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Oscillation of Second-Order Nonlinear Mixed Neutral Dynamic Equations with Non Positive Neutral Term on Time Scales

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Abstract: In this work, we establish some new oscillation results for the second-order nonlinear mixed neutral dynamic equation

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0,$$

where $z(t)=x(t)-p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t))$. Our results not only complement and generalize some existing results in [4], but also can be applied to some oscillation problems that do not covered before, we also give some examples to illustrate our main results.

Keywords: Oscillation; Mixed neutral dynamic equations; Time scales; Generalized Riccati technique.

Mathematics Subject Classification (2010): 34A21, 34C15, 34K40, 34N05.

1. Introduction

The theory of time scales was introduced by Hilger [1] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. A time scale T is a nonempty closed subset of the real numbers R. When the time scale equals to the real numbers, the obtained results represent the classical theories of differential equations while when time scale equals to the integer numbers, the results represent the theories of difference equations. Many other interesting time scales exist and give arise to many applications. The new theory of the so - called " dynamic equation" not only unify the theories of differential equations and difference equations, but also extends these classical cases to the so - called q- difference equations (when $\mathrm{T}=q^{\mathrm{N}_0}:=\{q^t:t\in\mathrm{N}_0 \text{ for } q>1\}$ or $\mathrm{T}=q^{\bar{\mathrm{Z}}}=q^{\mathrm{Z}}\cup\{0\}$) which have important applications in quantum theory (see [2]). Also, it can be applied to different types of time scales like $\mathrm{T}=h\mathrm{Z},\mathrm{T}=\mathrm{N}_0^2$, and the space of the harmonic numbers $\mathrm{T}=\mathrm{T}_n=\{t_n:n\in\mathrm{N}_0\}$.

For an introduction to time scale calculus and dynamic equations, see Bohner and Peterson books [3, 4]. In recent years, there has been much activities concerning oscillation and nonoscillation of the solution of various equations on time scales. We refer the reader to the papers [5-13] and references cited therein.

In this paper, we deal with oscillation of the second order mixed nonlinear neutral dynamic equation with negative neutral term on time scales

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0, \quad (1.1)$$

where

$$z(t)=x(t)-p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)) \quad (1.2)$$

subject to the following hypotheses:

(H_1) T is an unbounded above time scale and $t_0\in\mathrm{T}$ with $t_0>0$. We define the time scale interval $[t_0,\infty)_{\mathrm{T}}$ by $[t_0,\infty)_{\mathrm{T}}=[t_0,\infty)\bigcap\mathrm{T}$.

(H_2) η_1, τ_1 and $\tau_2:\mathrm{T}\to\mathrm{T}$ are rd-continuous such that $\eta_1(t)\le t$, $\tau_1(t)\le t$, $\tau_2(t)\ge t$, $\lim_{t\to\infty}\tau_1(t)=\infty=\lim_{t\to\infty}\eta_1(t)=\infty$. The function $\eta_2:\mathrm{T}\to\mathrm{T}$ is an injective rd-continuous increasing function such that $\eta_2(t)\ge t$.

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(H_3) p_1, and p_2 are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$ where

$$0 \le p_1(t) \le p_1 < 1.$$

(H_4) r is a positive rd-continuous function such that

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \infty, \qquad (1.3)$$

where γ is a quotient of odd positive integers.

(H_5) $f, g \in C(\mathbb{P} \times \mathbb{T}, \mathbb{P})$ such that $uf(t,u) \ge 0$, $ug(t,u) \ge 0$, $f(t,u) \ge q_1(t)u^{\alpha}$ and $g(t,u) \ge q_2(t)u^{\beta}$ for $u \ne 0$, where q_1 and q_2 are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$. α and β are quotients of odd positive integers.

Through out this paper we assume that

$$\theta(a,b;u) := [\int_u^a \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1}[\int_u^b \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}],\ d_+(t) = \max\{0, d(t)\},\ d_-(t) = \max\{0, -d(t)\},$$

and for sufficiently large t_1

$$\psi(t,t_1) := \theta(t, \tau_1(t); t_1), \quad \phi_i(t,t_1) := \theta(t, \eta_2^{-1}(\tau_i(t)); t_1), \quad i = 1,2,$$

$$A(t) := \begin{cases} b_0^{\alpha-\beta} & \alpha \ge \beta, \\ b_1^{\alpha-\beta}[\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{\alpha-\beta} & \alpha < \beta, \end{cases} \qquad (1.4)$$

$$C(t) := \begin{cases} b_0^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} \ge 1, \\ b_1^{\frac{\beta}{\gamma}-1}[\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} < 1, \end{cases} \qquad (1.5)$$

where b_0 and b_1 are positive constants, $\sigma(t)$ is the forward jump operator which is defined by $\sigma(t) = inf\{s \in \mathbb{T}, s > t\}$.

By a solution of (1.1), we mean a nontrivial real valued function x satisfies (1.1) for $t \in \mathbb{T}$. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory. A nontrivial solution $x(t)$ is said to be almost oscillatory if either $x(t)$ is oscillatory or $x^{\Delta}(t)$ is oscillatory.

We note that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$ and (1.1) becomes the second-order nonlinear mixed neutral differential equation

$$(r(t)(z'(t))^{\gamma})' + f(t, x(\tau_1(t))) + g(t, x(\tau_2(t))) = 0, \quad t \in [t_o, \infty). \quad (1.6)$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ and (1.1) becomes the second-order nonlinear mixed neutral difference equation

$$\Delta(r(t)(\Delta z(t))^{\gamma}) + f(t, x(\tau_1(t))) + g(t, x(\tau_2(t))) = 0, \quad t \in [t_o, \infty). \quad (1.7)$$

As a special case of (1.1), L. Erbe et al. [4] considered the second-order nonlinear functional dynamic equation

$$(r(t)[(x(t) - p(t)x(\eta(t)))^{\Delta}]^{\gamma})^{\Delta} + f(t, x(g(t))) = 0, \quad (1.8)$$

where $\eta(t) \le t$ and either $g(t) \ge t$ or $g(t) < t$ and proved that for all sufficiently large T_* and $T > T_*$ if

$$\limsup_{t\to\infty}\int_T^t [M(s,T_*)\rho(s)q(s)-\frac{r(s)(\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(s)}]\Delta s=\infty, \quad (1.9)$$

where $\rho(s)$ is a positive Δ-differentiable function, $q(t)\in C_{rd}(\mathrm{T},\mathrm{R}^+)$ such that $|uf(t,u)|\geq q(t)|u|^{\gamma+1}$ for all $u\neq 0$,

$$M(t,T_*):=\begin{cases}1 & g(t)\geq t\\ \omega^\gamma(t,T_*) & g(t)<t,\end{cases}$$

and $\omega(t,T_*)=\theta(t,g(t),T_*)$, then every solution of (1.8) is either oscillatory on $[t_0,\infty)_{\mathrm{T}}$ or tends to zero.

Meanwhile, as a special case of (1.6), Qi Li, *et al.* [14] obtained oscillation criteria for the delay differential equation

$$(r(t)((y(t)-p(t)y(\tau(t)))')^\gamma)'+q(t)f(y(\delta(t)))=0.$$

Arul and Shobha [15] improved the results obtained in Qi Li, *et al.* [14] and Thandapani, *et al.* [16] obtained some results on oscillatory behavior of the second order neutral difference equation:

$$\Delta(a_n(\Delta(x_n-p_nx_{n-\tau}))^\alpha)+q_nf(x_{n-\sigma})=0,$$

which is a special case of (1.7).

This paper is organized as follows: In Section 2, we give some lemmas that we need through our work. In Section 3, we establish some new sufficient conditions for oscillation of (1.1). Finally, in Section 4, we present some examples to illustrate our results.

2. Basic lemmas

In this section, we give some lemmas that play important roles in the proof of our results.

Lemma 2.1 *Let conditions* H_1-H_5 *be satisfied and* $x(t)$ *is a positive solution of (1.1). Then* $z(t)$ *satisfies one of the following two cases:*

(C_1) $z(t)>0, z^\Delta(t)>0$ and $(r(t)(z^\Delta(t))^\gamma)^\Delta\leq 0$

(C_2) $z(t)<0, z^\Delta(t)>0$ and $(r(t)(z^\Delta(t))^\gamma)^\Delta\leq 0$,

for $t\geq t_1$ where $t_1\geq t_0$ is sufficiently large.

Proof. Suppose that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau_i(t))>0$ and $x(\eta_i(t))>0, i=1,2$ on $[t_1,\infty)_{\mathrm{T}}$. (when $x(t)$ is negative the proof is similar, because the transformation $x(t)=-y(t)$ transforms (1.1) into the same form). From (1.1) and H_5, it follows that

$$(r(t)(z^\Delta(t))^\gamma)^\Delta\leq -q_1(t)\,x^\alpha(\tau_1(t))-q_2(t)x^\beta(\tau_2(t))<0 \quad \text{for } t\in[t_1,\infty)_{\mathrm{T}}. \quad (2.1)$$

Then, $r(t)(z^\Delta(t))^\gamma$ is strictly decreasing and of one sign on $[t_1,\infty)_{\mathrm{T}}$. Hence, there exists a $t_2\geq t_1$ such that $z^\Delta(t)>0$ or $z^\Delta(t)<0$ for $t\geq t_2$.

If $z^\Delta(t)>0$ for $t\geq t_2$, then we have (C_1) or (C_2). Now, we prove that $z^\Delta(t)<0$ cannot occur.

If $z^\Delta(t)<0$ for $t\geq t_2$, then $r(t)(z^\Delta(t))^\gamma\leq -c$ for $t\geq t_2$, where $c:=-r(t_2)(z^\Delta(t_2))^\gamma>0$. Thus we conclude that

$$z(t)\leq z(t_2)-c^{\frac{1}{\gamma}}\int_{t_2}^t\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

using (1.3), we have $\lim_{t\to\infty}z(t)=-\infty$. Then we have the following two possiblities

Case (a)**:** If $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty}t_k=\infty$ and $\lim_{k\to\infty}x(t_k)=\infty$. Assume that

$$x(t_k)=\max\{x(s):t_0\leq s\leq t_k\}.$$

Since $\lim_{t\to\infty}\eta_1(t)=\infty$, $\eta_1(t_k)>t_0$ for all sufficiently large k and $\eta_1(t)\leq t$, then

$$x(\eta_1(t_k))=max\{x(s):t_0\leq s\leq\eta_1(t_k)\}\leq max\{x(s):t_0\leq s\leq t_k\}=x(t_k), \quad (2.2)$$

therefore from(2.2) into (1.2), we have for all large k

$$z(t_k) = x(t_k) - p_1(t_k)x(\eta_1(t_k)) + p_2(t_k)x(\eta_2(t_k))$$
$$\geq x(t_k) - p_1(t_k)x(\eta_1(t_k))$$
$$\geq x(t_k) - p_1x(t_k) = (1-p_1)x(t_k) > 0,$$

which contradicts that $\lim_{t\to\infty} z(t) = -\infty$

Case (b)**:** If $x(t)$ is bounded, then $z(t)$ is also bounded which contradicts $\lim_{t\to\infty} z(t) = -\infty$.

Hence, $z(t)$ satisfies one of the two cases (C_1) or (C_2). This completes the proof.

Lemma 2.2 *Assume that $x(t)$ is a positive solution of (1.1) and $z(t)$ satisfies case (C_2). Then $\lim_{t\to\infty} x(t) = 0$*

proof. By $z(t) < 0$ and $z^{\Delta}(t) > 0$, we deduce that

$\lim_{t\to\infty} z(t) = l \leq 0$.

As in the proof of Case (a) of the previous lemma, $x(t)$ is bounded, then $\lim_{t\to\infty} x(t) = a \geq 0$. Now, if $a > 0$, then there exists $t_k \subseteq [t_2,\infty)_{\mathbb{T}}$ such that $\lim_{k\to\infty} t_k = \infty$, $\lim_{k\to\infty} x(t_k) = a > 0$ and

$$x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\},$$

then $z(t_k) \geq x(t_k) - p_1(t_k)x(\eta_1(t_k)) \geq x(t_k) - p_1x(t_k) = (1-p_1)x(t_k)$

thus, $0 > \lim_{k\to\infty} z(t_k) > (1-p_1)a > 0$, which is a contradiction. Therefore, $a = 0$ and $\lim_{t\to\infty} x(t) = 0.$

Lemma 2.3 *If $f(u) = bu - au^{\frac{\gamma+1}{\gamma}}$, where $a > 0$ and b are constants, then f attains its maximum value on $\mathbb{R}$ at $u^* = (\frac{b\gamma}{a(\gamma+1)})^{\gamma}$, and*

$$\max_{u\in\mathbb{R}} f = f(u^*) = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{b^{\gamma+1}}{a^{\gamma}}.$$

3. Main Results

Theorem 3.1 *Assume that H_1-H_5 hold, $\tau_2(t) \geq \eta_2(t)$ for all $t \geq t_0$, and there exists a positive real-valued Δ -differentiable function $\delta(t)$ such that for all sufficiently large $T > t_1$, we have*

$$\limsup_{t\to\infty} \int_T^t [\delta(s)\xi(s)[q_1(s)L^{\alpha}(s,t_1)A(s) + q_2(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s = \infty, \quad (3.1)$$

where

$$L(s,t_1) = \min\{\psi(s,t_1), \phi_1(s,t_1)\},$$

$$\xi(t) = \min\{\frac{1}{(1+p_2(\tau_1(t))^{\alpha}}, \frac{1}{(1+p_2(\tau_2(t))^{\beta}}, \frac{1}{(1+p_2(\eta_2^{-1}(\tau_1(t)))^{\alpha}}, \frac{1}{(1+p_2(\eta_2^{-1}(\tau_2(t)))^{\beta}}\}$$

Then, every solution of (1.1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$.

Proof. Assume that $x(t)$ is not almost oscillatory solution of (1.1). Then without loss of generality, there exists $t_3 \geq t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_3,\infty)_{\mathbb{T}}$. (when $x(t)$ is negative, the proof is similar). Then from lemma 2.1, $z(t)$ satisfies one of C_1 or C_2. Since $x(t)$ is not almost oscillatory, we have the two possibilities:

(I) $x^{\Delta}(t) < 0$ for $t \geq t_3$

(II) $x^{\Delta}(t) > 0$ for $t \geq t_3$

Case 1. Suppose that condition C_1 holds and $x^{\Delta}(t) < 0$, then we have

$$z(t) < x(t) + p_2(t)x(\eta_2(t))$$

$$\leq (1+p_2(t))x(t) \quad for\ t \geq t_3,$$

choosing $t_4 > t_3$ such that $\tau_1(t) \geq t_3$ for all $t \geq t_4$, then

$$x(\tau_1(t)) \geq \frac{1}{1+p_2(\tau_1(t))} z(\tau_1(t)) \quad \text{and} \quad x(\tau_2(t)) \geq \frac{1}{1+p_2(\tau_2(t))} z(\tau_2(t)),\ t \geq t_4, \quad (3.2)$$

substituting from (3.2) into (2.1), we have

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq \frac{-q_1(t)}{(1+p_2(\tau_1(t)))^{\alpha}} z^{\alpha}(\tau_1(t)) - \frac{q_2(t)}{(1+p_2(\tau_2(t)))^{\beta}} z^{\beta}(\tau_2(t))$$

$$\leq -N(t)[q_1(t) z^{\alpha}(\tau_1(t)) + q_2(t) z^{\beta}(\tau_2(t))] \quad for \quad t \geq t_4 \quad (3.3)$$

where $N(t) = \min\{\frac{1}{(1+p_2(\tau_1(t)))^{\alpha}}, \frac{1}{(1+p_2(\tau_2(t)))^{\beta}}\}$.

Defining the function w by

$$w = \delta(t) \frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(t)}, \quad (3.4)$$

then $w(t) > 0$ and

$$w^{\Delta}(t) = (\frac{\delta(t)}{z^{\beta}(t)})(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + r(\sigma(t))(z^{\Delta}(\sigma(t))^{\gamma} (\frac{\delta(t)}{z^{\beta}(t)})^{\Delta}.$$

$$= (\frac{\delta(t)}{z^{\beta}(t)})(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma} \frac{z^{\beta}(t)\delta^{\Delta}(t) - \delta(t)(z^{\beta}(t))^{\Delta}}{z^{\beta}(t) z^{\beta}(\sigma(t))}. \quad (3.5)$$

Substituting from (3.3) and (3.4) into (3.5), we obtained

$$w^{\Delta}(t) \leq -\delta(t) N(t)[q_1(t)(\frac{z(\tau_1(t))}{z(t)})^{\alpha} z^{\alpha-\beta}(t) + q_2(t)(\frac{z(\tau_2(t))}{z(t)})^{\beta}] +$$

$$\frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) - \frac{\delta(t) r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma} (z^{\beta}(t))^{\Delta}}{z^{\beta}(t) z^{\beta}(\sigma(t))},\ t \geq t_4. \quad (3.6)$$

Since $\tau_1(t) > t_1$ for all $t \geq t_4$, then integrating $z^{\Delta}(t)$ from $\tau_1(t)$ to t and using the fact that $r(t)(z^{\Delta}(t))^{\gamma}$ is a decreasing function, we have

$$z(t) - z(\tau_1(t)) = \int_{\tau_1(t)}^{t} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \leq r^{\frac{1}{\gamma}}(\tau_1(t)) z^{\Delta}(\tau_1(t)) \int_{\tau_1(t)}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Hence,

$$\frac{z(\tau_1(t))}{z(t)} \geq [1 - \frac{r^{\frac{1}{\gamma}}(\tau_1(t)) z^{\Delta}(\tau_1(t))}{z(t)} \int_{\tau_1(t)}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}] = 1 - \frac{r^{\frac{1}{\gamma}}(\tau_1(t)) z^{\Delta}(\tau_1(t))}{z(t)} [\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} - \int_{t_1}^{\tau_1(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]. \quad (3.7)$$

Integrating $z^{\Delta}(t)$ from t_1 to $\tau_1(t)$, we get

$$z(\tau_1(t)) \geq z(\tau_1(t)) - z(t_1) \geq r^{\frac{1}{\gamma}}(\tau_1(t)) z^{\Delta}(\tau_1(t)) \int_{t_1}^{\tau_1(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

hence

$$r^{\frac{1}{\gamma}}(\tau_1(t)) z^{\Delta}(\tau_1(t)) \leq z(\tau_1(t)) [\int_{t_1}^{\tau_1(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1}. \quad (3.8)$$

Substituting from (3.8) into (3.7), we have

$$\frac{z(\tau_1(t))}{z(t)} \geq 1 - \frac{z(\tau_1(t))}{z(t)} [\int_{t_1}^{\tau_1(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1} [\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} - \int_{t_1}^{\tau_1(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}],$$

then

$$\frac{z(\tau_1(t))}{z(t)} \geq \psi(t,t_1). \quad \text{for all} \quad t \geq t_4 \qquad (3.9)$$

Using $z(t)>0, z^{\Delta}(t)>0$ and $(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq 0$, then there exists $t_5 \in [t_4,\infty)_{\mathrm{T}}$ and positive constants b_0 and b_1 such that

$$z(t_0) := b_0 \leq z(t) \leq b_1 \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}, t \geq t_5, \qquad (3.10)$$

hence, we have

$$z^{\alpha-\beta}(t) \geq A(t) := \begin{cases} b_0^{\alpha-\beta} & \alpha \geq \beta \\ b_1^{\alpha-\beta}[\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{\alpha-\beta} & \alpha < \beta. \end{cases} \quad (3.11)$$

Using the chain rule, we get

$$(z^{\beta}(t))^{\Delta} \geq \begin{cases} \beta z^{\Delta}(t) z^{\beta-1}(t), & \beta \geq 1 \\ \beta z^{\Delta}(t)(z(\sigma(t)))^{\beta-1}, & \beta < 1 \end{cases} \qquad (3.12)$$

since, $\sigma(t) \geq t$ and $r(t)(z^{\Delta}(t))^{\gamma}$ is a decreasing function, then

$$z^{\Delta}(t) > \frac{(r(\sigma(t)))^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} z^{\Delta}(\sigma(t)). \quad (3.13)$$

Using (3.4), (3.12) and (3.13), we have

$$\frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))} \geq \frac{\beta\delta(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} (z(\sigma(t)))^{\frac{\beta}{\gamma}-1} w^{\lambda}(\sigma(t)), \quad (3.14)$$

where $\lambda = \dfrac{\gamma+1}{\gamma}$. Using (3.10), we have

$$(z(\sigma(t)))^{\frac{\beta}{\gamma}-1} \geq C(t) := \begin{cases} b_0^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} \geq 1 \\ b_1^{\frac{\beta}{\gamma}-1}[\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} < 1, \end{cases}$$

consequently (3.14) becomes

$$\frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))} \geq \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} w^{\lambda}(\sigma(t)), t \geq t_5. \qquad (3.15)$$

Since $\tau_2(t) \geq t$ and $z^{\Delta}(t) > 0$, then $\dfrac{z(\tau_2(t))}{z(t)} \geq 1$.

Substituting from the above inequality, (3.9), (3.11) and (3.15) into (3.6), we obtain

$$w^{\Delta}(t) \leq -\delta(t)N(t)[q_1(t)\psi^{\alpha}(t,t_1)A(t) + q_2(t)] + \frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} w^{\lambda}(\sigma(t)), \quad (3.16)$$

using lemma (2.3) and taking

$$b := \frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))} \quad \text{and} \quad a := \frac{\beta\delta(t)C(t)}{(\delta^{\sigma}(t))^{\lambda} r^{\frac{1}{\gamma}}(t)},$$

then

$$\frac{\delta_+^\Delta(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^\lambda r^{\frac{1}{\gamma}}(t)}w^\lambda(\sigma(t))\le\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(t)(\delta_+^\Delta(t))^{\gamma+1}}{\delta^\gamma(t)C^\gamma(t)}. \quad (3.17)$$

Substituting from (3.17) into (3.16), we obtain

$$w^\Delta(t)\le-\delta(t)N(t)[q_1(t)\psi^\alpha(t,t_1)A(t)+q_2(t)]+\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(t)(\delta_+^\Delta(t))^{\gamma+1}}{\delta^\gamma(t)C^\gamma(t)},\ t\ge t_5.$$

Integrating the above inequality from t_5 to t, we get

$$\int_{t_5}^{t}[\delta(s)N(s)[q_1(s)\psi^\alpha(s,t_1)A(s)+q_2(s)]-\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^\Delta(s))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}\le w(t_5)-w(t)<w(t_5)$$

which is a contradiction with (3.1).

Case 2. Suppose that condition C_1 holds and $x^\Delta(t)>0$. Then we have

$$\begin{aligned}z(t)&<x(t)+p_2(t)x(\eta_2(t))\\&\le(1+p_2(t))x(\eta_2(t))\quad\forall t\ge t_5.\end{aligned}$$

Choosing t_6 sufficiently large such that $t_6>t_5$ and $\eta_2^{-1}(t)>t_5$ for all $t\ge t_6$, then

$$x(t)\ge\frac{1}{1+p_2(\eta_2^{-1}(t))}z(\eta_2^{-1}(t))\quad t\ge t_6.$$

Taking $t_7>t_6$ such that $\tau_1(t)>t_6$ for all $t\ge t_7$, then

$$x(\tau_1(t))\ge\frac{1}{1+p_2(\eta_2^{-1}(\tau_1(t)))}z(\eta_2^{-1}(\tau_1(t)))\text{ and }x(\tau_2(t))\ge\frac{1}{1+p_2(\eta_2^{-1}(\tau_2(t)))}z(\eta_2^{-1}(\tau_2(t))),\quad t\ge t_7. \quad (3.18)$$

substituting from (3.18) into (2.1), we have

$$(r(t)(z^\Delta(t))^\gamma)^\Delta\le\frac{-q_1(t)}{(1+p_2(\eta_2^{-1}(\tau_1(t))))^\alpha}z^\alpha(\eta_2^{-1}(\tau_1(t)))-\frac{q_2(t)}{(1+p_2(\eta_2^{-1}(\tau_2(t))))^\beta}z^\beta(\eta_2^{-1}(\tau_2(t)))\ \forall t\ge t_7. \quad (3.19)$$

Using the same technique we used in Case 1, we obtain

$$\begin{aligned}w^\Delta(t)\le&-\delta(t)M(t)[q_1(t)(\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)})^\alpha z^{\alpha-\beta}(t)+q_2(t)(\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)})^\beta]+\\&\frac{\delta_+^\Delta(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^\lambda r^{\frac{1}{\gamma}}(t)}w^\lambda(\sigma(t))\quad\forall t\ge t_7, \quad (3.20)\end{aligned}$$

where $M(t)=\min\{\frac{1}{(1+p_2(\eta_2^{-1}(\tau_1(t))))^\alpha},\frac{1}{(1+p_2(\eta_2^{-1}(\tau_2(t))))^\beta}\}$.

Since $t\ge\eta_2^{-1}(\tau_1(t))>t_1$ for all $t\ge t_7$, then integrate $z^\Delta(t)$ from $\eta_2^{-1}(\tau_1(t))$ to t and using the fact that $r(t)(z^\Delta(t))^\gamma$ is a decreasing function, we have

$$z(t)-z(\eta_2^{-1}(\tau_1(t)))=\int_{\eta_2^{-1}(\tau_1(t))}^{t}\frac{(r(s)(z^\Delta(s))^\gamma)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s\le r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau_1(t)))z^\Delta(\eta_2^{-1}(\tau_1(t)))\int_{\eta_2^{-1}(\tau_1(t))}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Hence,

$$\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)}\ge[1-\frac{r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau_1(t)))z^\Delta(\eta_2^{-1}(\tau_1(t)))}{z(t)}\int_{\eta_2^{-1}(\tau_1(t))}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]$$

$$\geq 1-\frac{r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau_1(t)))z^{\Delta}(\eta_2^{-1}(\tau_1(t)))}{z(t)}[\int_{t_1}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}-\int_{t_1}^{\eta_2^{-1}(\tau_1(t))}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]. \qquad (3.21)$$

Integrating $z^{\Delta}(t)$ from t_1 to $\eta_2^{-1}(\tau_1(t))$, we get

$$z(\eta_2^{-1}(\tau_1(t)))\geq z(\eta_2^{-1}(\tau_1(t)))-z(t_1)\geq r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau_1(t)))z^{\Delta}(\eta_2^{-1}(\tau_1(t)))\int_{t_1}^{\eta_2^{-1}(\tau_1(t))}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

hence

$$r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau_1(t)))z^{\Delta}(\eta_2^{-1}(\tau_1(t)))\leq z(\eta_2^{-1}(\tau_1(t)))[\int_{t_1}^{\eta_2^{-1}(\tau_1(t))}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1}. \qquad (3.22)$$

Substituting from (3.22) into (3.21), we have

$$\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)}\geq 1-\frac{z(\eta_2^{-1}(\tau_1(t))))}{z(t)}[\int_{t_1}^{\eta_2^{-1}(\tau_1(t))}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1}[\int_{t_1}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}-\int_{t_1}^{\eta_2^{-1}(\tau_1(t))}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}],$$

then

$$\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)}\geq\frac{\int_{t_1}^{\eta_2^{-1}(\tau_1(t))}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_{t_1}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}:=\phi_1(t,t_1)\quad\forall t\geq t_7. \qquad (3.23)$$

Since $\tau_2(t)\geq\eta_2(t)$, then

$$\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)}\geq 1\quad\forall\, t\geq t_7. \qquad (3.24)$$

Substituting from (3.23) and (3.24) into (3.20), we obtain

$$w^{\Delta}(t)\leq-\delta(t)M(t)[q_1(t)\phi_1^{\alpha}(t,t_1)A(t)+q_2(t)]+\frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}. \qquad (3.25)$$

Using Lemma 2.3 and integrating from t_7 to t, we get

$$\int_{t_7}^{t}[\delta(s)M(s)[q_1(s)\phi_1^{\alpha}(s,t_1)A(s)+q_2(s)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\leq w(t_7)-w(t)<w(t_7) \qquad (3.26)$$

which is a contradiction with (3.1).

Finally, suppose that condition C_2 holds, then according to lemma 2.3, we have $\lim_{t\to\infty}x(t)=0$. This completes the proof

Theorem 3.2 *Assume that H_1-H_5 hold and $\eta_2(t)\geq\tau_2(t)$ for all $t\geq t_0$. Furthermore suppose that there exists a positive real-valued Δ -differentiable function $\delta(t)$ such that for all sufficiently large $T\geq t_1>t_0$, we have*

$$\limsup_{t\to\infty}\int_{T}^{t}[\delta(s)\xi(s)[q_1(s)L^{\alpha}(s,t_1)A(s)+q_2(s)v^{\beta}(s,t_1)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s=\infty, \qquad (3.27)$$

where

$$L(s,t_1)=\min\{\psi(s,t_1),\phi_1(s,t_1)\}\text{ and }v(s,t_1)=\min\{1,\phi_2(s,t_1)\}$$

Then, every solution of (1.1) is almost oscillatory on $[t_0,\infty)_{\mathrm{T}}$ or converges to zero as $t\to\infty$

Proof. Assume that $x(t)$ is not almost oscillatory solution of (1.1). Then without loss of generality, there exists $t_3 \geq t_0$ such that $x(t) > 0, x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_3, \infty)_T$. (when $x(t)$ is negative, the proof is similar). Then from lemma 2.1, $z(t)$ satisfies one of C_1 or C_2. Since $x(t)$ is not almost oscillatory, we have the two possibilities (I), (II) in the proof of Theorem 3.1

Case 1. Suppose that condition C_1 holds and $x^{\Delta}(t) < 0$, then the proof is similar to that of case 1 in Theorem 3.1. So, it is omitted.

Case 2. Suppose that condition C_1 holds and $x^{\Delta}(t) > 0$, then using the same technique that used in Theorem 3.1 in Case 2, untill we reach to (3.20). Hence

$$w^{\Delta}(t) \leq -\delta(t)M(t)[q_1(t)(\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)})^{\alpha} z^{\alpha-\beta}(t) + q_2(t)(\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)})^{\beta}] + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} w(\sigma(t)) - \frac{\delta(t) r^{\sigma}(t)(z^{\Delta\sigma}(t))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t) z^{\beta}(\sigma(t))} \quad \forall t \geq t_7. \qquad (3.28)$$

Since, $\eta_2(t) \geq \tau_2(t) > t_1$, then $t \geq \eta_2^{-1}(\tau_2(t)) > t_1$ for all $t \geq t_7$. Using the fact that $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing, we can prove that

$$\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)} \geq \frac{\int_{t_1}^{\eta_2^{-1}(\tau_2(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}} := \phi_2(t, t_1) \quad \forall t \geq t_7 \qquad (3.29)$$

Substituting from (3.11), (3.15), (3.23) and (3.29) into (3.28), we obtain

$$w^{\Delta}(t) \leq -\delta(t)M(t)[q_1(t)\phi_1^{\alpha}(t,t_1)A(t) + q_2(t)\phi_2^{\beta}(t,t_1)] + \frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} (w(\sigma(t)))^{\lambda}, \qquad (3.30)$$

Using Lemma 2.3 and integrating from t_7 to t, we get

$$\int_{t_7}^{t} [\delta(s)M(s)[q_1(s)\phi_1^{\alpha}(s,t_1)A(s) + q_2(s)\phi_2^{\beta}(s,t_1)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}] \leq w(t_7) - w(t) < w(t_7) \qquad (3.31)$$

which is a contradiction with (3.27).

Finally, if condition C_2 holds, then according to lemma 2.3, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof

Theorem 3.3 *Assume that* H_1-H_5, $\tau_2(t) \geq \eta_2(t)$ *for all* $t \geq t_0$ *and there exist functions* H, h *such that for each fixed* t, $H(t,s)$ *and* $h(t,s)$ *are rd-continuous with respect to* s *on* $\Delta \equiv \{(t,s) : t \geq s \geq t_0\}$ *such that*

$$H(t,t) = 0, t \geq t_0, \quad H(t,s) > 0, t > s \geq t_0, \qquad (3.32)$$

and H has a non-positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ satisfying

$$H^{\Delta_s}(t,s) + H(t,s)\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)} = -\frac{h(t,s)}{\delta^{\sigma}(t)}(H(t,s))^{\frac{\gamma}{\gamma+1}}. \qquad (3.33)$$

Assume that there exists a positive real-valued Δ-differentiable function $\delta(t)$ such that for all sufficiently large $T \geq t_1 > t_0$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t [\delta(s)\xi(s)H(t,s)[q_1(s)L^{\alpha}(s,t_1)A(s) + q_2(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s = \infty, \qquad (3.34)$$

Then, every solution of (1.1) is almost oscillatory on $[t_0, \infty)_T$ or converges to zero as $t \to \infty$.

Proof. Assume that $x(t)$ is not almost oscillatory solution of (1.1). Then without loss of generality, there exists $t_3 \geq t_0$ such that $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_3, \infty)_{\mathbb{T}}$. (when $x(t)$ is negative, the proof is similar). Then from lemma 2.1, $z(t)$ satisfies one of C_1 or C_2. Since $x(t)$ is not almost oscillatory, we have the two possibilities (I), (II) in the proof of Theorem 3.2

Case 1. Suppose that condition C_1 holds and $x^\Delta(t) < 0$. Proceeding as in the proof of Case 1 in Theorem 3.1 until we get (3.16), therefore

$$\delta(t)N(t)[q_1(t)\psi^\alpha(t,t_1)A(t)+q_2(t)] \leq -w^\Delta(t) + \frac{\delta_+^\Delta(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^\lambda r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^\lambda,$$

Multiplying both sides of the previous inequality by $H(t,s)$ and integrating from t_5 to t, we get

$$\int_{t_5}^{t}[H(t,s)\delta(s)N(s)[q_1(s)\psi^\alpha(s,t_1)A(s)+q_2(s)]\Delta s$$

$$\leq -\int_{t_5}^{t}H(t,s)w^\Delta(s)\Delta s + \int_{t_5}^{t}H(t,s)\frac{\delta_+^\Delta(s)}{\delta^\sigma(s)}w^\sigma(s)\Delta s - \int_{t_5}^{t}\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda\Delta s$$

$$\leq H(t,t_5)w(t_5) + \int_{t_5}^{t}[\frac{-h(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^\sigma(s)}w^\sigma(s)\Delta s - \int_{t_5}^{t}\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda\Delta s$$

$$\leq H(t,t_5)w(t_5) + \int_{t_5}^{t}[\frac{h\_(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^\sigma(s)}w^\sigma(s)\Delta s - \int_{t_5}^{t}\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda\Delta s. \quad (3.35)$$

Using lemma 2.3, with

$$a := \frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda} \quad \text{and} \quad b := \frac{h\_(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^\sigma(s)},$$

we get:

$$\frac{h\_(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^\sigma(s)}w^\sigma(s) - \frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda \leq \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(s)(h\_(s,t))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}. \quad (3.36)$$

Substituting (3.36) into (3.35), we get

$$\int_{t_5}^{t}[H(t,s)\delta(s)N(s)[q_1(s)\psi^\alpha(s,t_1)A(s)+q_2(s)]]\Delta s$$

$$\leq H(t,t_5)w(t_5) + \int_{t_5}^{t}\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(s)(h\_(s,t))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}\Delta s,$$

which implies

$$\frac{1}{H(t,t_5)}\int_{t_5}^{t}[\delta(s)N(s)H(t,s)[q_1(s)\psi^\alpha(s,t_1)A(s)+q_2(s)] -$$

$$\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(s)(h\_(s,t))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}]\Delta s \leq w(t_5),$$

this is a contradiction with (3.34).

Case 2. Suppose that condition C_1 holds and $x^\Delta(t) > 0$. Proceeding as in the proof of Case 2 in Theorem 3.1 until Eq. (3.25), we get:

$$\delta(t)M(t)[q_1(t)\phi_1^{\alpha}(t,t_1)A(t)+q_2(t)]\le -w^{\Delta}(t)+\frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}.$$

Multiplying both sides of the above inequality by $H(t,s)$, integrating from t_7 to t and following the same proof of the above Case, we obtain

$$\frac{1}{H(t,t_7)}\int_{t_7}^{t}[\delta(s)M(s)H(t,s)[q_1(s)\phi_1^{\alpha}(s,t_1)A(s)+q_2(s)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h\_(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s\le w(t_7)$$

which is a contradiction with (3.34).

Finally, suppose that condition C_2 holds. Then, according to lemma 2.3, we get $\lim_{t\to\infty}x(t)=0$. This completes the proof

Theorem 3.4 *Assume that* H_1-H_5 *hold,* $\eta_2(t)\ge\tau_2(t)$ *for all* $t\ge t_0$*. Also, assume thst there exist functions* H,h *and* δ *defined as in Theorem 3.3 and satisfying Eqs. (3.32), (3.33) and*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t[\delta(s)\xi(s)H(t,s)[q_1(s)L^{\alpha}(s,t_1)A(s)+\upsilon^{\beta}(s,t_1)q_2(s)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h\_(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s=\infty,\quad (3.37)$$

Then, every solution of (1.1) is almost oscillatory on $[t_0,\infty)_{\mathrm{T}}$ or converges to zero as $t\to\infty$.

Proof. Assume that $x(t)$ is not almost oscillatory solution of (1.1). Then without loss of generality, there exists $t_3\ge t_0$ such that $x(t)>0,\ x(\tau_i(t))>0$ and $x(\eta_i(t))>0, i=1,2$ on $[t_3,\infty)_{\mathrm{T}}$. (when $x(t)$ is negative, the proof is similar). Then from lemma 2.1, $z(t)$ satisfies one of C_1 or C_2. Since $x(t)$ is not almost oscillatory, we have the two possibilities (I), (II) in the proof of Theorem 3.1

Case 1. Suppose that condition C_1 holds and $x^{\Delta}(t)<0$. Then the proof is similar to that of Case 1 in Theorem 3.3. So it is omitted.

Case 2. Suppose that condition C_1 holds and $x^{\Delta}(t)>0$. Proceeding as in the proof of Case 2 in Theorem 3.2 until Eq. (3.30), we get

$$\delta(t)M(t)[q_1(t)\phi_1^{\alpha}(t,t_1)A(t)+q_2(t)\phi_2^{\beta}(t,t_1)]\le -w^{\Delta}(t)+\frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t))^{\lambda},$$

Multiplying both sides of the above inequality by $H(t,s)$, integrating from t_7 to t and following the same technique as in Case 1 Theorem 3.3, we obtain

$$\frac{1}{H(t,t_7)}\int_{t_7}^{t}[\delta(s)M(s)H(t,s)[q_1(s)\phi_1^{\alpha}(s,t_1)A(s)+q_2(s)\phi_2^{\beta}(s,t_1)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h\_(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s\le w(t_7)$$

which is a contradiction with (3.37).

Finally, suppose that condition C_2 holds, then according to lemma 2.3, we get $\lim_{t\to\infty}x(t)=0$. This completes the proof

4 Examples

In this section, we give some examples to illustrate our main results.

Example 4.1 *Take* $\mathrm{T}=[t_1+\Pi,\infty)_{\mathrm{P}}$ *where* $t_1\ge 0$ *and consider the equation*

$$[x(t)-\frac{1}{2}x(t-\frac{\Pi}{2})+2x(t+\frac{\Pi}{2})]''+32x(t-\frac{\Pi}{2})+8x(t+\Pi)=0 \text{ forall } \quad t\geq t_0+\Pi. \tag{4.1}$$

Here

$$\alpha=\beta=\gamma=1,\ r(s)=1,\ \eta_1(t)=\tau_1(t)=t-\frac{\Pi}{2},\ \eta_2(t)=t+\frac{\Pi}{2},\ \tau_2(t)=t+\Pi,\ p_1(t)=\frac{1}{2},$$
$$p_2(t)=2, q_1(t)=32 \text{ and } q_2(t)=8$$

then substituting in (1.4) and (1.5), we obtain $A(s)=C(s)=1$. Also

$$\int_{t_1+\Pi}^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}=\int_{t_1+\Pi}^{\infty}\Delta s=\infty \quad \text{and} \quad \eta_2^{-1}(\tau_1(t))=t-\Pi.$$

hence $t_1<\eta_2^{-1}(\tau_1(t))<\tau_1(t)$ and $\phi_1(s,t_1)<\psi(s,t_1)$. Consequently

$$L^{\alpha}(s,t_1)=\phi_1(s,t_1)=\frac{\int_{t_1}^{t-\Pi}\Delta s}{\int_{t_1}^{t}\Delta s}=\frac{t-\Pi-t_1}{t-t_1}.$$

Also $\xi(s)=\frac{1}{3}$. Since $\eta_2(t)\leq\tau_2(t)$, then subistituting in (3.1) with $\delta(t)=1$, we get

$$\limsup_{t\to\infty}\int_T^t\frac{1}{3}[32\frac{s-\Pi-t_1}{s-t_1}+8]\Delta s=\infty \tag{4.2}$$

using Theorem 3.1, we obtain that every solution of (4.1) is almost oscillatory or converges to zero as $t\to\infty$. Note that $x(t)=\sin 4t$ is an olmost oscillatory solution to Eq. (4.1).

Remark 4.1 *The results of Arul and Shobha [15], Erbe, et al. [7] and Qi Li, et al. [14] can not be applied to (4.1) as* $p_2(t)\neq 0$ *and* $f(t,x(\tau_1(t)))\neq 0\neq g(t,x(\tau_2(t)))$, *but according to Theorem 3.1 we obtain that every solution of (4.1) is almost oscillatory or converges to zero as* $t\to\infty$.

Example 4.2 *Take* $\mathrm{T}=[2t_1,\infty)_{\mathrm{T}}$ *where* $t_1\geq 0$ *and consider the equation*

$$[[[x(t)-\frac{1}{2}x(\eta_1(t))+\frac{1}{4}x(2t)]^{\Delta}]^4]^{\Delta}+q_1(t)x^9(t)+q_2(t)x^8(2t+1)=0. \tag{4.3}$$

Here

$$\alpha=9,\ \beta=8,\ \gamma=4,\ r(s)=1,\ \eta_1(t)\leq t,\ \eta_2(t)=2t,\ \tau_1(t)=t,\ \tau_2(t)=2t+1,\ p_1(t)=\frac{1}{2},$$
$$p_2(t)=\frac{1}{4}, q_1(t)=\frac{2^9(t-t_1)^9}{(t-2t_1)^9 b_0 t} \text{ and } q_2(t)=\frac{2}{t}$$

then substituting in (1.4) and (1.5), we obtain $A(s)=C(s)=b_0$,

$$\int_{2t_1}^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}=\int_{2t_1}^{\infty}\Delta s=\infty \text{ and } \eta_2^{-1}(\tau_1(t))=\frac{t}{2}$$

hence $t_1<\eta_2^{-1}(\tau_1(t))<\tau_1(t)$ and so $\phi_1(s,t_1)<\psi(s,t_1)$, consequently

$$L^{\alpha}(s,t_1)=\phi_1^9(s,t_1)=\left(\frac{\int_{t_1}^{\frac{t}{2}}\Delta s}{\int_{t_1}^{t}\Delta s}\right)^9=\left(\frac{\frac{t}{2}-t_1}{t-t_1}\right)^9>0.$$

Also $\xi(s) = (\frac{4}{5})^9$, since $\eta_2(t) \le \tau_2(t)$, then substitute in (3.1) with $\delta(t) = 1$ we have

$$\limsup_{t\to\infty} \int_T^t (\frac{4}{5})^9 \frac{3}{s} \Delta s = \infty \tag{4.4}$$

using Theorem 3.1, we find that every solution of (4.3) is almost oscillatory or converges to zero as $t \to \infty$.

Remark 4.2 *The results of Erbe, et al. [7] can not be applied to (4.3) as* $p_2(t) \ne 0, \alpha \ne \beta \ne \gamma$ *and both* $f(t, x(\tau_1(t))) \ne 0 \ne g(t, x(\tau_2(t)))$. *But according to Theorem 3.1, we obtain that every solution of (4.3) is almost oscillatory or converges to zero as* $t \to \infty$.

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