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Simulation of Riccati Differential Equations by Nonlocal Approximation of Nonlinear Terms and Reconstruction of Denominator Functions

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Abstract: Two ways to efficiently construct a Non-Standard Finite Difference Method (NSFDM) is to approximate the nonlinear term(s) of the differential equation nonlocally and also to reconstruct the denominator function(s). In this research, we shall simulate a special class of nonlinear differential equations called the Riccati Differential Equations (RDEs) by nonlocally approximating the nonlinear terms and also reconstructing the denominator functions. The need for this approach came up due to some shortcomings of existing methods in which the qualitative properties of the exact solutions are not usually transferred to the numerical (approximate) solutions. The approach developed in this research has the property that its solution does not exhibit numerical instabilities in view of the results generated.

Keywords: Denominator functions; Nonlinear; Nonlocal; RDEs; Simulation.
AMS Subject Classification: 65L05; 65L06; 65D30.

1. Introduction

Named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754), the RDEs find applications in random processes, optimal control and diffusion problem, Reid [1]. Besides its applications in engineering and science that today are considered classical, the RDE is also applied in financial mathematics Anderson and Moore [2], robust stabilization, stochastic realization theory, network synthesis and optimal control Riaz, *et al.* [3]. Also, according to Vahidi and Didgar [4], the RDE is an essential tool for modeling many physical situations such as spring mass systems, resistor-capacitor-induction circuits, bending of beams, chemical reactions, pendulum, the motion of rotating mass around body and so on.

In this paper, we shall develop a NSFDM for the simulation of RDEs of the form,

$$y'(t)=a(t)+b(t)y(t)+c(t)y^2(t) \tag{1}$$

with initial conditions,

$$y(t_0)=y_0 \tag{2}$$

where $a(t), b(t), c(t)$ are continuous with $c(t)\neq 0$ and $t_0,\ y_0$ are arbitrary constants for $y(t)$ which is an unknown function.

It is important to state that equation (1) must satisfy the existence and uniqueness theorem stated below.

Theorem 1.1 Henrici [5]

Let $f(t,y)$ be defined and continuous for all points (t,y) in the region D defined by $a\le t\le b, -\infty<y<\infty$, a *and* b finite, and let there exist a constant L such that, for every t,y,y^* such that (t,y) *and* (t,y^*) are both in D;

$$\left|f(t,y)-f(t,y^*)\right|\le L\left|y-y*\right| \tag{3}$$

Then, if η is any given number, there exists a unique solution $y(t)$ of the problem (1), where $y(t)$ is continuous and differentiable for all $(t,y)\, in\, D$. The requirement (3) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

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In recent years, to get reliable results with less effort, researchers have applied NSFDMs and obtained competitive results than those of the existing methods. These authors include, [6-8], [9, 10], Ibijola and Sunday [11], Sunday [12], [13, 14], among others.

It is important to note that the general form of NSFDM can be written as,

$$y_{n+1} = F(h, y_n) \tag{4}$$

Definition 1.1 Anguelov and Lubuma [9]

A finite difference scheme is called **non-standard finite difference method**, if at least one of the following conditions is met;

i) in the discrete derivative, the traditional denominator is replaced by a non-negative function ϕ such that,

$$\phi(h) = h + o(h^2),\ as\, h \to 0 \tag{5}$$

ii) non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable functions of several points of the mesh. For example,
$y^2 \approx y_n y_{n+1}, y_{n-1} y_n,\ y^3 \approx y_{n-1} y_n y_{n+1}, y_n^2 y_{n+1}$

We shall employ the following collection of rules set by Mickens [7] in developing NSFDM for RDEs.

- The order of the discrete derivatives must be exactly equal to the order of the corresponding derivatives of the differential equation.
- Denominator function for the discrete derivatives must be expressed in terms of more complicated function of the step-sizes than those conventionally used.
- The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many different ways. The non-linear terms y^2, y^3 can be modeled as follows

$$y^2 \cong y_n y_{n+1} \tag{6}$$

$$y^2 \cong y_n \left(\frac{y_{n+1} + y_n}{2}\right) \tag{7}$$

$$y^3 \cong y_n^2 y_{n+1} \tag{8}$$

$$y^3 \cong y_n^2 \left(\frac{y_{n+1} + y_n}{2}\right) \tag{9}$$

The particular form selected from equations (6) to (9) depends on the full discrete model.

- Special solutions of the differential equations should also be accompanied by special discrete solutions of the finite-difference models. For instance, an ordinary differential equation for which the substitution, $t \to -t$, leaves the equation invariant. If the discrete model does not also have this property, then numerical instabilities may occur.
- The finite-difference equation should not have solutions that do not correspond exactly to the solution of the differential equations.

For the purpose of this work, we shall assume that the function $F(h, y)$ in (4) has continuous derivatives with respect to both variables for h >0, $y \in R$ and that;

$$F(0, y) = y,\ \frac{\partial F(0, y)}{\partial h} = f(y) \tag{10}$$

It is necessary to note that consistency implies (10) if y is the solution of the differential equation (1).

Theorem 1.2 Anguelov and Lubuma [10]

The finite difference scheme (4) is stable with respect to monotone dependence on initial value, if

$$\frac{\partial F(h, y)}{\partial y} \geq 0, y \in R, h > 0 \tag{11}$$

2. Construction of Non-Standard Finite Difference Method for Riccati Differential Equations

We shall construct a NSFDM for RDEs of the form (1). This is achieved by representing the nonlinear term y^2 in equation (1) as follows,

$$\frac{y_{n+1} - y_n}{h} = a(t) + b(t) y_n + c(t) y_n y_{n+1} \tag{12}$$

That is, the nonlinear term y^2 in equation (1) is approximated by $y_n y_{n+1}$. From equation (12),

$$y_{n+1} - c(t)hy_n y_{n+1} = a(t)h + y_n + b(t)hy_n$$

Thus,

$$y_{n+1} = \frac{a(t)h + [1+b(t)h]y_n}{[1-c(t)hy_n]} \tag{13}$$

It is important to note that y_{n+1} is the value of the solution at $(n+1)^{th}$ time step, y_n is the value of the solution at n^{th} time step and h is the time stepping parameter. Equation (13) is a NSFDM (with trivial denominator) capable of solving RDEs of the form (1).

A more efficient method can be developed by replacing the denominator h in (13) with a denominator function $\phi(h)$ so that $\phi(h) \to 0$ as $h \to 0$. This nontrivial denominator helps in maintaining the positivity and stability of the solution. For the problems of the form (1), we approximate the denominator h as,

$$\phi(h) = 1 - e^{-h} \tag{14}$$

Substituting (14) in (13), we get

$$y_{n+1} = \frac{a(t)\left(1-e^{-h}\right) + \left[1+b(t)\left(1-e^{-h}\right)\right]y_n}{\left[1-c(t)\left(1-e^{-h}\right)y_n\right]} \tag{15}$$

Equation (15) is the NSFDM (with nontrivial denominator) capable of solving RDEs of the form (1). The nontrivial denominator function introduced in (15) helps in overcoming the unstable behavior of the NSFDM (with trivial denominator) in (13).

3. Results: Implementation of the NSFDM on RDEs

The NSFDM derived shall be used to simulate some RDEs to test its reliability and efficiency. The following notations shall be used in the Tables below:

ERR= Absolute error in NSFDM
Eval t =Evaluation time per seconds
EYH-Absolute error in Yang, *et al.* [15]
EFA-Absolute error in File and Aga [16]
ENB-Absolute error in Naeem, *et al.* [17]

Problem 3.1:

Consider the Riccati differential equation,

$$y'(t) = 1 - y^2(t) \tag{16}$$

with initial conditions,

$$y(0) = 0 \tag{17}$$

The exact solution to the problem is

$$y(t) = \frac{e^{2t}-1}{e^{2t}+1} \tag{18}$$

Source: Yang *et. al.* [15]

Applying equation (12) on (16), we get

$$\frac{y_{n+1} - y_n}{\phi(h)} = 1 - y_n y_{n+1} \tag{19}$$

Substituting equation (14) in (19) gives,

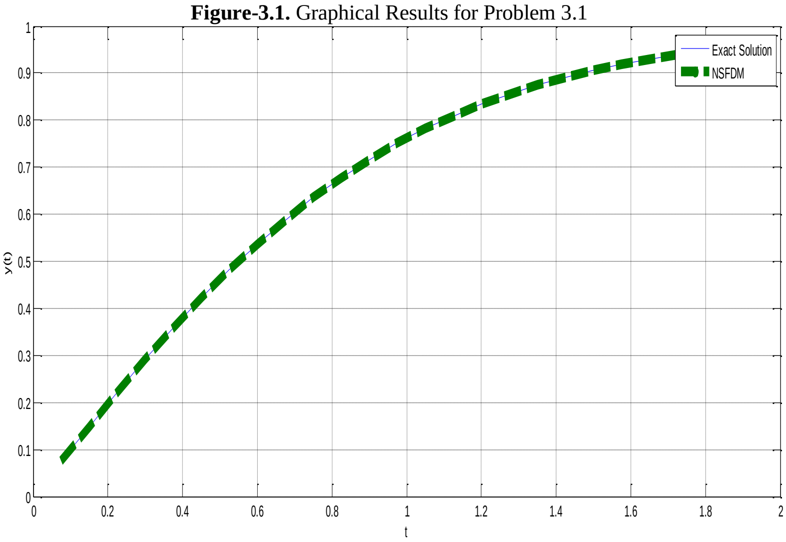
$$y_{n+1} = \frac{1-e^{-h} + y_n}{1+\left(1-e^{-h}\right)y_n} \tag{20}$$

Equation (20) is the NSFDM for the RDE in (16). On the application of equation (20) on (16), we obtain the numerical and graphical results presented in Table 3.1 and Figure 3.1 respectively.

Table-3.1. Showing the result for problem 3.1

t	Exact Solution	NSFDM	ERR	EYH	Eval t
0.1000	0.0996679946249558	0.0996679946249558	0.000000e+000	4.1401e-07	0.1259
0.2000	0.1973753202249040	0.1973753202249040	0.000000e+000	6.0186e-07	0.1277
0.3000	0.2913126124515909	0.2913126124515909	0.000000e+000	7.3747e-07	0.1294
0.4000	0.3799489622552250	0.3799489622552250	0.000000e+000	1.7322e-07	0.1311
0.5000	0.4621171572600099	0.4621171572600099	0.000000e+000	6.8524e-07	0.1328
0.6000	0.5370495669980354	0.5370495669980354	0.000000e+000	7.9810e-07	0.1453
0.7000	0.6043677771171637	0.6043677771171637	0.000000e+000	9.2621e-07	0.1470
0.8000	0.6640367702678491	0.6640367702678491	0.000000e+000	2.8318e-07	0.1487
0.9000	0.7162978701990247	0.7162978701990247	0.000000e+000	6.6469e-07	0.1504
1.0000	0.7615941559557652	0.7615941559557652	0.000000e+000	7.2660e-07	0.1521

Figure-3.1. Graphical Results for Problem 3.1



Problem 3.2

Consider the Riccati differential equation,

$$y'(t) = 1 + 2y(t) - y^2(t) \tag{21}$$

with the initial conditions,

$$y(0) = 0 \tag{22}$$

The exact solution is given by,

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{23}$$

Source: File and Aga [16]

Applying equation (12) on (21), we get

$$\frac{y_{n+1} - y_n}{\phi(h)} = 1 + 2y_n - y_n y_{n+1} \tag{24}$$

Substituting equation (14) in (24) gives,

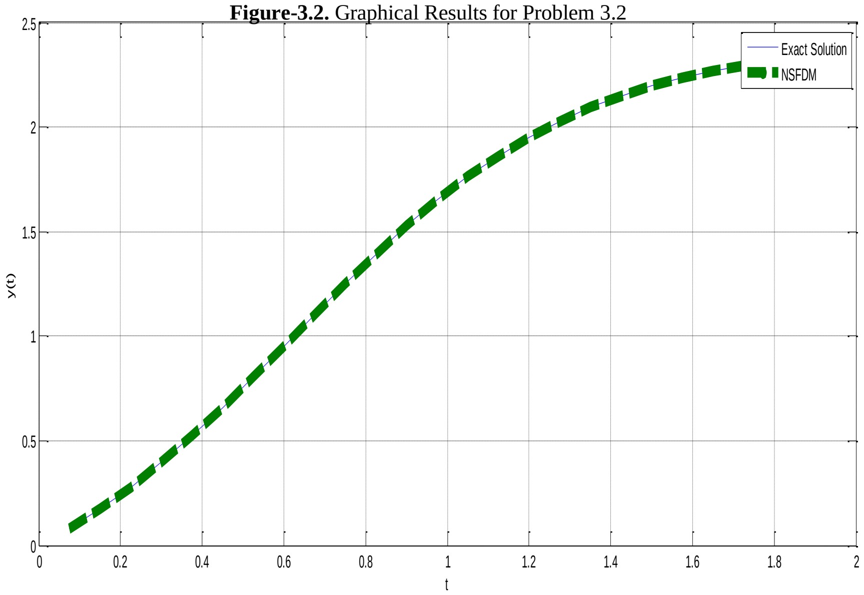
$$y_{n+1} = \frac{\left(1-e^{-h}\right)\left[1+2\left(1-e^{-h}\right)\right]y_n}{1+\left(1-e^{-h}\right)y_n} \tag{25}$$

Equation (25) is the NSFDM for the RDE in (21). On the application of equation (25) on (21), we obtain the numerical and graphical results presented in Table 3.2 and Figure 3.2 respectively.

Table-3.2. Showing the result for problem 3.2

t	Exact Solution	NSFDM	ERR	EFA	Eval t
0.1000	0.1102951969169624	0.1102951969169624	0.000000e+000	2.2551e-06	0.2436
0.2000	0.2419767996211093	0.2419767996211093	0.000000e+000	4.7763e-06	0.2454
0.3000	0.3951048486603785	0.3951048486603785	0.000000e+000	7.3083e-06	0.2472
0.4000	0.5678121662929389	0.5678121662929389	0.000000e+000	9.5635e-06	0.2490
0.5000	0.7560143934313761	0.7560143934313761	0.000000e+000	1.1301e-05	0.2508
0.6000	0.9535662164719235	0.9535662164719235	0.000000e+000	1.1301e-05	0.2526
0.7000	1.1529489669796242	1.1529489669796242	0.000000e+000	1.2408e-05	0.2545
0.8000	1.3463636553683762	1.3463636553683762	0.000000e+000	1.2940e-05	0.2565
0.9000	1.5269113132806256	1.5269113132806256	0.000000e+000	1.3100e-05	0.2584
1.0000	1.6894983915943840	1.6894983915943840	0.000000e+000	1.3245e-05	0.2602

Figure-3.2. Graphical Results for Problem 3.2



Problem 3.3

Consider the Riccati differential equation,

$$y'(t) = 10 + 3y(t) - y^2(t) \tag{26}$$

whose initial conditions are,

$$y(0) = 0 \tag{27}$$

The exact solution is given by,

$$y(t) = -2 + \frac{14e^{7t}}{5 + 2e^{7t}} \tag{28}$$

Source: Naeem, *et al.* [17]

Applying equation (12) on (26), we get

$$\frac{y_{n+1} - y_n}{\phi(h)} = 10 + 3y_n - y_n y_{n+1} \tag{29}$$

Substituting equation (14) in (29) gives,

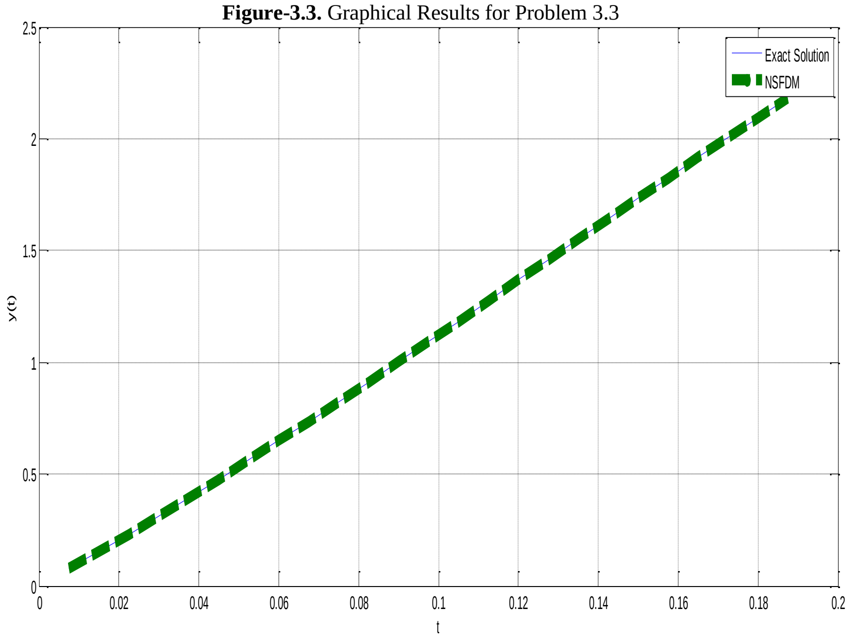
$$y_{n+1} = \frac{10\left(1-e^{-h}\right) + \left[3\left(1-e^{-h}\right)+1\right] y_n}{1+\left(1-e^{-h}\right) y_n} \tag{30}$$

Equation (30) is the NSFDM for the RDE in (26). On the application of equation (30) on (26), we obtain the numerical and graphical results presented in Table 3.3 and Figure 3.3 respectively.

Table-3.3. Showing the result for problem 3.3

t	Exact Solution	NSFDM	ERR	ENB	Eval t
0.1000	1.1229599550199856	1.1229599550199856	0.000000e+000	1.5×10^{-6}	0.0321
0.2000	2.3303636672393440	2.3303636672393440	0.000000e+000	3.2×10^{-6}	0.0493
0.3000	3.3592985913921902	3.3592985913921902	0.000000e+000	8.0×10^{-7}	0.0667
0.4000	4.0762561998939519	4.0762561998939519	0.000000e+000	3.2×10^{-6}	0.1056
0.5000	4.5086402379423145	4.5086402379423145	0.000000e+000	3.7×10^{-6}	0.1229
0.6000	4.7470598637518648	4.7470598637518648	0.000000e+000	9.7×10^{-7}	0.1419
0.7000	4.8720664654895440	4.8720664654895440	0.000000e+000	1.0×10^{-6}	0.1594
0.8000	4.9358801511182619	4.9358801511182619	0.000000e+000	8.5×10^{-7}	0.1766
0.9000	4.9680115179081801	4.9680115179081801	0.000000e+000	2.1×10^{-7}	0.1939
1.0000	4.9840783622386367	4.9840783622386367	0.000000e+000	1.4×10^{-6}	0.2985

Figure-3.3. Graphical Results for Problem 3.3



4. Conclusion

Conclusively, it is important to note that there are different ways of constructing the NSFDM for the solution of differential equations. In fact, according to Paridar [18], the construction of NSFDMs is not always straight forward and there are no general criteria for doing so.

It is clear from the results (numerical and graphical) generated above, that the NSFDM is a reliable and efficient method for the simulation of RDEs of the form (1). The results also show that the approximate solutions (obtained using the NSFDM) converges closely to the exact solutions. The evaluation time per seconds obtained were very small, showing that the method derived generates results very fast.

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