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A Result on the Behavior of Solutions of Second Order Delay Differential Equations

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Abstract: A wide class of second order linear autonomous delay differential equations with distributed type delay is considered. By the use of two distinet real roots of the corresponding characteristic equation, a new result on the behavior of the solutions is obtained.
Keywords: Delay differential equation; Characteristic equation; Roots; Asymptotic behavior.

1. Introduction

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations [1-3]. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. The theory of linear delay differential equations has been developed in the fundamental monographs [1], [2-6].

Our aim in this paper is to establish a new result for the solutions to second order linear delay differential equations with constant coefficients and constant delay. Analogous results for the solutions to second order linear delay differential equations has recently been obtained by the authors [7], [8-10] and Yeniçerioğlu [11]. Our work in the present paper is essentially motivated by the results in the papers by Philos and Purnaras [12-17].

Let us consider initial value problem for second order delay differential equation

$$y''(t) = p_1 y'(t) + p_2 y'(t-\tau) + q_1 y(t) + q_2 y(t-\tau)\ , \quad t \ge 0\,, \tag{1.1}$$

$$y(t) = \phi(t)\ , \quad -\tau \le t \le 0\,, \tag{1.2}$$

where p_1, p_2, q_1, q_2 are real numbers, τ is positive real number and $\phi(t)$ is a given continuously differentiable initial function on the interval $[-\tau, 0]$.

The equation of form of (1.1) is of interest in biology in explaining self-balancing of the human body and in robotics in consructing biped robots (see [18], [19]). These are illustrations of inverted pendulum problems. A typical example is the balancing of a stick (see [20]).

As usual, a twice continuously differentiable real-valued function y defined on the interval $[-\tau, \infty)$ is said to be a *solution* of the initial value problem (1.1) and (1.2) if y satisfies (1.1) for all $t \ge 0$ and (1.2) for all $-\tau \le t \le 0$.

It is known that (see, for example, [3]), for any given initial function ϕ, there exists a unique solution of the initial problem (1.1)-(1.2) or, more briefly, the solution of (1.1)-(1.2).

Along with the second order delay differential equation (1.1), we associate the following equation

$$\lambda^2 = p_1\lambda + p_2\lambda e^{-\lambda\tau} + q_1 + q_2 e^{-\lambda\tau}\ , \tag{1.3}$$

which will be called the *characteristic equation* of (1.1). Equation (1.3) is obtained from (1.1) by looking for solutions of the form $y(t) = e^{\lambda t}$ for $t \in IR$, where λ is a root of the equation (1.3).

For a given solution λ of the characteristic equation (1.3), we consider the (first order) delay differential equation

$$z'(t) = (p_1 - 2\lambda_0) z(t) + p_2 e^{-\lambda_0\tau} z(t-\tau) - (p_2\lambda_0 + q_2) e^{-\lambda_0\tau} \int_{t-\tau}^{t} z(s)ds\,. \tag{1.4}$$

By a *solution* of the first order delay differential equation (1.4), we mean a continuous real-valued function z defined on the interval $[-\tau, \infty)$ and satisfies (1.4) for all $t \ge 0$.

With the first order delay differential equation (1.4), we associate the equation

$$\delta = p_1 - 2\lambda_0 + p_2 e^{-\lambda_0\tau} e^{-\delta\tau} - \delta^{-1}(1 - e^{-\delta\tau})(p_2\lambda_0 + q_2) e^{-\lambda_0\tau}\,, \tag{1.5}$$

which is said to be the characteristic equation of (1.4). The last equation is obtained from (1.4) by seeking solutions of the form $z(t) = e^{\delta t}$ for $t \in IR$, where δ is a root of the equation (1.4).

The paper is organized as follows. A known (see Yeniçerioğlu [11]) asymptotic result for the solutions of the second order delay differential equation (1.1) is presented in Section 2. Section 3 is devoted to three lemma which concerns the real roots of the characteristic equation (1.5). The main result of the paper will be given in Section 4.

2. A Known Asymptotic Result

In this section, we will present an asymptotic result for the solutions of the second order delay differential equation (1.1), which is closely related to the main result of this paper. This asymptotic criterion has recently been obtained by Yeniçerioğlu [11].

Theorem 2.1. . *Let* λ_0 *be real root of the characteristic equation* (1.3) *and let* δ_0 *be real root of the characteristic equation* (1.5), *and set*

$$\beta_{\lambda_0} \equiv (p_2\lambda_0 + q_2)\tau e^{-\lambda_0\tau} + 2\lambda_0 - p_1 - p_2 e^{-\lambda_0\tau} \neq 0$$

and

$$\eta_{\lambda_0,\delta_0} \equiv 1 + p_2 e^{-(\lambda_0+\delta_0)\tau}\tau - \delta_0^{-2}\left(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau}\right)(p_2\lambda_0 + q_2)e^{-\lambda_0\tau}.$$

Also, define

$$L(\lambda_0;\phi) = \phi'(0) + (2\lambda_0 - p_1)\phi(0) - p_2\phi(-\tau) + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)ds$$

and

$$R(\lambda_0,\delta_0;\phi) = \phi(0) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} + p_2 e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0} e^{-\delta_0 s}\left(e^{-\lambda_0 s}\phi(s) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)ds$$
$$-(p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_{0}^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} e^{-\delta_0 u}\left(e^{-\lambda_0 u}\phi(u) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)du\right\}ds.$$

(Note that, because of $\beta_{\lambda_0} \neq 0$, *we always have* $\delta_0 \neq 0$.*) Assume that*

$$\mu_{\lambda_0,\delta_0} \equiv |\, p_2 \,|\, e^{-(\lambda_0+\delta_0)\tau}\tau + \delta_0^{-2}\left(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau}\right)|\, p_2\lambda_0 + q_2 \,|\, e^{-\lambda_0\tau} < 1 . \tag{2.1}$$

(This assumption guarantees that $\eta_{\lambda_0,\delta_0} > 0$.*) Then, for any* $\phi \in C([-\tau,0], IR)$, *the solution* y *of* (1.1)-(1.2) *satisfies*

$$\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\} = \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}. \tag{2.2}$$

3. Three lemma

Here, we will give three lemma that is concerned with the real roots of the characteristic equation (1.5).

Lemma 3.1. *Let* λ_0 *and* δ_0 *be real roots of the characteristic equations* (1.3) *and* (1.5), *respectively, and let* $\eta_{\lambda_0,\delta_0}$ *be defined as in Theorem 2.1. Suppose that*

$$p_2 < 0 \quad \text{and} \quad p_2\lambda_0 + q_2 > 0 . \tag{3.1}$$

Then $\eta_{\lambda_0,\delta_0} > 0$ *if* (1.5) *has another real root less than* δ_0, *and* $\eta_{\lambda_0,\delta_0} < 0$ *if* (1.5) *has another real root greater than* δ_0.

Proof of Lemma 3.1. Let $F(\delta)$ denote the characteristic function of (1.5), i.e.,

$$F(\delta) = \delta - p_1 + 2\lambda_0 - p_2 e^{-\lambda_0\tau}e^{-\delta\tau} + \delta^{-1}(1 - e^{-\delta\tau})(p_2\lambda_0 + q_2)e^{-\lambda_0\tau}$$

or

$$F(\delta) = \delta - p_1 + 2\lambda_0 - p_2 e^{-\lambda_0\tau}e^{-\delta\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_{0}^{\tau} e^{-\delta s}ds \tag{3.2}$$

for $\delta \in \mathbb{R}$. We obtain immediately

$$F'(\delta) = 1 + p_2\tau e^{-\lambda_0\tau}e^{-\delta\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_{0}^{\tau} s e^{-\delta s}ds \tag{3.3}$$

for $\delta \in \mathbb{R}$. Furthermore,

$$F''(\delta) = -p_2\tau^2 e^{-\lambda_0\tau}e^{-\delta\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau s^2 e^{-\delta s}ds$$

for $\delta \in \mathbb{R}$. So, taking into account (3.1), we conclude that

$$F''(\delta) > 0 \text{ for } \delta \in \mathbb{R}. \tag{3.4}$$

Now, assume that (1.5) has another real root δ_1 with $\delta_1 < \delta_0$ (respectively, $\delta_1 > \delta_0$). From the definition of the function F by (3.2) it follows that $F(\delta_1) = F(\delta_0) = 0$, and consequently Rolle's Theorem guarantees the existence of a point α with $\delta_1 < \alpha < \delta_0$ (resp., $\delta_1 > \alpha > \delta_0$) such that $F'(\alpha) = 0$. But, (3.4) implies that F' is positive on (α,∞) (resp., F' is negative on
(-∞,α)). Thus we must have $F'(\delta_0) > 0$ (resp., $F'(\delta_0) < 0$). The proof of lemma 3.1 can be completed, by observing that

$$F'(\delta_0) = \eta_{\lambda_0,\delta_0}.$$

Lemma 3.2. *Let* λ_0 *be real root of the characteristic equations* (1.3). *Assume that*

$$-p_2 e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds < \frac{1}{\tau} \tag{3.5}$$

and

$$|p_2|\tau\, e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} + |p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\int_0^\tau s e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds \le 1. \tag{3.6}$$

Then, in the interval $\left(p_1 - 2\lambda_0 - \frac{1}{\tau}, \infty\right)$, *the characteristic equation* (1.5) *has a unique root* δ_0*; this root satisfies* (2.1), *and the root* δ_0 *is less than* $p_1 - 2\lambda_0 + \frac{1}{\tau}$, *provided that*

$$-p_2 e^{-\left(p_1-\lambda_0+\frac{1}{\tau}\right)\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0+\frac{1}{\tau}\right)s}ds > -\frac{1}{\tau}. \tag{3.7}$$

Proof of Lemma 3.2. Consider the real-valued function F defined by (3.2). The derivative F' of F is given by (3.3). It follows from (3.2) that

$$\begin{aligned}
F\left(p_1 - 2\lambda_0 - \tfrac{1}{\tau}\right) &= \left(p_1 - 2\lambda_0 - \tfrac{1}{\tau}\right) - (p_1 - 2\lambda_0) \\
&\quad - p_2 e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds \\
&= -\frac{1}{\tau} - p_2 e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds \\
&< -\frac{1}{\tau} + \frac{1}{\tau} = 0
\end{aligned}$$

and consequently, by (3.5), it holds

$$F\left(p_1 - 2\lambda_0 - \tfrac{1}{\tau}\right) < 0. \tag{3.8}$$

Moreover, from (3.2) we obtain, for $\ge p_1 - 2\lambda_0 - \frac{1}{\tau}$,

$$\begin{aligned}
F(\delta) &\ge \delta - (p_1 - 2\lambda_0) - |p_2|e^{-\lambda_0\tau}e^{-\delta\tau} - |p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\int_0^\tau e^{-\delta s}ds \\
&\ge \delta - (p_1 - 2\lambda_0) - |p_2|e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} - |p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds.
\end{aligned}$$

Therefore,

$$F(\infty) = \infty. \tag{3.9}$$

Furthermore, using (3.3), we have, for every $\delta > p_1 - 2\lambda_0 - \frac{1}{\tau}$,

$$\begin{aligned}
F'(\delta) &\ge 1 - |p_2|\tau e^{-\lambda_0\tau}e^{-\delta\tau} - |p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\int_0^\tau s e^{-\delta s}ds \\
&> 1 - |p_2|\tau e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} - |p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds \\
&\ge 1 - 1 = 0.
\end{aligned}$$

Consequently, in view of (3.6), it holds

$$F'(\delta) > 0 \quad \text{for all } \delta > p_1 - 2\lambda_0 - \frac{1}{\tau},$$

which implies that F is strictly increasing on $\left(p_1 - 2\lambda_0 - \frac{1}{\tau}, \infty\right)$. By using this fact as well as (3.8) and (3.9), we conclude that, in the intreval $\left(p_1 - 2\lambda_0 - \frac{1}{\tau}, \infty\right)$, the equation $F(\delta) = 0$ (which coincides with (1.5)) has a unique real root δ_0. This root satisfies (2.1). Indeed, by using again (3.6), we have

$$\mu_{\lambda_0,\delta_0} < |p_2|\tau e^{-\left(p_1-\lambda_0-\frac{1}{\tau}\right)\tau} + |p_2\lambda_0 + q_2|e^{-\lambda_0\tau}\int_0^\tau s e^{-\left(p_1-2\lambda_0-\frac{1}{\tau}\right)s}ds$$
$$\leq 1 .$$

Finally, let us assume that (3.7) holds. Then it follows from (3.2) that

$$F\left(p_1 - 2\lambda_0 + \frac{1}{\tau}\right) = \frac{1}{\tau} - p_2 e^{-\left(p_1-\lambda_0+\frac{1}{\tau}\right)\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-\left(p_1-2\lambda_0+\frac{1}{\tau}\right)s}ds$$
$$> \frac{1}{\tau} - \frac{1}{\tau} = 0 .$$

As $F\left(p_1 - 2\lambda_0 + \frac{1}{\tau}\right) > 0$, we see that δ_0 must be less than $p_1 - 2\lambda_0 + \frac{1}{\tau}$. This completes the proof of the lemma 3.2.

Lemma 3.3. *Let* λ_0 *be real root of the characteristic equations* (1.3). *Suppose that statement* (3.1) *is true. Then we have:*

a) *In the interval* $[p_1 - 2\lambda_0, \infty)$, *the characteristic equation* (1.5) *has no roots.*

b) *Assume that* (3.5) *holds. Then: (i)* $\delta = p_1 - 2\lambda_0 - \frac{1}{\tau}$ *is not a root of the characteristic equation* (1.5). *(ii) In the interval* $\left(p_1 - 2\lambda_0 - \frac{1}{\tau}, p_1 - 2\lambda_0\right)$, (1.5) *has a unique root.*

(iii) In the interval $\left(-\infty, p_1 - 2\lambda_0 - \frac{1}{\tau}\right)$, (1.5) *has a unique root.*

Proof of Lemma 3.3. a) Let $\hat{\delta}$ be real root of the characteristic equation (1.5). Using (3.1), we can immediately see that

$$p_2 e^{-(\lambda_0+\hat{\delta})\tau} - (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-\hat{\delta}s}ds < 0 .$$

Hence, from (1.5) it follows that $\hat{\delta} - (p_1 - 2\lambda_0) < 0$, i.e., $\hat{\delta} < (p_1 - 2\lambda_0)$. We have thus proved that every real root of (1.5) is always less than $p_1 - 2\lambda_0$.

b) Consider the real-valued function F defined by (3.2). As in the proof of Lemma 3.1, we see that (3.4) holds and consequently

$$F \text{ is convex on } \mathbb{R} . \tag{3.10}$$

Next, we observe that, as in the proof of Lemma 3.2, assumption (3.5) means that (3.8) holds true. Inequality (3.8) implies, in particular, that $\delta = p_1 - 2\lambda_0 - \frac{1}{\tau}$ is not a root of the characteristic equation (1.5). From (3.2) we obtain

$$F(p_1 - 2\lambda_0) = -p_2 e^{-(p_1-\lambda_0)\tau} + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^\tau e^{-(p_1-2\lambda_0)s}ds .$$

So, by using (3.1), we conclude that

$$F(p_1 - 2\lambda_0) > 0 . \tag{3.11}$$

Furthermore, from (3.2) we get

$$F(\delta) \geq \delta - (p_1 - 2\lambda_0) - p_2 e^{-(\lambda_0+\delta)\tau} \quad \text{for } \delta \in \mathbb{R} .$$

Using this inequality, it is not difficult to show that

$$F(-\infty) = \infty . \tag{3.12}$$

From (3.8), (3.10) and (3.11) it follows that, in the interval $\left(p_1 - 2\lambda_0 - \frac{1}{\tau}, p_1 - 2\lambda_0\right)$, the characteristic equation (1.5) has a unique root. Moreover, (3.8), (3.10) and (3.12) guarantee that, in the interval $\left(-\infty, p_1 - 2\lambda_0 - \frac{1}{\tau}\right)$, (1.5) has also a unique root. The proof of the lemma 3.3 is complete.

4. The Main Result

Theorem 4.1. *Let* λ_0 *and* δ_0 *be real roots of the characteristic equation* (1.3) *and* (1.5), *respectively, and let* β_{λ_0}, $\eta_{\lambda_0,\delta_0}$, $L(\lambda_0;\phi)$ *and* $R(\lambda_0,\delta_0;\phi)$ *be defined as in Theorem 2.1. Suppose that statement* (3.1) *is true. Also, let* δ_1 *be real root of* (1.5) *with* $\delta_1 \neq \delta_0$. *(Note that, because of* $\beta_{\lambda_0} \neq 0$, *we have* $\delta_0 \neq 0$ *and* $\delta_1 \neq 0$. *Moreover, Lemma 3.1. guarantees that* $\eta_{\lambda_0,\delta_0} \neq 0$. *) Then the solution* y *of the IVP* (1.1) *and* (1.2) *satisfies*

$$C_1(\lambda_0,\delta_0,\delta_1;\phi) \leq e^{-\delta_1 t}\left[e^{-\lambda_0 t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} - e^{-\delta_0 t}\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right]$$
$$\leq C_2(\lambda_0,\delta_0,\delta_1;\phi) \tag{4.1}$$

for all $t \geq 0$, *where*

$$C_1(\lambda_0,\delta_0,\delta_1;\phi) = \min_{-\tau\le t\le 0}\left\{e^{-\delta_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} - e^{-\delta_0 t}\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right]\right\} \quad (4.2)$$

and

$$C_2(\lambda_0,\delta_0,\delta_1;\phi) = \max_{-\tau\le t\le 0}\left\{e^{-\delta_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} - e^{-\delta_0 t}\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right]\right\}. \quad (4.3)$$

We see immediately that inequalities (4.1) can equivalently be written as follows

$$C_1(\lambda_0,\delta_0,\delta_1;\phi)e^{(\delta_1-\delta_0)t} \le e^{-\delta_0 t}\left[e^{-\lambda_0 t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right] - \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}$$

$$\le C_2(\lambda_0,\delta_0,\delta_1;\phi)e^{(\delta_1-\delta_0)t}\quad,\quad t\ge 0\,.$$

Hence, if $\boldsymbol{\delta_1} < \boldsymbol{\delta_0}$, then the solution $\boldsymbol{y}$ of the IVP (1.1) and (1.2) satisfies (2.2).

Also, we observe that (4.1) is equivalent to

$$e^{\lambda_0 t}\left[C_1(\lambda_0,\delta_0,\delta_1;\phi)e^{\delta_1 t} + \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} + \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}e^{\delta_0 t}\right]$$
$$\le y(t)$$
$$e^{\lambda_0 t}\left[C_2(\lambda_0,\delta_0,\delta_1;\phi)e^{\delta_1 t} + \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} + \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}e^{\delta_0 t}\right]$$

for all $t \ge 0$.

Proof of Theorem 4.1. Consider an arbitrary initial function $\phi \in C([-\tau,0],IR)$ and let y be the solution of the initial problem (1.1)–(1.2). Define

$$x(t) = e^{-\lambda_0 t}y(t) \quad \text{for} \quad t \ge -\tau$$

and next, set

$$w(t) = e^{-\delta_0 t}\left(x(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right) - \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \quad \text{for} \quad t \ge -\tau\,.$$

As it has been shown by Yeniçerioğlu [11], the fact that y satisfies (1.1) for $t \ge 0$ is equivalent to the fact that w satisfies

$$w(t) = -p_2 e^{-(\lambda_0+\delta_0)\tau}\int_{t-\tau}^{t} w(s)ds + (p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{t-s}^{t} w(u)du\right\}ds \quad (4.4)$$

for $t \ge 0$, while the initial condition (1.2) takes the equivalent form

$$w(t) = e^{-\delta_0 t}\left(e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right) - \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \quad \text{for} \quad t \in [-\tau, 0]. \quad (4.5)$$

Now we define

$$h(t) = e^{(\delta_0-\delta_1)\,t}w(t) \quad \text{for } t \ge -\tau\,. \quad (4.6)$$

Because of the definitions of x and w, we have the following expression for the function h:

$$h(t) = e^{-\delta_1 t}\left[e^{-\lambda_0 t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} - e^{\delta_0 t}\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right] \quad \text{for } t \ge -\tau\,. \quad (4.7)$$

Furthermore, by the use of the function h, (4.4) can equivalently be written as

$$h(t) = -p_2 e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0} e^{(\delta_1-\delta_0)\,s}h(s+t)ds$$
$$+(p_2\lambda_0 + q_2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} e^{(\delta_1-\delta_0)\,u}h(u+t)du\right\}ds \quad (4.8)$$

for $t \ge 0$ and (4.5) becomes

$$h(t) = e^{-\delta_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} - e^{\delta_0 t}\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right] \quad \text{for} \quad t \in [-\tau, 0]. \quad (4.9)$$

As solution y satisfies the initial condition (1.2), we can use (4.7) as well as the definitions of $C_1(\lambda_0,\delta_0,\delta_1;\phi)$ and $C_2(\lambda_0,\delta_0,\delta_1;\phi)$ by (4.2) and (4.3), respectively, to see that

$$C_1(\lambda_0,\delta_0,\delta_1;\phi) = \min_{-\tau\le t\le 0} h(t) \quad \text{and} \quad C_2(\lambda_0,\delta_0,\delta_1;\phi) = \max_{-\tau\le t\le 0} h(t). \quad (4.10)$$

In view of (4.7) and (4.10), the double inequality (4.1) can equivalently written as follows

$$\min_{-\tau\le s\le 0} h(s) \le h(t) \le \max_{-\tau\le s\le 0} h(s) \quad \text{for all } t \ge 0. \quad (4.11)$$

All we have to prove that (4.11) hold. We will use the fact that h satisfies (4.8) for all $t \ge 0$ in order to show that (4.11) is valid. We restrict ourselves to proving that

$$h(t) \ge \min_{-\tau\le s\le 0} h(s) \quad \text{for every } t \ge 0\,. \quad (4.12)$$

The proof of the inequality

$$h(t) \le \max_{-\tau\le s\le 0} h(s) \quad \text{for every } t \ge 0$$

can be obtained in a similar way, and so it is omitted. In the rest of the proof we will establish (4.12). In order to so, we consider an arbitrary real number A with $A < \min_{-\tau \le s \le 0} h(s)$, i.e., with

$$h(t) > A \quad \text{for } -\tau \le t \le 0. \tag{4.13}$$

We will show that

$$h(t) > A \quad \text{for all } t \ge 0. \tag{4.14}$$

To this end, let us assume that (4.14) fails to hold. Then, because of (4.13), there exists a point $t_0 > 0$ so that

$$h(t) > A \quad \text{for } -\tau \le t < t_0\,, \quad \text{and} \quad h(t_0) = A.$$

Thus, by using (3.11) and (1.5), from (4.8) we obtain

$$\begin{aligned}
A &= h(t_0) \\
&= -p_2 e^{-(\lambda_0+\delta_0)\tau} \int_{-\tau}^{0} e^{(\delta_1-\delta_0)s} h(s+t_0)ds \\
&\quad +(p_2\lambda_0+q_2)e^{-\lambda_0\tau} \int_0^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} e^{(\delta_1-\delta_0)\,u} h(u+t_0)du\right\} ds \\
&> A\left(-p_2 e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0} e^{(\delta_1-\delta_0)s}ds + (p_2\lambda_0+q_2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} e^{(\delta_1-\delta_0)\,u}du\right\}ds\right) \\
&= \frac{A}{\delta_1-\delta_0}\left(-p_2 e^{-(\lambda_0+\delta_0)\tau}\left[1-e^{-(\delta_1-\delta_0)\tau}\right] + (p_2\lambda_0+q_2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left[1-e^{-(\delta_1-\delta_0)s}\right]ds\right) \\
&= \frac{A}{\delta_1-\delta_0}\left(p_2\left[e^{-(\lambda_0+\delta_1)\tau} - e^{-(\lambda_0+\delta_0)\tau}\right] + (p_2\lambda_0+q_2)e^{-\lambda_0\tau}\int_0^{\tau}\left[e^{-\delta_0 s}-e^{-\delta_1 s}\right]ds\right) \\
&= \frac{A}{\delta_1-\delta_0}\Big(p_2\left[e^{-(\lambda_0+\delta_1)\tau} - e^{-(\lambda_0+\delta_0)\tau}\right] \\
&\qquad +(p_2\lambda_0+q_2)e^{-\lambda_0\tau}\left[\delta_1^{-1}\left(e^{-\delta_1\tau}-1\right)-\delta_0^{-1}\left(e^{-\delta_0\tau}-1\right)\right]\Big) \\
&= \frac{A}{\delta_1-\delta_0}\left(p_1-2\lambda_0-\delta_0+\delta_1-p_1+2\lambda_0\right) = A\,.
\end{aligned}$$

We have thus arrived at a contradiction and so (4.14) is true. Since (4.14) is satisfied for all real numbers A with $A < \min_{-\tau \le s \le 0} h(s)$, it follows that (4.12) is always fulfilled. The proof of the theorem 4.1 is complete.

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