.6, 0.3)}, 0.5 \right), \left(\frac{x_3}{(0.5, 0.4)}, 0.6 \right) \right\},$$

$$F_p(e_2) = \left\{ \left(\frac{x_1}{(0.7, 0.2)}, 0.5 \right), \left(\frac{x_2}{(0.6, 0.4)}, 0.6 \right), \left(\frac{x_3}{(0.8, 0.2)}, 0.6 \right) \right\},$$

$$F_p(e_3) = \left\{ \left(\frac{x_1}{(0, 0.7)}, 0.1 \right), \left(\frac{x_2}{(0.5, 0.1)}, 0.3 \right), \left(\frac{x_3}{(0.3, 0.3)}, 0.1 \right) \right\}.$$

Let $G_q : F \to (I \times I)^U \times I^U$ be another PIFSS over (U,F) defined as follows:

$$G_q(f_1) = \left\{ \left(\frac{x_1}{(0.3, 0)}, 0.6 \right), \left(\frac{x_2}{(0.7, 0.2)}, 0.6 \right), \left(\frac{x_3}{(0.6, 0.3)}, 0.7 \right) \right\},$$

$$G_q(f_2) = \left\{ \left(\frac{x_1}{(0.8, 0.1)}, 0.6 \right), \left(\frac{x_2}{(0.7, 0.3)}, 0.7 \right), \left(\frac{x_3}{(0.9, 0.1)}, 0.8 \right) \right\},$$

$$G_q(f_3) = \left\{ \left(\frac{x_1}{(0.1, 0.6)}, 0.2 \right), \left(\frac{x_2}{(0.6, 0)}, 0.5 \right), \left(\frac{x_3}{(0.5, 0.1)}, 0.2 \right) \right\}.$$

Now we take $K = \{(e_1, f_1), (e_1, f_2), (e_2, f_1), (e_2, f_2), (e_2, f_3), (e_3, f_1)\}$. Then RPIFSSs between F_p and G_q is

$$R_t = \{R_t(e_1, f_1) = \left\{\left(\frac{x_1}{(0.2, 0.1)}, 0.4\right), \left(\frac{x_2}{(0.6, 0.3)}, 0.5\right), \left(\frac{x_3}{(0.5, 0.4)}, 0.6\right)\right\},$$

$$R_t(e_1, f_2) = \left\{\left(\frac{x_1}{(0.2, 0.1)}, 0.4\right), \left(\frac{x_2}{(0.6, 0.3)}, 0.5\right), \left(\frac{x_3}{(0.5, 0.4)}, 0.6\right)\right\},$$

$$R_t(e_2, f_1) = \left\{\left(\frac{x_1}{(0.3, 0.2)}, 0.5\right), \left(\frac{x_2}{(0.6, 0.4)}, 0.6\right), \left(\frac{x_3}{(0.6, 0.3)}, 0.6\right)\right\},$$

$$R_t(e_2, f_2) = \left\{\left(\frac{x_1}{(0.7, 0.2)}, 0.5\right), \left(\frac{x_2}{(0.6, 0.4)}, 0.6\right), \left(\frac{x_3}{(0.6, 0.3)}, 0.6\right)\right\},$$

$$R_t(e_2, f_3) = \left\{\left(\frac{x_1}{(0.1, 0.6)}, 0.2\right), \left(\frac{x_2}{(0.6, 0.4)}, 0.5\right), \left(\frac{x_3}{(0.5, 0.2)}, 0.2\right)\right\},$$

$$R_t(e_3, f_1) = \left\{\left(\frac{x_1}{(0, 0.7)}, 0.1\right), \left(\frac{x_2}{(0.5, 0.1)}, 0.3\right), \left(\frac{x_3}{(0.3, 0.3)}, 0.1\right)\right\}\}.$$

Definition 5.8.

Let R_{1_t} be a PIFSR between F_p and G_q over soft universes (U,E) and (U,K) respectively. Let R_{2_t} be a PIFSR between G_q and L_r over soft universes (U,K) and (U,L) respectively. Then composition of R_{t_1} and R_{t_2} is defined as:

$$R_{1_t} o R_{2_t}(e,l) = R_{1_t}(e,k) \tilde{\cap} R_{2_t}(k,l)$$

where

$$R_{1_t} : E \times K \to (I \times I)^U \times I^U,\ R_{2_t} : K \times L \to (I \times I)^U \times I^U.$$

Definition 5.9.

Let R_{1_t} and R_{2_t} be two PIFSRs between F_p and G_q over soft universe (U,E) and (U,F) respectively. Then the standard operations of union, intersection, inverse and containment are defined as follows:

i. Union: $(R_{1_t} \cup R_{2_t})(e,f) = \max(R_{1_t}(e,f), R_{2_t}(e,f)),\ \forall (e,f) \in E \times F,$

ii. Intersection: $(R_{1_t} \cap R_{2_t})(e,f) = \min(R_{1_t}(e,f), R_{2_t}(e,f)),\ \forall (e,f) \in E \times F,$

iii. Inverse : $R^{-1}{}_{1_t}(e,f) = R_{1_t}(f,e),\quad \forall (f,e) \in F \times E.$

iv. Containment: $R_{1_t} \subseteq R_{2_t} \Rightarrow R_{1_t}(e,f) \le R_{2_t}(e,f).$

Definition 5.10.

Let R_{t_1} and R_{t_2} be two PIFSRs between F_p and G_q over soft universe (U,E) **and** (U,F) respectively. Then the following relationships are true:

i. $\left(R_{1_t}{}^{-1}\right)^{-1} = R_{1_t},$

ii. IF $R_{1_t} \subseteq R_{1_t}$ then $R_{1_t}{}^{-1} \subseteq R_{2_t}{}^{-1}.$

Proof .i. $\left(R_{t_1}{}^{-1}\right)^{-1}(e,f) = R_{t_1}{}^{-1}(f,e) = R_{t_1}(e,f).$ Hence $\left(R_{t_1}{}^{-1}\right)^{-1} = R_{t_1}.$

ii. Since we have $R_{1_t} \subseteq R_{2_t} \Rightarrow R_{1_t}(e,f) \le R_{2_t}(e,f)$ then

$$R_{1_t}(e,f) \le R_{2_t}(e,f) \Rightarrow R_{1_t}{}^{-1}(e,f) \le R_{2_t}{}^{-1}(e,f) \Rightarrow R_{1_t}{}^{-1} \subseteq R_{2_t}{}^{-1}.$$

Theorem 5.11.

Let R_{1_t} be a PIFSR between F_p and G_q and R_{2_t}, R_{3_t} be PIFSR between G_q and L_r. Then

i. $R_{1_t} o\left(R_{2_t} \cup R_{3_t}\right) = \left(R_{1_t} oR_{2_t}\right) \cup \left(R_{1_t} oR_{3_t}\right),$

ii. $R_{1_t} o\left(R_{2_t} \cap R_{3_t}\right) = \left(R_{1_t} oR_{2_t}\right) \cap \left(R_{1_t} oR_{3_t}\right),$

iii. $\left(R_{1_t} oR_{2_t}\right)^{-1} = R^{-1}{}_{2_t} oR^{1}{}_{1_t}.$

Proof i.

Suppose $e \in E,\ f \in F,\ l \in L$ then

$$\begin{aligned}
R_{1_t} o\left(R_{2_t} \cup R_{3_t}\right) &= R_{1_t}(e, f) o\left(R_{2_t}(f,l) \cup R_{3_t}(f,l)\right) \\
&= R_{1_t}(e, f) \tilde{\cap} \max\left\{R_{2_t}(f,l), R_{3_t}(f,l)\right\} \\
&= \max\left\{\left(R_{1_t}(e, f) \tilde{\cap} R_{2_t}(f,l)\right), \left(R_{1_t}(e, f) \tilde{\cap} R_{3_t}(f,l)\right)\right\} \\
&= \max\left\{R_{1_t} oR_{2_t}(e,l), R_{1_1} oR_{3_t}(e,l)\right\} \\
&= \left(R_{1_t} oR_{2_t}\right)(e,l) \cup \left(R_{1_t} oR_{3_t}\right)(e,l).
\end{aligned}$$

Therefore $R_{1_t} o\left(R_{2_t} \cup R_{3_t}\right) = \left(R_{1_t} oR_{2_t}\right) \cup \left(R_{1_t} oR_{3_t}\right).$

ii. Following the same reasoning of (i) it is proved that

$$R_{1_t} o\left(R_{2_t} \cap R_{3_t}\right) = \left(R_{1_t} oR_{2_t}\right) \cap \left(R_{1_t} oR_{3_t}\right),$$

iii. Suppose $e \in E,\ f \in F$ then we have

$$\left(R_{1_t} oR_{2_t}\right)^{-1}(e,l) = \left(R_{1_t} oR_{2_t}\right)(l,e) = R_{1_t}(l,k) \tilde{\cap} R_{2_t}(k,e) = R_{2_t}(k,e) \tilde{\cap} R_{1_t}(l,k) =$$
$$R_{2_t}{}^{-1}(e,k) \tilde{\cap} R_{1_t}{}^{-1}(k,l) = \left(R_{1_t}{}^{-1} oR_{2_t}{}^{-1}\right)(e,l).$$

Therefore $\left(R_1 oR_2\right)^{-1} = R^{-1}{}_2 oR^{1}{}_1.$

6. Mapping on Possibility Intuitionistic Fuzzy Soft Classes

In this section we generalize the concept of mapping on possibility intuitionistic fuzzy soft set classes and study some properties of possibility intuitionistic fuzzy soft images and inverse images, some examples for these concepts are also given.

Definition 6.1.

Let X be a universe and E be a set of parameters. Then the collection of all possibility intuitionistic fuzzy soft sets over X , with parameters from E is called an possibility intuitionistic fuzzy soft class and is denoted by (X, E).

Definition 6.2.

Let (X, E) and (Y, E') be possibility intuitionistic fuzzy soft classes. Let $r: X \to Y$ and $s: E \to E'$ be mappings. Then a mapping $f:(X, E) \to (Y, E')$ is defined as follows: for possibility intuitionistic fuzzy soft set (F_p, A) in (X, E), $f(F_p, A)$ is a possibility intuitionistic fuzzy soft set in (Y, E') obtained as follows :

$$f(F_p,A)(\beta)(y)=\begin{cases}\left(\underset{x\in r^{-1}(y)}{\vee}(\underset{\alpha}{\vee}F(\alpha)),\ \underset{x\in r^{-1}(y)}{\vee}\left(\vee p(\alpha)\right)\right) & \text{if } r^{-1}(y)\neq\phi,\ s^{-1}(\beta)\cap A\neq\phi\\ \left((0,0),0\right) & ,\quad \text{otherwise}\end{cases}$$

For $\beta\in s(E)\subseteq E'$, $y\in Y$and $\forall\alpha\in s^{-1}(\beta)\cap A$, $f\left(F_p,A\right)$ is called possibility intuitionistic fuzzysoft image for the possibility intuitionistic fuzzy soft set $\left(F_p,A\right)$.

Definition 6.3.

Let (X,E) and (Y,E') be possibility intuitionistic fuzzy soft classes. Let $r:X\to Y$ and $s:E\to E'$ be mappings. Then a mapping $f^{-1}:(Y,E')\to(X,E)$ is defined as follows: for possibility intuitionistic fuzzy soft set (G_q,B) in (Y,E'), $f^{-1}(G_q,B)$ is an possibility intuitionistic fuzzy soft set in (X,E) obtained as follows for $\alpha\in s^{-1}(B)\subseteq E$, and $x\in X$

$$f^{-1}(G_q,B)(\alpha)(x)=\begin{cases}G(s(\alpha))(r(x)),q(s(\alpha))(r(x)) & \text{for } s(\alpha)\in B\\ \left((0,0),0\right) & ,\quad \text{otherwise}\end{cases}$$

$f^{-1}(G_q,B)$ is called possibility intuitionistic fuzzy soft inverse image of possibility intuitionistic fuzzy soft set (G_q,B).

Example 6.4

Let $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $E=\{e_1,e_2,e_3\}$, $E'=\{e'_1,e'_2,e'_3\}$ and (X,E), (Y,E'), classes of an possibility intuitionistic fuzzy soft sets. Let $r:X\to Y$ and $s:E\to E'$ be mappings defined as :
$r(x_1)=y_2$, $r(x_2)=y_1$, $r(x_3)=y_3$, $s(e_1)=e'_3$, $s(e_2)=e'_1$ and $s(e_3)=e'_1$.
Let (F_p,A) and (G_q,B) be two possibility intuitionistic fuzzy soft sets over X and Y respectively such that

$$\begin{aligned}(F_p,A)=\Bigg\{&\left(e_1,\left\{\left(\frac{x_1}{(0.5,0)},0.4\right),\left(\frac{x_2}{(0.1,0.2)},0.2\right),\left(\frac{x_3}{(0.3,0.1)},0\right)\right\}\right),\\ &\left(e_2,\left\{\left(\frac{x_1}{(0,0.4)},0.1\right),\left(\frac{x_2}{(0.9,0.1)},0.3\right),\left(\frac{x_3}{(0.3,0.2)},0.2\right)\right\}\right),\\ &\left(e_3,\left\{\left(\frac{x_1}{(0.6,0.2)},0.2\right),\left(\frac{x_2}{(0.5,0.3)},0.4\right),\left(\frac{x_3}{(0.8,0.1)},0.3\right)\right\}\right)\Bigg\}.\end{aligned}$$

$$\begin{aligned}(G_q,B)=\Bigg\{&\left(e'_1,\left\{\left(\frac{y_1}{(0.8,0)},0.3\right),\left(\frac{y_2}{(0.3,0.1)},0.1\right),\left(\frac{y_3}{(0,0.4)},0\right)\right\}\right),\\ &\left(e'_2,\left\{\left(\frac{y_1}{(0.6,0.2)},0.4\right),\left(\frac{y_2}{(0,0)},0.3\right),\left(\frac{y_3}{(0.5,0)},0.5\right)\right\}\right),\\ &\left(e'_3,\left\{\left(\frac{y_1}{(0.8,0.1)},0.2\right),\left(\frac{y_2}{(0.6,0.1)},0.6\right),\left(\frac{y_3}{(0.9,0)}0.4\right)\right\}\right)\Bigg\}.\end{aligned}$$

Then we can define a mapping $f:(X,E)\to(Y,E')$ as follows: for possibility intuitionistic fuzzy soft set (F_p,A) in (X,E), $f(F_p,A)$ is possibility intuitionistic fuzzy soft set in (Y,E') is obtained as

$$f(F_p,A)(e'_1)(y_1)=\left(\underset{x\in r^{-1}(y_1)}{\vee}\left(\underset{\alpha}{\vee}F(\alpha)\right),\underset{x\in r^{-1}(y_1)}{\vee}\left(\underset{\alpha}{\vee}p(\alpha)\right)\right)$$
$$=\left(\underset{x\in\{x_2\}}{\vee}\left(\underset{\alpha\in\{e_2,e_3\}}{\vee}F(\alpha)\right),\underset{x\in\{x_2\}}{\vee}\left(\underset{\alpha\in\{e_2,e_3\}}{\vee}p(\alpha)\right)\right)$$
$$=\left(\underset{x\in\{x_2\}}{\vee}\left(F(e_2)\vee F(e_3)\right)(x),\underset{x\in\{x_2\}}{\vee}\left(p(e_2)\vee p(e_3)\right)(x)\right)$$
$$=\left(\underset{x\in\{x_2\}}{\vee}\left\{\frac{x_1}{(0.6,0.2)},\frac{x_2}{(0.9,0.1)},\frac{x_3}{(0.8,0.1)}\right\},\underset{x\in\{x_2\}}{\vee}\{0.2,0.4,0.3\}(x)\right)$$
$$=\left((0.9,0.1),0.4\right).$$

$$f(F_p,A)(e'_1)(y_2)=\left(\underset{x\in r^{-1}(y_2)}{\vee}\left(\underset{\alpha}{\vee}F(\alpha)\right),\underset{x\in r^{-1}(y_2)}{\vee}\left(\underset{\alpha}{\vee}p(\alpha)\right)\right)$$
$$=\left(\underset{x\in\{x_1\}}{\vee}\left(\underset{\alpha\in\{e_2,e_3\}}{\vee}F(\alpha)\right),\underset{x\in\{x_1\}}{\vee}\left(\underset{\alpha\in\{e_2,e_3\}}{\vee}p(\alpha)\right)\right)$$
$$=\left(\underset{x\in\{x_1\}}{\vee}\left(F(e_2)\vee F(e_3)\right)(x),\underset{x\in\{x_1\}}{\vee}\left(p(e_2)\vee p(e_3)\right)(x)\right)$$
$$=\left(\underset{x\in\{x_1\}}{\vee}\left\{\frac{x_1}{(0.6,0.2)},\frac{x_2}{(0.9,0.1)},\frac{x_3}{(0.8,0.1)}\right\},\underset{x\in\{x_1\}}{\vee}\{0.2,0.4,0.3\}(x)\right)$$
$$=\left((0.6,0.2),0.2\right).$$

$$f(F_p,A)(e'_1)(y_3)=\left(\underset{x\in r^{-1}(y_3)}{\vee}\left(\underset{\alpha}{\vee}F(\alpha)\right),\underset{x\in r^{-1}(y_3)}{\vee}\left(\underset{\alpha}{\vee}p(\alpha)\right)\right)$$
$$=\left(\underset{x\in\{x_3\}}{\vee}\left(\underset{\alpha\in\{e_2,e_3\}}{\vee}F(\alpha)\right),\underset{x\in\{x_3\}}{\vee}\left(\underset{\alpha\in\{e_2,e_3\}}{\vee}p(\alpha)\right)\right)$$
$$=\left(\underset{x\in\{x_3\}}{\vee}\left(F(e_2)\vee F(e_3)\right)(x),\underset{x\in\{x_3\}}{\vee}\left(p(e_2)\vee p(e_3)\right)(x)\right)$$
$$=\left(\underset{x\in\{x_3\}}{\vee}\left\{\frac{x_1}{(0.6,0.2)},\frac{x_2}{(0.9,0.1)},\frac{x_3}{(0.8,0.1)}\right\},\underset{x\in\{x_3\}}{\vee}\{0.2,0.4,0.3\}(x)\right)$$
$$=\left((0.8,0.1),0.3\right)$$

By similar calculations, consequently, we get

$$(f(F_p,A),B)=\left\{\left(e'_1=\left\{\frac{y_1}{((0.9,0.1),0.4)},\frac{y_2}{((0.6,0.2),0.2)},\frac{y_3}{((0.8,0.1),0.3)}\right\}\right)\right.,$$
$$\left(e'_2=\left\{\frac{y_1}{((0,0),0)},\frac{y_2}{((0,0),0)},\frac{y_3}{((0,0),0)}\right\}\right),$$
$$\left.\left(e'_3=\left\{\frac{y_1}{((0.1,0.2),0.2)},\frac{y_2}{((0.5,0),0.4)},\frac{y_3}{((0.3,0.1),0)}\right\}\right)\right\}.$$

Next for a possibility intuitionistic fuzzy soft inverse images, the mapping $f^{-1}:(Y,E')\to(X,E)$ is defined as follows: for a possibility intuitionistic fuzzy soft set (G_q,B) in (Y,E'), $f^{-1}(G_q,B)$ is a possibility intuitionistic fuzzy soft set in (X,E) obtained as follows:

$$f^{-1}(G_q,B)(e_1)(x_1)=\left(G\left(s(e_1)\right)\left(r(x_1)\right),q\left(s(e_1)\right)\left(r(x_1)\right)\right)$$
$$=\left(G(e'_3)(y_2),q(e'_3)(y_2)\right)$$
$$=\left((0.6,0.1),0.6\right)$$

$$f^{-1}(G_q,B)(e_1)(x_2)=\left(G\left(s(e_1)\right)\left(r(x_2)\right),q\left(s(e_1)\right)\left(r(x_2)\right)\right)$$
$$=\left(G(e'_3)(y_1),q(e'_3)(y_1)\right)$$
$$=\left((0.8,0.1),0.2\right)$$

$$f^{-1}(G_q,B)(e_1)(x_3)=\left(G\left(s(e_1)\right)\left(r(x_3)\right),q\left(s(e_1)\right)\left(r(x_3)\right)\right)$$
$$=\left(G(e'_3)(y_3),q(e'_3)(y_3)\right)$$
$$=\left((0.9,0),0.4\right)$$

By similar calculations, consequently, we get

$$(f^{-1}(G_q,B),A)=\left\{\left(e_1=\left\{\frac{x_1}{((0.6,0.1),0.6)},\frac{x_2}{((0.8,0.1),0.2)},\frac{x_3}{((0.9,0),0.4)}\right\}\right),\right.$$
$$\left(e_2=\left\{\frac{x_1}{((0.3,0.1),0.1)},\frac{x_2}{((0.8,0),0.3)},\frac{x_3}{((0,0.4),0)}\right\}\right),$$
$$\left.\left(e_3=\left\{\frac{x_1}{((0.3,0.1),0.1)},\frac{x_2}{((0.8,0),0.3)},\frac{x_3}{((0,0.4),0)}\right\}\right)\right\}.$$

Definition 6.5.

Let $f:(X,E)\to(Y,E')$ be mapping and (F_p,A) and (G_q,B) intuitionistic fuzzy soft sets in (X,E). Then for $\beta\in E'$, $y\in Y$, the union and intersection of possibility intuitionistic fuzzy soft images $f(F_p,A)$ and $f(G_q,B)$ in (Y,E') are defined as follows:

$$(f((F_p,A)\vee f(G_q,B)))(\beta)(y)=f(F_p,A)(\beta)(y)\vee f(G_q,B)(\beta)(y).$$
$$(f((F_p,A)\wedge f(G_q,B)))(\beta)(y)=f(F_p,A)(\beta)(y)\wedge f(G_q,B)(\beta)(y).$$

Definition 6.6.

Let $f:(X,E)\to(Y,E')$ be a mapping and (F_p,A) and (G_q,B) possibility intuitionistic fuzzy soft sets in (Y,E'). Then for $\alpha\in E, x\in X$, the union and intersection of possibility intuitionistic fuzzy soft inverse images $f^{-1}(F_p,A)$ and $f^{-1}(G_q,B)$ in (X,E) are define as follows:

$$(f^{-1}(F_p,A)\vee f^{-1}(G_q,B))(\alpha)(x)=f^{-1}(F_p,A)(\alpha)(x)\vee f^{-1}(G_q,B)(\alpha)(x).$$
$$(f^{-1}(F_p,A)\wedge f^{-1}(G_q,B))(\alpha)(x)=f^{-1}(F_p,A)(\alpha)(x)\wedge f^{-1}(G_q,B)(\alpha)(x).$$

Theorem. 6.7.

Let $f:(X,E)\to(Y,E')$ be a mapping. Then for possibility intuitionistic fuzzy soft sets (F_p,A) and (G_q,B) in the possibility intuitionistic fuzzy soft class (X,E), we have

i. $f(\varnothing)=\varnothing,$

ii. $f(X)\subseteq Y,$

iii. $f((F_p,A)\vee(G_q,B))=f(F_p,A)\vee f(G_q,B),$

iv. $f((F_p,A)\wedge(G_q,B))\subseteq f(F_p,A)\wedge f(G_q,B),$

v. If $(F_p,A)\subseteq(G_q,B)$, then $f(F_p,A)\subseteq f(G_q,B).$

Proof. For (i) and (ii) the proof is trivial, so we just give the proof of (iii), (iv) and (v). By using the Atanassov union which is the basic intuitionistic fuzzy union we have

(iii) For $\beta\in E'$ and $y\in Y$, we want to prove that

$$f((F_p,A)\vee(G_q,B))(\beta)(y)=f(F_p,A)(\beta)(y)\vee f(G_q,B)(\beta)(y).$$

For left hand side, consider $f\left((F_p,A)\vee f(G_q,B)\right)(\beta)(y)=f(H_w,A\cup B)(\beta)(y).$

Then

$$f(H_w,A\cup B)(\beta)(y)=\begin{cases}\left(\underset{x\in r^{-1}(y)}{\vee}\left(\underset{\alpha}{\vee}H(\alpha)\right),\ \underset{x\in r^{-1}(y)}{\vee}\left(\underset{\alpha}{\vee}w(\alpha)\right)\right)(x), \\ \text{if } r^{-1}(y)\neq\phi, s^{-1}(\beta)\cap(A\cup B)\neq\varnothing. \\ ((0,0),0) \qquad \text{otherwise.}\end{cases}$$

Where $H(\alpha)=\underset{Atan}{\cup}\left(F(\alpha),G(\alpha)\right)$ and $w(\alpha)=\max\left(p(\alpha),q(\alpha)\right)$. For $\alpha\in(A\cup B)\cap s^{-1}(\beta)$.

Such that $F(\alpha)=\left(\mu(\alpha),\gamma(\alpha)\right), G(\alpha)=\left(\theta(\alpha),\varphi(\alpha)\right)$ then

$$\begin{aligned}H(\alpha)&=\underset{Atan}{\cup}\left(F(\alpha),G(\alpha)\right)\\&=\left\{\max\left(\mu(\alpha),\theta(\alpha)\right),\min\left(\gamma(\alpha),\varphi(\alpha)\right)\right\}\end{aligned}$$

Considering only the non-trivial case, we have

$$f(H_w,A\cup B)(\beta)(y)=\begin{pmatrix}\underset{x\in r^{-1}(y)}{\vee}\left(\vee\left\{\max\left(\mu(\alpha),\theta(\alpha)\right),\min\left(\gamma(\alpha),\varphi(\alpha)\right)\right\}\right), \\ \underset{x\in r^{-1}(y)}{\vee}\left(\vee\max\ \left(p(\alpha),q(\alpha)\right)\right)\end{pmatrix}. \text{(I)}$$

For right hand side and by using Definition 6.5,We have

$$
\begin{aligned}
(f(F_p,A)\vee f(G_q,B))(\beta)(y) &= f(F_p,A)(\beta)(y)\vee f(G_q,B)(\beta)(y)\\
&=\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}F(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}\mu(\alpha)\right)\right)\vee\\
&\quad\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}G(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}\mu(\alpha)\right)\right)\\
&=\left(\bigvee_{x\in r^{-1}(y)}\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}\left(F(\alpha)\vee G(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cup B)}\left(p(\alpha)\vee q(\alpha)\right)\right)\\
&=\begin{pmatrix}\bigvee_{x\in r^{-1}(y)}\left(\vee\left\{\max\left(\mu(\alpha),\theta(\alpha)\right),\min\left(\gamma(\alpha),\varphi(\alpha)\right)\right\}\right),\\ \bigvee_{x\in r^{-1}(y)}\left(\vee\max\left(p(\alpha),q(\alpha)\right)\right)\end{pmatrix}\qquad \text{(II)}
\end{aligned}
$$

By (I) and (II), we get (iii).

Remark 6.8.
The Atanassove union can be replaced by any *S-norm* which is a general intuitionistic fuzzy union (see Fathi [16]).

(iv) For $\beta\in E'$ and $y\in Y$, and using Definition 6.5, we have

$$
\begin{aligned}
f\left((F_p,A)\wedge(G_q,B)\right)(\beta)(y) &= f(H_w,A\cap B)(\beta)(y)\\
&=\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cap B)}H(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cap B)}w(\alpha)\right)\right)\\
&=\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cap B)}F(\alpha)\wedge G(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cap B)}p(\alpha)\wedge q(\alpha)\right)\right)\\
&=\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cap B)}F(\alpha)\wedge G(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap(A\cap B)}p(\alpha)\wedge q(\alpha)\right)\right)\\
&\subseteq\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}F(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}p(\alpha)\right)\right)\wedge\\
&\quad\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}G(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap B}q(\alpha)\right)\right)\\
&=\left(f(F_p,A)\wedge f(G_q,B)\right)(\beta)(y)
\end{aligned}
$$

This gives (iv).

(v) To prove (v) considering only the non-trivial case, for $\beta\in E'$ and $y\in Y$, and since we have $(F,A)\subseteq(G,B)$. Then

$$
\begin{aligned}
f((F_p,A))\beta(y) &=\left(\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}F(\alpha)\right),\bigvee_{x\in r^{-1}(y)}\left(\bigvee_{\alpha\in s^{-1}(\beta)\cap A}p(\alpha)\right)\right)\\
&=\bigvee_{x\in r^{-1}(y)}\bigvee_{\alpha\in s^{-1}(\beta)\cap A}\left(F(\alpha),p(\alpha)\right)\\
&\subseteq\bigvee_{x\in r^{-1}(y)}\bigvee_{\alpha\in s^{-1}(\beta)\cap B}\left(G(\alpha),q(\alpha)\right)\qquad \text{since } \left(F_p,A\right)\subseteq\left(G_q,B\right)\\
&= f((G_q,B))(\beta)(y).
\end{aligned}
$$

This gives (v).

Remark 6.9.
In the above theorem we can see that the reversal of the inequalities (ii), (iv) and implication (v) cannot be hold, as is shown in the following example.

Example 6.10.

Let $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $E=\{e_1,e_2,e_3\}$ and $E^{'}=\{e^{'}_1,e^{'}_2,e^{'}_3\}$. Suppose that (X,E) and $(Y,E^{'})$ are possibility intuitionistic fuzzy soft classes. Define $r:X\to Y$ and $s:E\to E^{'}$ as follows: $r(x_1)=y_1$, $r(x_2)=y_3$, $r(x_3)=y_3$, $s(e_1)=e^{'}_1$, $s(e_2)=e^{'}_3$, and $s(e_3)=e^{'}_2$.

Let (F_p,A) and (G_q,B) be two possibility intuitionistic fuzzy soft sets over X,Y respectively. Such that

$$(F_p,A)=\left\{\left(e_1,\left\{\left(\frac{x_1}{(0.3,0.1)},0.2\right),\left(\frac{x_2}{(1,0)},0\right),\left(\frac{x_3}{(0,0)},0.3\right)\right\}\right),\right.$$
$$\left(e_2,\left\{\left(\frac{x_1}{(0.5,0)},0.4\right),\left(\frac{x_2}{(0.7,0)},0.7\right),\left(\frac{x_3}{(0,0)}\right),0.6\right\}\right),$$
$$\left.\left(e_3,\left\{\left(\frac{x_1}{(0.9,0.1)},0.1\right),\left(\frac{x_2}{(0.4,0.1)},0.9\right),\left(\frac{x_3}{(1,0)},0.3\right)\right\}\right)\right\}.$$

$$(G_q,B)=\left\{\left(e^{'}_1,\left\{\left(\frac{y_1}{(1,0)},0.2\right),\left(\frac{y_2}{(0.5,0.2)},0\right),\left(\frac{y_3}{(0.6,0.1)},0.5\right)\right\}\right),\right.$$
$$\left(e^{'}_2,\left\{\left(\frac{y_1}{(0.8,0.1)},0.1\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0.2,0.1)},0.6\right)\right\}\right),$$
$$\left.\left(e^{'}_2,\left\{\left(\frac{y_1}{(0.6,0.1)},0.5\right),\left(\frac{y_2}{(0.3,0)},0.2\right),\left(\frac{y_3}{(0,0)},0.1\right)\right\}\right)\right\}.$$

Then calculations give

$$Y\not\subset\left\{\left(e^{'}_1,\left\{\left(\frac{y_1}{(0.3,0.1)},0.2\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(1,0)},0.3\right)\right\}\right),\right.$$
$$\left(e^{'}_2,\left\{\left(\frac{y_1}{(0.9,0.1)},0.1\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(1,0)},0.9\right)\right\}\right),$$
$$\left.\left(e^{'}_3,\left\{\left(\frac{y_1}{(0.5,0)},0.4\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0.7,0)},0.7\right)\right\}\right)\right\}.$$

Since Y is the collection of all possibility intuitionistic fuzzy soft sets over Y with the parameters from $E^{'}$.

For (iv) and (v),we define mappings $r:X\to Y$ and $s:E\to E^{'}$ as follows:

$r(x_1)=y_2$, $r(x_2)=y_2$, $r(x_3)=y_2$, $s(e_1)=e^{'}_3$, $s(e_2)=e^{'}_2$, $s(e_3)=e^{'}_2$, and $s(e_4)=e^{'}_1$.

Let $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $E=\{e_1,e_2,e_3,e_4\}$, $E^{'}=\{e^{'}_1,e^{'}_2,e^{'}_3\}$, and (X,E), $(Y,E^{'})$ classes of possibility intuitionistic fuzzy soft set.

Now we choose two possibility intuitionistic fuzzy soft sets in (X,E) as

$$(F_p,A)=\left\{e_3,\left\{\left(\frac{x_1}{(0.6,0)},0.1\right),\left(\frac{x_2}{(0.9,0)},0.3\right),\left(\frac{x_3}{(0.6,0.3)},0\right)\right\}\right\},$$
$$(G_q,B)=\left\{e_3,\left\{\left(\frac{x_1}{(0.4,0.5)},0.1\right),\left(\frac{x_2}{(0.3,0.5)},0\right),\left(\frac{x_3}{(1,0)},0.3\right)\right\}\right\}.$$

Then calculations gives

$$f((F_p,A))\wedge f((G_q,B))=\left\{\left(e_1^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),\right.$$

$$\left(e_2^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0.9,0)},0.3\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),$$

$$\left.\left(e_3^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right)\right\}.$$

$$\not\subset\left\{\left(e_1^{'}=\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),\right.$$

$$\left(e_2^{'}=\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0.6,0.3)},0.1\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),$$

$$\left.\left(e_3^{'}=\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0.6,0.3)},0.1\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right)\right\}.$$

$$= f((F_p,A)\wedge(G_q,B))$$

Also we have

$$f(F_p,A)=\left\{\left(e_1^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),\right.$$

$$\left(e_2^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0.9,0)},0.3\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),$$

$$\left.\left(e_3^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right)\right\}.$$

$$\subseteq\left\{\left(e_1^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),\right.$$

$$\left(e_2^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(1,0)},0.3\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right),$$

$$\left.\left(e_3^{'},\left\{\left(\frac{y_1}{(0,0)},0\right),\left(\frac{y_2}{(0,0)},0\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right)\right\}.$$

$$=f\left(G_q,B\right)$$

But $(F_p,A)\not\subset(G_q,B)$.

Theorem 6.11.

Let $f:(X,E)\to(Y,E^{'})$ be mapping. Then for possibility intuitionistic fuzzy soft sets (F_p,A) and (G_q,B) in the possibility intuitionistic fuzzy soft class $(Y,E^{'})$, we have

i. $f^{-1}(\varnothing)=\varnothing$.

ii. $f^{-1}(Y)=X$

iii. $f^{-1}((F_p, A)\vee(G_q, B)) = f^{-1}(F_p, A)\vee f^{-1}(G_q, B)$.

iv. $f^{-1}((F_p, A)\wedge(G_q, B)) = f^{-1}(F_p, A)\wedge f^{-1}(G_q, B)$.

v. If $(F_p, A)\subseteq(G_q, B)$, then $f^{-1}(F_p, A)\subseteq f^{-1}(G_q, B)$.

Proof. For (i) and (ii) the proof is trivial so we just give the proof of (iii), (iv) and (v).
By using the Atanassov union which is the basic intuitionistic fuzzy union we have

(iii) For $\alpha\in E$ and $x\in X$, we want to show
$f^{-1}\big((F_p, A)\vee(G_q, B)\big)(\alpha)(x) = f^{-1}\big(F_p, A\big)(\beta)(y)\vee f^{-1}\big(G_q, B\big)(\alpha)(x)$. So

$$
\begin{aligned}
f^{-1}\big((F_p, A)\vee(G_q, B)\big)(\alpha)(x) &= f^{-1}\big((H_w, A\cup B)\big)(\alpha)(x)\\
&= \big(H(s(\alpha))(r(x)), w(s(\alpha))(r(x))\big)\ , \quad s(\alpha)\in A\cup B,\ r(x)\in Y\\
&= \big(H(\beta)(r(x)), w(\beta)(r(x))\ \big) \qquad \text{where } \beta = S(\alpha)\\
&= \Big(\bigcup_{Atan}\big(F(\beta), G(\beta)\big), \max\big(p(\alpha), q(\alpha)\big)\Big)\big(r(x)\big)\\
&= \begin{pmatrix}\big\{\max\big(\mu(\alpha),\theta(\alpha)\big), \min\big(\gamma(\alpha),\varphi(\alpha)\big)\big\},\\ \max\big(p(\alpha), q(\alpha)\big)\end{pmatrix}\big(r(x)\big). \text{(I)}
\end{aligned}
$$

For right hand side and by using Definition 6.6, we have

$$
\begin{aligned}
(f^{-1}(F_p, A)\vee f^{-1}(G_q, B))(\alpha)(x) &= f^{-1}(F_p, A)(\alpha)(x)\vee f^{-1}(G_q, B)(\alpha)(x)\\
&= \big(F(s(\alpha))(r(x)), p(s(\alpha))(r(x))\big)\vee\\
&\quad \big(G(s(\alpha))(r(x)), q(s(\alpha))(r(x))\big)\ , \quad \beta = s(\alpha)\in A\cup B\\
&= \big(F(\beta)(r(x)), p(\beta)(r(x))\ \big)\vee\big(G(\beta)(r(x)), q(\beta)(r(x))\ \big),\ \beta = s(\alpha)\\
&= \begin{pmatrix}\big\{\max\big(\mu(\alpha),\theta(\alpha)\big), \min\big(\gamma(\alpha),\varphi(\alpha)\big)\big\},\\ \max\big(p(\alpha), q(\alpha)\big)\end{pmatrix}\big(r(x)\big) \quad \text{(II)}
\end{aligned}
$$

From (I) and (II), we get (iii).

(iv) For $\alpha\in E$, $x\in X$ and using Definition 6.6, we have

$$
\begin{aligned}
f^{-1}((F_p, A)\wedge(G_q, B))(\alpha)(x) &= f^{-1}((H_w, A\cap B))(\alpha)(x)\\
&= \big(H(s(\alpha))(r(x)), w(s(\alpha))(r(x))\big), \qquad s(\alpha)\in A\cap B\\
&= \big(H(\beta)(r(x)), w(\beta)(r(x))\big) \qquad \beta = s(\alpha)\\
&= \big((F(\beta)\wedge G(\beta))(r(x)), (p(\beta)\wedge q(\beta))(r(x))\big)\\
&= \big(F(\beta), p(\beta)\big)(r(x))\wedge\big(G(\beta), q(\beta)\big)(r(x))\\
&= \big(F(s(\alpha))(r(x)), p(s(\alpha))(r(x))\big)\wedge\big(G(s(\alpha))(r(x)), q(s(\alpha))(r(x))\big)\\
&= f^{-1}(F_p, A)(\alpha)(x)\wedge f^{-1}(G_q, B)(\alpha)(x)\\
&= (f^{-1}(F_p, A)\wedge f^{-1}(G_q, B))(\alpha)(x)
\end{aligned}
$$

This gives (iv).
(v) Since we have $(F_p, A)\subseteq(G_q, B)$ then

$$
\begin{aligned}
f^{-1}((F_p,A))(\alpha)(x) &= \big(F(s(\alpha))(r(x)),\, p(s(\alpha))(r(x))\big)\\
&= \big(F(\beta)(r(x)),\, p(\beta)(r(x))\big)\ ,\ s(\alpha)=\beta\\
&\subseteq \big(G(\beta)(r(x)), q(\beta)(r(x))\big)\\
&= \big(G(s(\alpha))(r(x)), q(s(\alpha))(r(x))\big)\\
&= f^{-1}((G_q,B))(\alpha)(x)
\end{aligned}
$$

Then this gives (v).

Remark 6.12.
In the previous theorem we can see that the reversal of an implication (v) cannot be hold, as is shown in the following example.

Example 6.13.
Let $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $E=\{e_1,e_2,e_3,e_4\}$, $E'=\{e'_1,e'_2,e'_3\}$ and (X,E), (Y,E') classes of possibility intuitionistic fuzzy soft sets. To show (v) define mappings $r:X\to Y$, $s:E\to E'$ as follows:

$r(x_1)=y_2$, $r(x_2)=y_2$, $r(x_3)=y_3$, $s(e_1)=e'_1$,
$s(e_2)=e'_2$, $s(e_3)=e'_2$, and $s(e_4)=e'_1$.

We choose two possibility intuitionistic fuzzy soft sets in (Y,E') as:

$$(F_p,A)=\left\{e'_3=\left\{\left(\frac{y_1}{(0.6,0.3)},0.2\right),\left(\frac{y_2}{(0.6,0)},0.4\right),\left(\frac{y_3}{(0,0)},0\right)\right\}\right\},$$

$$(G_q,B)=\left\{e'_3=\left\{\left(\frac{y_1}{(0.8,0.1)},0.3\right),\left(\frac{y_2}{(0.4,0.2)},0\right),\left(\frac{y_3}{(0.5,0.4)},0\right)\right\}\right\}.$$

for a possibility intuitionistic fuzzy soft set (F_p,A) in (Y,E'), $f^{-1}(F_p,A)$ is a possibility intuitionistic fuzzy soft set in (X,E) obtained as follows:

$$
\begin{aligned}
f^{-1}(F_p,A)(e_1)(x_1) &= \big(F\big(s(e_1)\big)\big(r(x_1)\big),\, p\big(s(e_1)\big)\big(r(x_1)\big)\big)\\
&= \big(F(e'_1)(y_2),\, p(e'_1)(y_2)\big)\\
&= \varnothing\\
f^{-1}(F_p,A)(e_1)(x_2) &= \big(F\big(s(e_1)\big)\big(r(x_2)\big),\, p\big(s(e_1)\big)\big(r(x_2)\big)\big)\\
&= \big(F(e'_1)(y_2),\, p(e'_1)(y_2)\big)\\
&= \varnothing\\
f^{-1}(F_p,A)(e_1)(x_3) &= \big(F\big(s(e_1)\big)\big(r(x_3)\big),\, p\big(s(e_1)\big)\big(r(x_3)\big)\big)\\
&= \big(F(e'_1)(y_3),\, p(e'_1)(y_3)\big)\\
&= \varnothing
\end{aligned}
$$

By similar calculations, consequently, we get

$$f^{-1}(F_p,A)=\varnothing.$$

Next for a possibility intuitionistic fuzzy soft set (G_q,B) in (Y,E'), $f^{-1}(G_q,B)$ is a possibility intuitionistic fuzzy soft set in (X,E) obtained as follows:

$$\begin{aligned} f^{-1}(G_q, B)(e_1)(x_1) &= \left(G\left(s(e_1)\right)\left(r(x_1)\right), q\left(s(e_1)\right)\left(r(x_1)\right)\right) \\ &= \left(G(e'_1)(y_2), q(e'_1)(y_2)\right) \\ &= \varnothing \\ f^{-1}(G_q, B)(e_1)(x_2) &= \left(G\left(s(e_1)\right)\left(r(x_2)\right), q\left(s(e_1)\right)\left(r(x_2)\right)\right) \\ &= \left(G(e'_1)(y_2), q(e'_1)(y_2)\right) \\ &= \varnothing \end{aligned}$$

$$\begin{aligned} f^{-1}(G_q, B)(e_1)(x_3) &= \left(G\left(s(e_1)\right)\left(r(x_3)\right), q\left(s(e_1)\right)\left(r(x_3)\right)\right) \\ &= \left(G(e'_1)(y_3), q(e'_1)(y_3)\right) \\ &= \varnothing \end{aligned}$$

By similar calculations, consequently, we get

$$f^{-1}(G_q, B) = \varnothing.$$

This implies $f^{-1}(F_p, A) = \varnothing \subseteq \varnothing = f^{-1}(G_q, B)$, but $(F_p, A) \not\subset (G_q, B)$.

8. Conclusion

We have introduced new operations on intuitionistic fuzzy soft set and some properties of these operations have also been established. A simple example has been presented as an application of this mathematical tool. In this paper, we have defined the notion of a mapping on the classes of fuzzy soft sets which is a pivotal notion for advanced development of any new area of mathematical sciences.We have studied the properties of fuzzy soft images and inverse images which have been supported by examples and counterexamples.We hope these fundamental results will help the researchers to enhance and promote the research on Fuzzy Soft Set Theory.

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