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Effect of Correlation of Brownian Motions on an Investor's Optimal Investment and Consumption Decision under Ornstein-Uhlenbeck Model

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Abstract: The aim of this paper is to investigate and give a closed form solution to an investment and consumption decision problem where the risk-free asset has a rate of return that is driven by the Ornstein-Uhlenbeck Stochastic interest rate of return model. The maximum principle is applied to obtain the HJB equation for the value function. Owing to the introduction of the consumption factor and the Ornstein-Uhlenbeck Stochastic interest rate of return, the HJB equation derived becomes much more difficult to deal with than the one obtained in literature. In the same spirit with the techniques literature, the nonlinear second-order partial differential equation was transformed into an ordinary differential equation; specifically, the Bernoulli equation, using elimination of dependency on variables for easy tackling.

Keywords: Elimination of variable dependency; Maximum principle; Optimal investment and consumption; Ornstein-Uhlenbeck; Utility maximization; Stochastic interest rate.

1. Introduction

The classical Merton's portfolio optimization problems shows that an investor dynamically allocates his wealth between onerisk asset and one risk-free asset and chooses an optimalconsumption rate to maximize total expected discountedutility of consumption [1, 2].In this Merton's model, thereare no, transaction costs, borrowing and shortingconstraints. Hundreds of literally extensions and applications on investment and consumption problems have been inspired by this pioneer work of Merton. For example,the introduction of transaction costs into the investment andconsumption problems, one can refer to [3], [4], and [5]. In investigating the optimal consumption problem with borrowingconstraints authors like, [6], [7], [8] and [9] have made very useful contributions.However, theabove mentioned models generally were studied under theassumption that the risky asset's price dynamics was driven by ageometric Brownian motion (GBM) and the risk-free asset with a rate of return that is assumed constant.Some authors have studied the problem under the extension of geometric Brownian motion (GBM) called the constant elasticity of variance (CEV) model which is anatural extension of the geometric Brownian motion (GBM). The constant elasticity of variance (CEV) model has an advantage that the volatility ratehas correlation with the risky asset price.Cox and Ross originally proposedthe use of constant elasticity of variance (CEV) model as an alternative diffusion process for pricing European option, [10]. [11], [12], [13], and [14] have appliedit to analyze the option pricing formula. Further applications of the constant elasticity of variance (CEV) model, in the recent years, has been in the areas of annuity contracts and theoptimal investment strategies in the utility framework using dynamic programming principle.

Detailed discussions can be found in the following; [15], [16, 17], [18], [19], [20], [21] and [22].

This paper aims at investigating and giving a closed form solution to an investmentand consumption decision problem where the risk-free asset has a rate of return that is driven by the Ornstein-Uhlenbeck Stochastic interest rate of return model. Dynamic programming principle, specifically, the maximum principle isapplied to obtain the HJB equation for the value function.

Owing to the introduction of the consumption factor and the Ornstein-Uhlenbeck Stochastic interest rate of return, the HJB equation derived is much more difficult to deal with than the one obtained by [16]. Inspired by the techniques of [16] and [23], the nonlinear second-order partial differential equation was transformed intoan ordinary

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differential equation, specifically the Bernoulli equation, using elimination of dependency on variables, which is easy to tackle.

The rest of this paper is organized as follows. In section 2 is the formulation of the financial market and the proposed optimizationproblem. In section 3, dynamic programming principle is applied to obtain the HJB equation and the optimal investment and consumption strategies in the power utility preference case investigated and the findings given. Section 4 concludes the paper.

2. Formulation of the Financial Market and the Proposed Optimization Problem

This section proposes the problem formulation of optimal investment and consumption decisions for an investor with Ornstein-Uhlenbeck Stochastic interest rate of return.

A financial market in which two assets are traded continuously over a time frame $[0, T]$ is considered. The first asset is a risk-free asset with price $P(t)$ time t,which price process $P(t)$ satisfies

$$dP(t) = r(t)P(t)dt: P(0) = p_0 > 0, \quad (1)$$

where $r(t) > 0$is governed by the Ornstein-Uhlenbeck Stochastic interest rate of return.
That is

$$r(t) = \alpha(\beta - r(t))dt + \sigma dZ_1(t): r(0) = r_{0,} \quad (2)$$

where α is the speed of mean reversion, β the mean level attracting the interest rate and σ the volatility constant of the interest rate. $Z_1$is a standard Brownian motion. The second asset is a risky asset with price $P_1(t)$ at time t whose price process is governed by the geometric Brownian motion.

$$dP_1(t) = P_1(t)[\mu dt + \lambda dZ_2(t)]; P_1(0) = p_{1_{0,}} \quad (3)$$

where μ and λ are constants and μ the drift parameter while λ is the diffusion parameter (volatility). $Z_2(t)$is another standard Brownian motion.

This work assumes a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and uncertainties in the models are generated by the Brownian motions $Z_1(t)$ and $Z_2(t)$. Assuming the investor invests $\pi(t)$,part of the total wealth $X(t)$on the risky asset at time t, $t \in [0, T]$, clearly, the amount invested on the risk-free asset is $[X(t) - \pi(t)]$.
Further assumptions are that

i. Transaction cost, tax and dividends are paid on the amount invested in the risky asset at constant rates ϑ, θ and $d$respectively.
ii. Consumption withdrawals are made from the risk-free account.

Therefore for any trading strategy $(\pi(t), K(t))$ the total wealth process of the investor follows the stochastic differential equation (SDE)

$$dX(t) = \pi(t)\frac{dP_1(t)}{P_1(t)} + [x(t) - \pi(t)]\frac{dP(t)}{P(t)} - [(\vartheta + \theta - d)\pi(t) + K(t)]dt, \quad (4)$$

where$K(t)$ is the rate of consumption.
Applying (2a) and (3a) in (4) obtains:

$$dX(t) = \pi(t)[\mu dt + \lambda dZ_2(t)] + \{X(t) - \pi(t)]r(t)dt - [(\vartheta + \theta - d)\pi(t) + K(t)]dt, \quad (5)$$

which becomes

$$dX(t) = \{[(\mu + d) - (r + \vartheta + \theta)]\pi(t) + r(t)X(t) - K(t)]dt + \lambda\pi(t)dZ_2(t). \quad (6)$$

Definition: (admissible strategy). An investment and consumption $(\pi(t), K(t))$ strategy is said to be admissible if the following conditions are satisfied:

i. $(\pi(t), k(t))$ is $\mathcal{F}_t$ −progressively measurable and
ii. $\int_0^T \pi^2(t)dt < \infty, \int_0^T k(t)dt < \infty \ ; \forall\, T > 0$ (7a)
iii. $E\left[\int_0^T (\lambda^2\pi^2(t))dt\right] < \infty$ (7b)
iv. For $\forall\, (\pi(t), k(t))$,the stochastic differential equation (6) has a unique solution,(Chang etal.,[24]).

Assuming the set of all admissible investment and consumption strategies $(\pi(t), k(t))$ is denoted by$B = [(\pi(t), k(t)): 0 \leq t \leq T]$,then the investor`s problem can be stated mathematically thus:

$$\mathrm{Max}_{[\pi(t),k(t)]\in B}\, E[(U(X(T)]. \quad (8)$$

where$U(.)$ is strictly concave and satisfies the condition $U(+\infty) = 0$ and $U'(0) = +\infty$ and T is the time horizon.
This study considers the power utility function given by

$$U(X(t)) = \frac{X^{1-b}}{1-b}; b \neq 1. \quad (9)$$

Using the classical tools of stochastic optimal control where consumption is involved, define the value function at time T as:

$$G(T, r(t), P_1(t), X(t)) = \sup_B E\left[\int_0^T e^{-\varrho\tau}\frac{K^{1-b}}{1-b}d\tau + e^{-\varrho T}\frac{X_T{}^{1-b}}{1-b}\right];\ P_1(t) = p_1; X(t) = x; r(t) = r, K(t) = k; 0 < t < T. \quad (10)$$

Therefore the investor`s problem becomes

$$G(t, r, p_1, x) = \sup_{[\pi(t),k(t)]B} E\left[\int_0^T e^{-\varrho\tau}\frac{k^{1-b}}{1-b}d\tau + e^{-\varrho T}\frac{x^{1-b}}{1-b}\right] \quad (11)$$

subject to (6).

3. The Optimization Programme

Two cases will be considered thus, namely; when the shocks do not correlate and when the shocks correlate.

3.1. CASE 1: When the Brownian Motions Do Not Correlate

In this section, it is assumed that the Brownian motions do not correlate. The theorem below follows.

Theorem 3.1: If the rate of the return of the risk-free asset in an investor`s portfolio who has a power utility preference given by $U(X) = \frac{X^{1-b}}{1-b}, b \neq 1$, is of Ornstein-Uhlenbeck model, then the optimal policy that maximizes the expected utility of terminal wealth and consumption at the terminal time T is to invest in the risky asset;

$$\pi^* = \frac{x}{b}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1-b)\right],$$

with optimal consumption

$$k^* = x(1-b)^{\frac{2}{b}}(rp_1)^{\frac{b-1}{b}}\left[(n-1)\int_t^T A(s)\, e^{\int_t^T (1-n)B(\tau)d\tau} ds + \left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}\right]^{\frac{1}{1-n}},$$

and the optimal value function

$$G^*(t,r,p_1,x) = \frac{(xp_1 r)^{1-b}}{(1-b)^2}\left[e^{(n-1)\int_t^T B(\tau)d\tau}\left[(n-1)\int_t^T A(s)\, e^{\int_t^T (1-n)B(\tau)d\tau} ds + \left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}\right]^{\frac{1}{1-n}}.$$

Proof:

The derivation of Hamilton-Jacobi-Bellman (HJB) partial differential starts with the Bellman;

$$G(t,r,p_1,x) = sup_\pi \left\{\frac{C^{1-k}}{1-k} + \frac{1}{1+\zeta} E[G(t+\Delta t, r', x')]\right\}. \quad (12)$$

The actual utility over time interval of length Δt is $\frac{C^{1-k}}{1-k}\Delta t$ and the discounting over such period is expressed as $\frac{1}{1+\zeta\Delta t}, \zeta > 0$.

Therefore, the Bellman equation becomes;

$$G(t,r,p_1,x) = sup_\pi \left\{\frac{C^{1-k}}{1-k}\Delta t + \frac{1}{1+\vartheta\Delta t} E[G(t+\Delta t, r', p_1{}', x')]\right\}. \quad (13)$$

The multiplication of (13) by $(1+\zeta\Delta t)$ and rearranging terms obtains;

$$\vartheta G(t,r,p_1,x)\Delta t = sup_\pi \left\{\frac{C^{1-k}}{1-k}\Delta t\,(1+\zeta\Delta t) + E(\Delta G)\right\}. \quad (14)$$

Dividing (14) by Δt and taking limit to zero, obtains the Bellman equation;

$$\zeta G = sup_\pi \left\{\frac{C^{1-k}}{1-k} + \frac{1}{dt} E(dG)\right\}. \quad (15)$$

Applying the dynamic programming maximum principle which states that;

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial r}dr + \frac{\partial G}{\partial p_1}dp_1 + \frac{\partial G}{\partial x}dx + \frac{\partial^2 G}{\partial r\partial p_1}drdp_1 + \frac{\partial^2 G}{\partial r\partial x}drdx\frac{\partial^2 G}{\partial p_1\partial x}d\rho_1 dx + \frac{1}{2}\left[\frac{\partial^2 G}{\partial r^2}(dr)^2 + \frac{\partial^2 G}{\partial p_1{}^2}(dp_1)^2 + \frac{\partial^2 G}{\partial x^2}(dx)^2\right], \quad (16)$$

and making use of (2), (3) and (6) one obtains

$$dG = G_t dt + G_r[\alpha(\beta-r)dt + \sigma dZ_1] + G_{p_1}[p_1(\mu dt + \lambda dZ_2)] + G_x\{[(\mu+d)-(r+\sigma+\theta)]\pi + rx - k]dt + \lambda\pi dz_2\} + G_{p_1 x}\lambda^2\pi^2 p_1 dt + \frac{1}{2}\left[G_{rr}\sigma^2 dt + G_{p_1p_1}\lambda^2 p_1{}^2 dt + G_{xx}x^2\pi^2 dt\right]. \quad (17)$$

Therefore the Hamilton-Jacobi-Bellman equation is

$$\frac{k^{1-b}}{1-b} + G_t + \mu p_1 G_{p_1} + \alpha(\beta-r)G_r + \{[(\mu+d)-(r+\vartheta+\theta)]\pi + rx - k\}G_x + \lambda^2\pi p_1 G_{p_1 x} + \frac{1}{2}\left[\sigma^2 G_{rr} + \lambda^2 p_1{}^2 G_{p_1p_1} + \lambda^2\pi^2 G_{xx}\right] - \zeta G = 0. \quad (18)$$

G_t, G_{ρ_1}, G_r *and* G_x are first partial derivatives with respect to t, p_1, r and x respectively and $G_{p_1x}, G_{rr}, G_{p_1p_1}$ and G_{xx} are second order partial derivatives. The boundary condition is such that

$$G(T,r,p_1,x) = U(X(T)).$$

The differentiation of (18) with respect to π gives

$$[(\mu+d)-(r+\vartheta+\theta)]G_x + \lambda^2 p_1 G_{p_1x} + \lambda^2\pi G_{xx} = 0, \quad (19)$$

from which the optimal value of π is obtained as

$$\pi^* = \frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} - \frac{p_1 G_{p_1x}}{G_{xx}}. \quad (20)$$

Differentiating (18) with respect to k gives

$$k^{-b} - G_x = 0, \quad (21)$$

from which the optimal rate of consumption is given as

$$k^* = (G_x)^{-\frac{1}{b}}. \quad (22)$$

Substituting (20) and (22) into (18) yields;

$$\frac{\left(G_x^{-\frac{1}{b}}\right)^{\frac{b-1}{b}}}{1-b} + G_t + \mu p_1 G_{p_1} + \alpha(\beta - r)G_r + \left[[(\mu + d) - (r + \vartheta + \theta)]\left[\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} - \frac{p_1 G_{p_1 x}}{G_{xx}}\right] + rx - (G_x)^{-\frac{1}{b}}\right]G_x - \lambda^2\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} - \frac{p_1 G_{p_1 x}}{G_{xx}}\right]^2 G_{xx}\Bigg] - \zeta G = 0, \quad (23)$$

which becomes

$$\frac{G_x^{1-\frac{1}{b}}}{1-b} + G_t + \mu p_1 G_{p_1} + \alpha(\beta - r)G_r + \left[\frac{-[(\mu+d)-(r+\vartheta+\theta)]^2 G_x}{\lambda^2 G_{xx}} - \frac{p_1[(\mu+d)-(r+\vartheta+\theta)]G_{p_1 x}}{G_{xx}} + rx\right]G_x - G_x^{1-\frac{1}{b}} - \lambda^2 p_1\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} + \frac{p_1 G_{p_1 x}}{G_{xx}}\right]G_{p_1} + \frac{\sigma^2}{2}G_{rr} + \frac{\lambda^2 p_1}{2}G_{p_1 p_1} + \frac{\lambda^2}{2}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} + \frac{p_1 G_{p_1 x}}{G_{xx}} + \frac{p_1[(\mu+d)-(r+\vartheta+\theta)]G_{p_1 x}}{\lambda^2 G_{xx}}\right] - \zeta G = 0. \quad (24)$$

Clearly equation (24) is second order partial differential equation. Therefore a solution of the structure

$$G(t, r, p_1, x) = \frac{x^{1-b}}{1-b}J(t, r, p_1) \quad (25a)$$

such that

$$J(T, r, p_1) = 1, \quad (25b)$$

is conjectured to eliminate dependency on x.
From (25a) we obtain the following

$$G_t = \frac{x^{1-b}}{1-b}J_t, G_x = x^{-b}J, G_{xx} = -bx^{-b-1}J, G_{p_1 x} = x^{-b}J_{p_1},$$
$$G_{p_1} = \frac{x^{1-b}}{1-b}J_{p_1}, G_{p_1 p_1} = \frac{x^{1-b}}{1-b}J_{p_1 p_1}, G_r = \frac{x^{1-b}}{1-b}J_r, G_{rr} = \frac{x^{1-b}}{1-b}J_{rr}. \quad (25c)$$

Using the equivalents of $G_x, G_{p_1 x}$ and G_{xx} from (25c) in (20) yields

$$\pi^* = \frac{x}{b}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + c\frac{J_{p_1}}{J}\right]. \quad (26)$$

Also applying the equivalent G_x from (25c) in (22) gives

$$k^* = (x^{-b}J)^{-\frac{1}{b}} = xJ^{-\frac{1}{b}}. \quad (27)$$

Using (25a) and (25c) in (24) gives

$$\frac{b}{1-b}x^{1-b}J^{\frac{b-1}{b}} + \frac{x^{1-b}}{1-b}J_t + \mu p_1\frac{x^{1-b}}{1-b}J_{p_1} + \alpha(\beta - r)\frac{x^{1-b}}{1-b}J_r + \frac{[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2}x^{-b}J + \frac{3}{2}p_1\frac{[(\mu+d)-(r+\vartheta+\theta)]}{b}\frac{x^{1-b}}{1-b}(1-b)J_{p_1} + \frac{\lambda^2 p_1^2}{2b}\frac{x^{1-b}}{1-b}(1-b)\frac{J_{p_1}^2}{J} + \frac{\sigma^2}{2}\frac{x^{1-b}}{1-b}J_{rr} + \frac{x^{1-b}}{1-b}\frac{\lambda^2 p_1^2}{2}J_{p_1 p_1} - \zeta\frac{x^{1-b}}{1-b}J = 0, \quad (28)$$

which simplifies to

$$\frac{b}{1-b}J^{\frac{b-1}{b}} + J_t + \mu p_1 J_{p_1} + \alpha(\beta - r)J_t + \frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2 J}{2b\lambda^2} + \frac{3}{2}\frac{(1-b)p_{1[(\mu+d)-(r+\vartheta+\theta)]}}{b}J_{p_1} + \frac{(1-b)\lambda^2 p_1^2}{2b}\frac{J_{p_1}^2}{J} + \frac{\sigma^2}{2}J_{rr} + \frac{\lambda^2 p_1^2}{2}J_{p_1 p_1} - \zeta J = 0. \quad (29)$$

Equation (28) further simplifies to

$$\frac{b}{1-b}J^{\frac{b-1}{b}} + J_t + \mu p_1 J_{p_1} + \alpha(\beta - r)J_t + \frac{3}{2}\frac{(1-b)p_{1[(\mu+d)-(r+\vartheta+\theta)]}}{b}J_{p_1} + \frac{(1-b)\lambda^2 p_1^2}{2b}\frac{J_{p_1}^2}{J} + \frac{\sigma^2}{2}J_{rr} + \frac{\lambda^2 p_1^2}{2}J_{p_1 p_1}\left[\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} - \zeta\right]J = 0, \quad (30)$$

which is a second order partial differential equation. Sequel to (30), the conjecture that

$$J(t, r, p_1) = H(t, r)\frac{p_1^{1-b}}{1-b}, \quad (31a)$$

is made to eliminate the dependency on p_1 such that

$$H(T, r) = \frac{1-b}{p_1^{1-b}}, \quad (31b)$$

at the terminal time, T. Obtained from (30a) are

$$J_t = \frac{p_1^{1-b}}{1-b}H_t; J_r = \frac{p_1^{1-b}}{1-b}H_r; J_{p_1} = p_1^{-b}H; J_{p_1 p_1} = -bp_1^{-b-1}H; J_{rr} = \frac{p_1^{1-b}}{1-b}H_{rr}. \quad (31c)$$

Applying the equivalent of J_{p_1} from (31c) and (31a) to (26) gives the optimal investment in the risky asset as

$$\pi^* = \frac{x}{b}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1-b)\right]. \quad (32)$$

Also applying (31a) to (27) yields;

$$k^* = x(1-b)^{\frac{1}{b}}H^{-\frac{1}{b}}p_1^{\frac{b-1}{b}}. \quad (33)$$

Using (31a) and (31c) in (30) gives

$$\frac{b}{1-b}\left[\frac{p_1^{1-b}}{1-b}H\right]^{\frac{b-1}{b}} + \frac{p_1^{1-b}}{1-b}H_t + \mu\, p_1\, \rho_1^{1-b}H + \alpha(\beta - r)\frac{p_1^{1-b}}{1-b}H_r + \frac{3}{2b}(1-b)p_{1[(\mu+d)-(r+\vartheta+\theta)]}p_1 p_1^{-b}H + \frac{(1-b)\lambda^2}{2b}\frac{p_1^2\, p_1^{-2b}}{\rho_1^{1-b}}\frac{H^2}{H}(1-b) + \frac{\sigma^2}{2}\frac{p_1^{1-b}}{1-b}H_{rr} + \frac{\lambda^2 p_1^2}{2}(-b)\,p_1^{-b-1}H + \left[\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} - \zeta\right]\frac{p_1^{1-b}}{1-b}H = 0, \quad (34)$$

that simplifies to

$$\left[\frac{p_1{}^{1-b}}{(1-b)}\right]^{-\frac{1}{b}}\frac{b}{1-b}H^{\frac{b-1}{b}}+H_t+[(1-b)\mu\frac{3}{2b}(1-b)^2[(\mu+d)-(r+\vartheta+\theta)]-\frac{b(1-b)\lambda^2}{2}+\frac{(1-b)^3\lambda^2}{2b}+\left[\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2}-\zeta\right]\Bigg]H+\alpha(\beta-r)H_r+\frac{\sigma^2}{2}H_{rr}=0. \quad (35)$$

Equation (35) is a second order partial differential equation in r. To contend with this situation, the conjecture that

$$H(t,r)=I(t)\frac{r^{1-b}}{1-b}, \quad (36a)$$

such that at terminal time, T

$$I(T)=\frac{(1-b)^2}{(rp_1)^{1-b}}, \quad (36b)$$

is made to eliminate the dependency on r.
From (36a) we obtain

$$H_t=\frac{r^{1-b}}{1-b}\frac{dI}{dt};H_r=r^{-b}I;H_{rr}=-br^{-b-1}I. \quad (36c)$$

Applying (36a) to (33) obtains

$$k^*=x(1-b)^{\frac{2}{b}}(p_1r)^{\frac{b-1}{b}}I^{-\frac{1}{b}}. \quad (37)$$

Using (36a) and (36c) in (35) gives

$$\frac{b}{1-b}\left[\frac{p_1{}^{1-b}}{(1-b)}\right]^{-\frac{1}{b}}\left[\frac{r^{1-b}}{1-b}\right]^{\frac{1-b}{b}}+\frac{r^{1-b}}{1-b}\frac{dI}{dt}+\Bigg[[(1-b)\mu\frac{3}{2b}(1-b)^2[(\mu+d)-(r+\vartheta+\theta)]-(1-b)\lambda^2\frac{(b^2+2b-1)}{2}+\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2}-\zeta\Bigg]\frac{r^{1-b}}{1-b}I+\alpha(\beta-r)r^{-b}I+\frac{\sigma^2}{2}(-b)r^{-b-1}I=0. \quad (38)$$

Equation (38) on further simplification gives

$$\frac{r^{\frac{b-1}{b}}}{(1-b)^{2b-3}}bp_1{}^{1-\frac{1}{b}}I^{\frac{b-1}{b}}+\frac{dI}{dt}+\Bigg[(1-b)\mu\frac{3}{2b}(1-b)^2[(\mu+d)-(r+\vartheta+\theta)]-(1-b)\lambda^2\frac{(b^2+2b-1)}{2}+\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2}-\zeta+\frac{\alpha(\beta-r)(1-b)}{r}-\frac{b\sigma^2}{2}\frac{(1-b)}{r^2}\Bigg]I=0, \quad (39)$$

which now takes the form

$$A(t)I^n+\frac{dI}{dt}+B(t)I=0, \quad (40a)$$

a Bernoulli equation, where

$$n=\frac{b-1}{b};A(t)=\frac{r^{\frac{b-1}{b}}}{(1-b)^{2b-3}}bp_1{}^{1-\frac{1}{b}};\ B(t)=\Bigg[(1-b)\mu\frac{3}{2b}(1-b)^2[(\mu+d)-(r+\vartheta+\theta)]-(1-b)\lambda^2\frac{(b^2+2b-1)}{2}+\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2}-\zeta+\frac{\alpha(\beta-r)(1-b)}{r}-\frac{b\sigma^2}{2}\frac{(1-b)}{r^2}\Bigg]. \quad (40b)$$

Putting

$$M=I^{1-n}, \quad (41a)$$

such that

$$\frac{1}{1-n}\frac{dM}{dt}=I^{-n}\frac{dI}{dt}. \quad (41b)$$

Equation (40a), on rearranging, becomes

$$\frac{dM}{dt}+(1-n)B(t)M=(n-1)A(t), \quad (42)$$

which is a first order ordinary differential equation in which the variable M has taken the place of I. The solution can be obtained using theorem below;

Theorem 3.2: If $Q(t) and R(t)$ are continuous on the interval $[t,T]$, then the general solution of $M(t)$ of $\frac{dM}{dt}+Q(t)M=R(t)$ on (t,T) is given by

$$M(t)=e^{-\int_t^T Q(t)dt}\left[\int_t^T R(s)\,e^{\int_t^T Q(t)dt}ds+C\right], \quad (43)$$

Myint [24].
Therefore, the solution to (40) is

$$M(t)=e^{-\int_t^T(1-n)B(\tau)d\tau}\left[\int_t^T(n-1)A(s)e^{\int_t^T(1-n)B(\tau)d\tau}ds+C\right], \quad (44a)$$

which further becomes

$$M(t)=e^{(n-1)\int_t^T B(\tau)d\tau}\left[(n-1)\int_t^T A(s)\,e^{(1-n)\int_t^T B(\tau)d\tau}ds+C\right]. \quad (44b)$$

Applying the boundary condition, equation (34b)

$$M(T)=I^{1-n}(T)=\left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}=C. \quad (45)$$

Therefore

$$M(t)=e^{(n-1)\int_t^T B(\tau)d\tau}\left[(n-1)\int_t^T A(s)\,e^{\int_t^T(1-n)B(\tau)d\tau}ds+\left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}\right]. \quad (46)$$

Using (25a), (31a), (35a) and (46) the optimal value function for the investor`s problem is given as

$$G^*(t,r,p_1,x)=\frac{(xp_1r)^{1-b}}{(1-b)^2}\left[e^{(n-1)\int_t^T B(\tau)d\tau}\left[(n-1)\int_t^T A(s)\,e^{\int_t^T(1-n)B(\tau)d\tau}ds+\left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}\right]\right]^{\frac{1}{1-n}}, \quad (47)$$

which at the terminal time T, becomes

$$G^*(T,r,p_1,x)=\frac{(xp_1r)^{1-b}}{(1-b)^3}\left[\left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}\right]^{\frac{1}{1-n}}$$
$$=\frac{(xp_1r)^{1-b}(1-b)^2}{(1-b)^3(rp_1)^{1-b}}$$
$$=\frac{x^{1-b}}{1-b}$$

as expected.
The optimal rate of consumption is

$$k^*=x(1-b)^{\frac{2}{b}}(rp_1)^{\frac{b-1}{b}}\left[(n-1)\int_t^T A(s)\,e^{\int_t^T(1-n)B(\tau)d\tau}ds+\left[\frac{(1-b)^2}{(rp_1)^{1-b}}\right]^{1-n}\right]^{\frac{1}{1-n}}. \qquad (48)$$

3.2. CASE 2: When the Brownian Motionscorrelate

In this section, the theorem below follows:

THEOREM 3.3: An investor who has a power utility preference;$U(x)-\frac{x^{1-b}}{1-b,b\neq 1}$ and a risk-free asset which rate of return is driven by the Ornstein-Uhlenbeck model has the optimal policy that maximizes his expected utility of terminal wealth and consumption at the terminal time T, investing at any time$t$, in the risky asset.

$$\pi^*=\frac{x}{b}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{b\lambda^2}+(1-b)\left(1+\frac{\rho\sigma}{r\lambda}\right)\right]$$

with optimal consumption

$$k^*=x(1-b)^{\frac{2}{b}}(r\rho_1)^{\frac{b-1}{b}}(n-1)\int_t^T C(s)\,e^{(1-n)\int_t^T D(t)dt}ds+\left[\frac{(1-b)^2}{(r\rho_1)^{1-b}}\right]^{1-n}\Bigg]^{\frac{1}{1-r}}$$

and value function given by

$$G^*(t,r,\rho_1,x)=\frac{(x\rho_1r)^{1-b}}{(1-b)^2}\left[e^{(n-)\int_t^T D(t)dt}\left[(n-1)\int_t^T C(s)\,e^{(1-n)\int_t^T D(t)dt}ds+\left[\frac{(1-b)^2}{(r\rho_1)^{1-b}}\right]^{1-n}\right]^{\frac{1}{1-n}}\right.$$

Proof: The Hamilton- Jacob-Bellman equation (HJB) corresponding to this situation is given as

$$\frac{k^{1-b}}{1-b}+G_t+\mu p_1G_{p_1}+\alpha(\beta-r)G_r+\{[(\mu+d)-(r+\vartheta+\theta)]\pi+rx-k\}G_x+\rho\sigma xp_1G_{rp_1}+\rho\sigma\lambda\pi G_{rx}+\lambda^2\pi_{p_1}G_{p_1}x+\frac{1}{2}\left[\sigma^2G_{rr}+\lambda^2{p_1}^2G_{p_1p_1}+\lambda^2\pi^2G_{xx}\right]-\zeta G=0, \qquad (49)$$

G_t, G_{ρ_1} *and* $G_x$are first partial derivatives $G_{r\rho}, G_{rx}, G_{\rho_1x}, G_{rr}, G_{\rho_1\rho_1}$ *and*G_{xx} are second order partial derivatives.
Differentiating (49) with respect to π gives the optimal investment in the risky asset as;

$$\pi^*=\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2G_{xx}}-\frac{\rho\sigma}{\lambda}\frac{G_{rw}}{G_{xx}}-\frac{p_1G_{p_1}}{G_{xx}}. \qquad (50)$$

Again differentiating(49) with respect to $k$gives

$$k^{-b}-G_x=0, \qquad (51a)$$

from which

$$k^*=(G_x)^{-\frac{1}{b}}. \qquad (51b)$$

The substitution in (49) using (51a) and (51b) yields.

$$\frac{(G_x^{-\frac{1}{b}})^{1-b}}{1-b}+G_t+\mu p_1+\alpha(\beta-r)G_r+\left\{[(\mu+d)-(r+\vartheta+\theta)]\left[\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2G_{xx}}-\frac{\rho\sigma}{\lambda}\frac{G_{rw}}{G_{xx}}-\frac{p_1G_{p_1}}{G_{xx}}\right]+rx-(G_x)^{-\frac{1}{b}}\right\}G_x+\rho\sigma\lambda_{p_1}G_{rp_1}+\rho\sigma\left[\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2G_{xx}}-\frac{\rho\sigma G_{rw}}{\lambda G_{xx}}-\frac{p_1G_{p_1}}{G_{xx}}\right]G_{rx}+\lambda^2p_1\left[\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2G_{xx}}-\frac{\rho\sigma G_{rw}}{\lambda G_{xx}}-\frac{p_1G_{p_1}}{G_{xx}}\right]G_{p_1x}+\frac{\sigma^2}{2}G_{rr}+\frac{\lambda^2{\rho_1}^2}{2}G_{p_1p_1}+\frac{\lambda^2}{2}\left[\frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2G_{xx}}-\frac{\rho\sigma G_{rw}}{\lambda G_{xx}}-\frac{\rho_1G_{\rho_1}}{G_{xx}}\right]^2G_{xx}-\zeta G=0. \qquad (52)$$

The simplification of (52) gives;

$$\left[\frac{(G_x^{-\frac{1}{b}})^{1-b}}{1-b}-{G_x}^{1-\frac{1}{b}}\right]+G_t+\mu p_1G_{\rho_1}+\alpha(\beta-r)G_r-\frac{[(\mu+d)-(r+\vartheta+\theta)]^2}{\lambda^2}\frac{{G_x}^2}{G_{xx}}-\frac{3\rho\sigma}{\lambda}[(\mu+d)-(r+\vartheta+\theta)]\frac{G_xG_{rx}}{G_{xx}}-3p_1[(\mu+d)-(r+\vartheta+\theta)]\frac{G_xG_{p_1x}}{G_{xx}}+rxG_x+\rho\sigma\lambda p_1G_{rp_1}-\frac{\rho^2\sigma^2}{2}\frac{G^2rx}{G_{xx}}-3\rho\sigma\lambda p_1\frac{G_{rx}G_{p_1}}{G_{xx}}-\frac{3\lambda^2{p_1}^2}{2}\frac{{G_{p_1}}^2x}{G_{xx}}-\frac{[(\mu+d)-(r+\vartheta+\theta)]}{2\lambda^2}\frac{{G_x}^2}{G_{xx}}+\frac{\sigma^2}{2}G_{rr}+\frac{\lambda^2{p_1}_1^2}{2}G_{p_1p_1}-\zeta G=0, \qquad (53)$$

which is clearly a second order partial equation. To cope with this, it is conjectured that a solution of the form

$$G(t,r,p_1,x)=\frac{x^{1-b}}{1-b}J(t,r,p_1), \qquad (54a)$$

such that

$$J(T,r,p_1,)=1. \qquad (54b)$$

This eliminate dependency on x
From (54a) obtain

$$G_t = \frac{x^{1-b}}{1-b}J_t, G_t = x^{-b}J, G_{xx} = -bx^{-b-1}J, G_{p_1x} = x^{-b}J_{p_1},$$
$$G_{\rho_1} = \frac{x^{1-b}}{1-b}J_{p_1}, G_{p_1p_1} = \frac{x^{1-b}}{1-b}J_{p_1p_1}, G_r = \frac{x^{1-b}}{1-b}J_r, G_{rr} = \frac{x^{1-b}}{1-b}J_{rr}, G_{rx} = x^{-b}J_r, G_{rp_1} = \frac{x^{1-b}}{1-b}J_{rp_1}. \quad (54c)$$

Applying the equivalents of G_x, G_{rx}, G_{ρ_1x} *and* G_{xx} from (54c) yields

$$\pi^* = \frac{[(\mu+d)-(r+\vartheta+\theta)]x}{b\lambda^2} + \frac{\rho\sigma x J_r}{b\lambda J} + \frac{p_1 x J_{p_1}}{b\ J}. \quad (55)$$

Using the equivalent of G_x from (54c) in (51b0 gives

$$k^* = xJ^{-\frac{1}{b}}. \quad (56)$$

The application of (54a) and (54c) in (53) gives

$$\left[\frac{(x^{-b}J)^{-\frac{1}{b}(1-b)}}{1-b} - (x^{-b}J)^{1-\frac{1}{b}}\right] + \frac{x^{1-b}}{1-b}J_t + \frac{x^{1-b}}{1-b}\mu p_1 + \frac{x^{1-b}}{1-b}\alpha(\beta - r)J_r + \frac{[(\mu+d)-(r+\vartheta+\theta)]^2}{2\lambda^2}\frac{(1-b)x^{1-b}}{(1-b)b}J + \frac{3\rho\sigma}{\lambda}[(\mu+d) - (r+\vartheta+\theta)]\frac{(1-b)x^{1-b}}{1-b}J_r + 3p_1[(\mu+d)-(r+\vartheta+\theta)]\frac{(1-b)x^{1-b}}{1-b}J_{p_1} + \frac{rx^{1-b}}{1-b}(1-b)J + \rho\sigma\lambda p_1\frac{x^{1-b}}{1-b}J_{rp_1} + \frac{\rho^2\sigma^2(1-b)}{2b(1-b)}\frac{x^{1-b}J_r{}^2}{J} + \frac{3\rho\sigma\lambda p_1(1-b)x^{1-b}}{b\ J}J_rJ_{p_1} + \frac{3}{2}\lambda^2 p_1{}^2\frac{(1-b)x^{1-b}J_{p_1}}{b(1-b)J} + \frac{\sigma^2}{2}\frac{x^{1-b}}{1-b}J_{rr} + \frac{\lambda^2 p_1{}^2}{2}\frac{x^{1-b}}{1-b}J_{p_1p_1} - \zeta\frac{x^{1-b}}{1-b}J = 0.$$
(57)

which simplifies to

$$\frac{b}{1-b}J^{\frac{b-1}{b}} + J_t + \left[\mu p_1 + \frac{\zeta[(\mu+d)-(r+\vartheta+\theta)]}{b}\right]J_{p_1} + \left[\alpha(\beta - r) + \zeta(1-b)\frac{\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda}\right]J_r + \left[\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} + (1-b)r - \zeta\right]J + \rho\sigma\lambda p_1 J_{rp_1} + \frac{(1-b)\rho^2\sigma^2}{2b}\frac{J_r{}^2}{J} - \frac{3(1-b)\rho\sigma\lambda p_1}{b}\frac{J_rJ_{\rho_1}}{J} + \frac{3}{2}\frac{\lambda^2 p_1{}^2(1-b)}{b}\frac{J_{p_1}{}^2}{J} + \frac{\sigma^2}{2}J_{rr} + \frac{\lambda^2 p_1{}^2}{2}J_{p_1p_1} = 0. \quad (58)$$

Equation (57b) is again second order differential equation. Therefore the conjecture that

$$J(t,r,\rho_1) = H(t,r)\frac{\rho^{1-b}}{1-b} \quad (59a)$$

is made such that

$$H(T,r) = \frac{1-b}{\rho^{1-b}}, \quad (59b)$$

at the terminal time t, and dependency on ρ_1 eliminated.
Obtained from (59a) are

$$J_t = \frac{p_1{}^{1-b}}{1-b}H_t; J_r = \frac{p_1{}^{1-b}}{1-b}H_r; J_{p_1} = p_1{}^{-b}H; J_{p_1p_1} = -bp_1{}^{-b-1}H, J_{rr} = \frac{p_1{}^{1-b}}{1-b}H_{rr}\ and\ J_{rp_1} = p_1{}^{-b}H_r \quad (59c)$$

The application of the equivalents of J_r *and* J_{ρ_1} from (59ac) and (59a) to (55) gives the following

$$\pi^* = \frac{[(\mu+d)-(r+\vartheta+\theta)]x}{b\lambda^2} + \frac{1-b}{b}x + \frac{\rho\sigma x}{b\lambda}\frac{H_r}{H}, \quad (60)$$

for the optimal investment is the risky asset.
Applying (59a) to (56) gives;

$$k^* = x\left[\frac{p_1{}^{1-b}}{1-b}H\right]^{-\frac{1}{b}} \quad (61)$$
$$= x(1-b)^{\frac{1}{b}}p_1{}^{\frac{b-1}{b}}H^{-\frac{1}{b}}. \quad (62)$$

Using (59a) and (59c) in (58) yields

$$\frac{b}{1-b}\left[\frac{p_1{}^{1-b}}{1-b}H\right]^{\frac{b-1}{b}} + \frac{p_1{}^{1-b}}{1-b}H_t + \left[\mu p_1 + \frac{3(1-b)[(\mu+d)-(r+\vartheta+\theta)]}{b}\right]\frac{p_1{}^{1-b}}{p_1(1-b)}(1-b)H + \left[\alpha(\beta - r)\frac{3(1-b)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda}\right] + \frac{p_1{}^{1-b}}{1-b}H + \rho\sigma\frac{\lambda p_1{}^{1-b}}{1-b}(1-b)H_r + \frac{(1-b)\rho^2\sigma^2}{2b}\frac{p_1{}^{1-b}}{1-b}\frac{H_r{}^2}{H} + \frac{3(1-b)\rho\sigma}{b}\frac{\lambda p_1{}^{1-b}}{(1-b)}(1-b)H_r + \frac{3}{2}\frac{\lambda^2 p_1{}^2(1-b)}{b}\frac{p_1{}^{-2b}}{p_1{}^{1-b}}H\frac{(1-b)}{(1-b)} + \frac{\sigma^2}{2}\frac{p_1{}^{1-b}}{1-b}H_{rr} + \frac{\lambda^2 p_1{}^2}{2}(-b)p_1{}^{-b-1}H{=}0. \quad (63)$$

This simplifies to

$$\left[\frac{p_1{}^{1-b}}{1-b}\right]^{-\frac{1}{b}}\frac{b}{1-b}H^{\frac{b-1}{b}} + H_t + \left[(1-b)\left[\mu p_1 + \frac{3(1-b)[(\mu+d)-(r+\vartheta+\theta)]}{b}\right] + \left[\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} + (1-b)r - \zeta\right] + \frac{3}{2}\frac{(1-b)^2\lambda^2 p_1{}^2}{b} - \frac{b(1-b)\lambda^2 p_1{}^2}{b} - \frac{b(1-b)\lambda^2 p_1{}^2}{2}\right]H + \left[\alpha(\beta - r)\frac{3(1-b)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda} + (1-b)\rho\sigma\lambda + \frac{3(1-b)^2\rho\sigma\lambda}{b}\right]H_r + \frac{(1-b)\rho^2\sigma^2}{2b}\frac{H_r{}^2}{H} + \frac{\sigma^2}{2}H_{rr} = 0. \quad (64)$$

Equation (64) is yet a second order partial differential equation unit. To eliminate the dependency on r, the conjecture that

$$H(t,r) = I(t)\frac{r^{1-b}}{1-b} \quad (65a)$$

is used such that

$$I(T) = \frac{(1-b)^2}{(r\rho_1)^{1-b}}, \quad (65b)$$

at the terminal time T
From (65a) one obtains

$$H_t = \frac{r^{1-b}}{1-b}\frac{dI}{dt}; H_r = r^{-b}I; H_{rr} = -br^{-b-1}I. \quad (65c)$$

Applying the equivalent of H_r from (65c) and (65a) to (60) yields.

$$\pi^* = \frac{[(\mu+d)-(r+\vartheta+\theta)]}{b\lambda^2} + \frac{1-b}{b}x + \frac{1-b}{r}\frac{\rho\sigma x}{b\lambda} = \frac{x}{b}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1-b)\left(1+\frac{\rho\sigma}{r\lambda}\right)\right]. \quad (66)$$

The optimal strategy investing in the risky asset

Using (65a) in (62) gives

$$k^* = x(1-b)^{-\frac{1}{b}}p_1^{\frac{b-1}{b}}\left[\frac{r^{1-b}}{1-b}I\right]^{-\frac{1}{b}} = x(1-b)^{\frac{2}{b}}(p_1 r)^{\frac{b-1}{b}}I^{-\frac{1}{b}}. \quad (67)$$

The application of (65a) and (65c) in (64) gives

$$\frac{b}{1-b}\left[\frac{p_1^{1-b}}{1-b}\right]^{-\frac{1}{b}}\left[\frac{r^{1-b}}{1-b}I\right]^{\frac{1-b}{b}} + \frac{r^{1-b}}{1-b}\frac{dI}{dt} + \left[(1-b)\left[\mu p_1 + \frac{3(1-b)[(\mu+d)-(r+\vartheta+\theta)]}{b}\right]\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} + +(1-b)r - \zeta + \frac{3}{2}\frac{(1-b)^2\lambda^2 p_1{}^2}{b} - \frac{b(1-b)\lambda^2 p_1{}^2}{b}\right]\frac{r^{1-b}}{1-b}I + \left[\alpha(\beta-r)\frac{3(1-b)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda} + (1-b)\rho\sigma\lambda + \frac{3(1-b)^2\rho\sigma\lambda}{b}\right]r^{-b}I + \frac{(1-b)^2\rho^2\sigma^2}{2b}\frac{r^{-2b}}{r^{1-b}}\frac{I^2}{I} - \frac{b\sigma^2 r^{-b-1}}{2}I = 0. \quad (68)$$

The above equation (68) simplifies to

$$\frac{r^{\frac{b-1}{b}}}{(1-b)^{2b-3}}bp_1^{\frac{b-1}{b}} + \frac{dI}{dt} + \left[(1-b)\left[\mu p_1 + \frac{3(1-b)[(\mu+d)-(r+\vartheta+\theta)]}{b}\right]\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} + (1-b)r - \zeta + \frac{3}{2}\frac{(1-b)^2\lambda^2 p_1{}^2}{b} - \frac{b(1-b)\lambda^2 p_1{}^2}{b} + \frac{(1-b)}{b}\left[\alpha(\beta-r)\frac{3(1-b)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda} + (1-b)\rho\sigma\lambda + \frac{3(1-b)^2\rho\sigma\lambda}{b}\right] + \frac{(1-b)^3}{r}\frac{\rho^2\sigma^2}{2b} + \frac{(1-b)b\sigma^2}{2r^2}\right]I = 0 \quad (69)$$

and further takes the form

$$C(t)I^n + \frac{dI}{dt} + D(t)I = 0, \quad (70a)$$

where $\frac{b-1}{b} = n, C(t) = \frac{r^{\frac{b-1}{b}}}{(1-b)^{2b-3}}b\rho_1^{\frac{b-1}{b}};$

$$D(t) = \left[(1-b)\left[\mu p_1 + \frac{3(1-b)[(\mu+d)-(r+\vartheta+\theta)]}{b}\right]\frac{(1-b)[(\mu+d)-(r+\vartheta+\theta)]^2}{2b\lambda^2} + (1-b)r - \zeta + \frac{3}{2}\frac{(1-b)^2\lambda^2 \rho_1{}^2}{b} - \frac{b(1-b)\lambda^2 p_1{}^2}{b} + \frac{(1-b)}{b}\left[\alpha(\beta-r)\frac{3(1-b)\rho\sigma[(\mu+d)-(r+\vartheta+\theta)]}{\lambda} + (1-b)\rho\sigma\lambda + \frac{3(1-b)^2\rho\sigma\lambda}{b}\right] + \frac{(1-b)^3}{r^2}\frac{\rho^2\sigma^2}{2b} - \frac{(1-b)b\sigma^2}{2r^2}\right].$$

(70b)

Let

$$Z = I^{1-n}, \quad (71a)$$

such that

$$\frac{1}{1-n}\frac{dZ}{dt} = I^{-n}\frac{dI}{dt}. \quad (71b)$$

Equation (70a) on rearranging becomes

$$\frac{dZ}{dt} + (1-n)D(t)z = (n-1)C(t), \quad (72)$$

which a first order ordinary differentiation equation where the variable z has taken the place of I.

Using theorem 2 above, the solution of (72) is

$$Z(t) = e^{-\int_t^T (1-n)D(t)dt}\left[\int_t^T (n-1)C(s)e^{\int_t^T (1-n)D(t)dt}\,ds + F\right], \quad (73)$$

and using the boundary condition (65a)

$$Z(T) = I^{1-n}(T) = \left[\frac{(1-b)^2}{(r\rho_1)^{1-b}}\right]^{1-n} = F. \quad (74)$$

Therefore

$$Z(t) = e^{(n-)\int_t^T D(t)dt}\left[(n-1)\int_t^T C(s)\,e^{(1-n)\int_t^T D(t)dt}ds + \left[\frac{(1-b)^2}{(r\rho_1)^{1-b}}\right]^{1-n}\right]. \quad (75)$$

The optimal value function using (54a), (59a), (65a) and (75)is

$$G^*(t,r,\rho_1,x) = \frac{(x\rho_1 r)^{1-b}}{(1-b)^2}\left[e^{(n-)\int_t^T D(t)dt}\left[(n-1)\int_t^T C(s)\,e^{(1-n)\int_t^T D(t)dt}ds + \left[\frac{(1-b)^2}{(r\rho_1)^{1-b}}\right]^{1-n}\right]\right]^{\frac{1}{1-n}}$$

which at the terminal time T is

$$G^*(T,r,\rho_1,x) = \frac{x^{1-b}}{1-b}.$$

The optimal rate of consumption is

$$k^* = x(1-b)^{\frac{2}{b}}(r\rho_1)^{\frac{b-1}{b}}(n-1)\int_t^T C(s)\,e^{(1-n)\int_t^T D(t)dt}ds + \left[\frac{(1-b)^2}{(r\rho_1)^{1-b}}\right]^{1-n}\Bigg]^{\frac{1}{1-r}} \quad (76)$$

3.3. The Effect of Correlation of Brownian Motions

Let π^{*c} be the investment into the risky asset when the Brownian motions correlate and π^{*NC} when the Brownian motions do not correlate we have

$$\pi^{*c} = \frac{x}{b}\left[\frac{[(\mu + d) - (r + \vartheta + \theta)]}{\lambda^2} + (1 - b)\left(1 + \frac{\rho\sigma}{r\lambda}\right)\right]$$

and

$$\pi^{*NC} = \frac{x}{b}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1 - b)\right].$$

From the above we get

$$\pi^{*}c = \pi^{*NC} + \frac{(1-b)\rho\sigma}{1+r\lambda}. \qquad (77)$$

Equation (77) shows that π^{*c} is greater (less) than π^{*NC} as long as b is less (greater) than unity.
Let

$$\frac{(1-b)\rho\sigma}{1+r\lambda} = f\,\pi^{*NC} \qquad (78)$$

then

$$\frac{\pi^{*c}}{\pi^{*NC}} = \frac{1}{1+f}, \qquad (79a)$$

or

$$\pi^{*c}{:}\,\pi^{*NC} = 1{:}\,1 + f. \qquad (79b)$$

f is positive, if $b < 1$ or $b > 1$. The case, $b = 1$, which is not allowed, however gives π^{*c} equals π^{*NC}.

4. Conclusions

We have obtained a closed form solution to an investment and consumption decision problem where the risk-free asset has a rate of return that is driven by the Ornstein-Uhlenbeck Stochastic interest rate of return model. It is observed that:

I. The case of Brownian motions not correlating; equation (32), clearly shows that if the sum of the drift parameter and dividend rate equals the sum of the tax rate, transaction cost rate and the rate of the return of the risk-free asset, then, the optimal investment strategy on the risky asset becomes totally dependent on the relative risk aversion coefficient $'b'$ and the total amount available for investment. Also, the investment strategy is horizon dependent as x, and r are horizon dependent.

The optimal consumption as shown by (48) is a function of the total amount available for investment, the relative risk aversion coefficient $'b'$ the rate of return of the risk free asset and the price of the risky asset. It is also horizon dependent.

II. The case of correlating Brownian motions, it can be seen from equation (60) that the optimal investment in the risky asset is horizon dependent and is also a ratio of the total amount available for investment and the relative risk aversion coefficient. It is a function of $\mu, d, r, \vartheta, \theta, \leftthreetimes, \rho$ *and* σ. if the sum $(\mu + d)$ equals $(r + \vartheta + \theta)$, then it equals a function of b, ρ, σ, r *and* $\leftthreetimes$.

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