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Formation of Multiple Off-Grid Points for the Treatment of Systems of Stiff Ordinary Differential Equations

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Abstract

This paper is concerned with the construction of two-step hybrid block Simpson's method with four off-grid points for the solutions of stiff systems of ordinary differential equations (ODEs). This is achieved by transforming a k-step multi-step method into continuous form and evaluating at various grid points to obtain the discrete schemes. The discrete schemes are applied as a block for simultaneous integration. The block matrix equation is A-stable and of order $[7, 7, 7, 7, 7, 7]^T$. This order 'p' is achieved by the aid of Maple13 software program. The performance of the method is demonstrated on some numerical experiments. The results revealed that the hybrid block Simpson's method is efficient, accurate and convergent on stiff problems.

Keywords: Hybrid method; Off-step point; Blocks method; First order system; Multi-step method.

1. Introduction

A considerable literature exists for the conventional *k-step* linear multi-step methods for the solution of ordinary differential equations (ODE's) of the form

$$y' = f(x, y), \quad y(a) = y_0, \quad x \in [a,b] \tag{1.1}$$

Where y satisfies a given set of initial condition [1], and we assume that the function f also satisfies the Lipschitz condition which guarantees existence, uniqueness and continuous differentiable solution, [2]. For the discrete solution of (1.1) linear multi-step methods has being studied by, Lambert [3], Lambert [4], and continuous solutions of (1.1) Lie and Norsett [5] and Onumanyi, *et al.* [6], Onumanyi, *et al.* [7]. One important advantage of the continuous over the discrete approach is the ability to provide discrete schemes for simultaneous integration. These discrete schemes can as well be reformulated as general linear methods (GLM) Butcher [8]. The block methods are self-starting and can directly be applied to both initial and boundary value problems by Skwame [9] and Donald, *et al.* [10]. Block methods for solving ordinary differential equations have initially been proposed by Milne [11] who advanced their use only as a means of obtaining starting values for predictor-corrector algorithms. Several authors Roser [12], Shampine and Watts [13], Fatunla [14], and Ngwane and Jator [15] among others] have modified it to be more efficient as a computational procedure for the integration 0f IVPs throughout the range of integration rather than just as a starting method for method for multistep methods [16].

From the results of this study a two-step hybrid block Simpson's method with four off-grid points will be presented. By using Onumanyi, *et al.* [6], Onumanyi, *et al.* [17] approach; the derived schemes will be applied in block form in order to achieve its order'p' and error constants; the region of absolute stability, and the results of absolute errors.

2. Derivation of the Method

Consider the collocation methods defined for the step $|x_n, x_{n+1}|$ by

$$y(x) = \sum_{j=0}^{t-1} \alpha(x) y_{n+1} + h \sum_{j=0}^{m-1} \beta_j(x) f(x_j, y(\bar{x}_j)), \tag{2.1}$$

Where t denotes the number of interpolation points x_{n+j}, $j = 0, \cdots, t-1$, and m denotes the number of distinct collocation points $\bar{x}_j \in [x_n, x_{n+k}]$, $j = 0,1,\cdots, m-1$ the points $\bar{x}_j$ are chosen from the step x_{n+j} as well as one or more off-step points.

The following assumptions are made;

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1. Although the step size can be variable, for simplicity in our presentation of the analysis in this paper, we assume it is constant $h = x_{n+1} - x_n$, $N = \dfrac{b-a}{h}$ with the steps given by $\{x_n / x_n = a + nh, n = 0,1,\cdots,N\}$,
2. That (1.1) has a unique solution and the coefficients $\alpha_j(x), \beta_j(x)$ in (2.1) can be represented by polynomial of the form

$$\alpha_j(x) = \sum_{j=0}^{t+m-1} \alpha_{j,i+1} x^i, \quad j\varepsilon(0,1,2,\cdots,t-1) \tag{2.2}$$

$$h\beta_j(x) = h\sum_{j=0}^{t+m-1} \beta_{j,i+1} x^i, \quad j\varepsilon(0,1,2,\cdots,m-1) \tag{2.3}$$

With constant coefficients $\alpha_{j,i+1}, h\beta_{j,i+1}$ and collocation conditions

$$\bar{y}(x_{n+j}) = y_{n+j}, \qquad j\varepsilon\{0,1,\cdots,t-1\} \tag{2.4}$$

$$\bar{y}'(\bar{x}_j) = f(\bar{x}_j, \bar{y}(\bar{x}_j)), \qquad j\varepsilon\{0,1,\cdots m-1\} \tag{2.5}$$

With these assumptions we obtained an MC polynomial in the form

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^i, \quad \alpha_j = \sum_{j=0}^{t-1} C_{i+1,j+1} + \sum_{j=0}^{m-1} C_{i+1,j+1} f_{n+j} \tag{2.6}$$

And also we get D Matrix as follows:

$$\underline{D} = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+2}^2 & \cdots & x_n^{t+m-1} \\ \vdots & & & & \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_0^{t+M-2} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_1^{t+M-2} \\ \vdots & & & & \\ 0 & 1 & 2\bar{x}_{m-1} & \cdots & (t+m-1)\bar{x}_{m-1}^{t+M-2} \end{bmatrix} \tag{2.7}$$

The parameters required for equation (2.7) to obtain two-step Block Hybrid Simpson's Methods with four off-grid points (BHSM4)are $K = 2, \quad t = 1, \quad m = k+5$; where

$$\bar{x}_0 = x_n, \bar{x}_{\frac{1}{2}} = x_{n+\frac{1}{2}}, \bar{x}_1 = x_{n+1}, \bar{x}_{\frac{5}{4}} = x_{n+\frac{5}{4}}, \bar{x}_{\frac{3}{2}} = x_{n+\frac{3}{2}}, \bar{x}_{\frac{7}{4}} = x_{n+\frac{7}{4}}, \bar{x}_2 = x_{n+2}$$

The matrix (2.7) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 & 6x_{n+\frac{1}{2}}^5 & 7x_{n+\frac{1}{2}}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+\frac{5}{4}} & 3x_{n+\frac{5}{4}}^2 & 4x_{n+\frac{5}{4}}^3 & 5x_{n+\frac{5}{4}}^4 & 6x_{n+\frac{5}{4}}^5 & 7x_{n+\frac{5}{4}}^6 \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 & 6x_{n+\frac{3}{2}}^5 & 7x_{n+\frac{3}{2}}^6 \\ 0 & 1 & 2x_{n+\frac{7}{4}} & 3x_{n+\frac{7}{4}}^2 & 4x_{n+\frac{7}{4}}^3 & 5x_{n+\frac{7}{4}}^4 & 6x_{n+\frac{7}{4}}^5 & 7x_{n+\frac{7}{4}}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \end{bmatrix} \tag{2.8}$$

By using the maple software program and evaluating (2.8) at the grid points $x = x_{n+\frac{1}{2}}$, $x = x_{n+1}$, $x = x_{n+\frac{5}{4}}$, $x = x_{n+\frac{3}{2}}$, $x = x_{n+\frac{7}{4}}$, $x = x_{n+2}$ we obtain six discrete schemes.

Therefore, the hybrid block methods are

$$
\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \tfrac{1}{211680}h[30585 f_n + 143290 f_{n+\frac{1}{2}} - 321888 f_{n+1} + 519232 f_{n+\frac{5}{4}} - 391818 f_{n+\frac{3}{2}} \\
&\quad + 149952 f_{n+\frac{7}{4}} - 23513 f_{n+2}] \\
y_{n+1} &= y_n + \tfrac{1}{26460}h[3735 f_n + 22372 f_{n+\frac{1}{2}} - 21672 f_{n+1} + 47488 f_{n+\frac{5}{4}} - 38052 f_{n+\frac{3}{2}} + 14976 f_{n+\frac{7}{4}} \\
&\quad - 2387 f_{n+2}] \\
y_{n+\frac{5}{4}} &= y_n + \tfrac{5}{677376}h[19137 f_n + 114310 f_{n+\frac{1}{2}} - 96600 f_{n+1} + 267232 f_{n+\frac{5}{4}} - 200550 f_{n+\frac{3}{2}} \\
&\quad + 78240 f_{n+\frac{7}{4}} - 12425 f_{n+2}] \\
y_{n+\frac{3}{2}} &= y_n + \tfrac{1}{7840}h[1107 f_n + 6622 f_{n+\frac{1}{2}} - 5712 f_{n+1} + 16576 f_{n+\frac{5}{4}} - 10542 f_{n+\frac{3}{2}} + 4416 f_{n+\frac{7}{4}} \\
&\quad - 707 f_{n+2}] \\
y_{n+\frac{7}{4}} &= y_n + \tfrac{7}{69120}h[1395 f_n + 8330 f_{n+\frac{1}{2}} - 7056 f_{n+1} + 20384 f_{n+\frac{5}{4}} - 11466 f_{n+\frac{3}{2}} + 6624 f_{n+\frac{7}{4}} \\
&\quad - 931 f_{n+2}] \\
y_{n+2} &= y_n + \tfrac{1}{6615}h[933 f_n + 5600 f_{n+\frac{1}{2}} - 4956 f_{n+1} + 14336 f_{n+\frac{5}{4}} - 8736 f_{n+\frac{3}{2}} + 6144 f_{n+\frac{7}{4}} - 91 f_{n+2}]
\end{aligned}
\tag{2.9}
$$

3. Stability of Block Method

The equations (2.9) when put together formed the block as

$$
\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{5}{4}}\\ y_{n+\frac{3}{2}}\\ y_{n+\frac{7}{4}}\\ y_{n+2} \end{bmatrix}
=
\begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} y_{n+2}\\ y_{n+\frac{7}{4}}\\ y_{n+\frac{3}{2}}\\ y_{n+\frac{5}{4}}\\ y_{n+1}\\ y_{n} \end{bmatrix}
$$

$$
+h\begin{bmatrix}
\frac{143290}{211680} & -\frac{321888}{211680} & \frac{519232}{211680} & -\frac{391818}{211680} & \frac{149952}{211680} & -\frac{23513}{211680}\\
\frac{22372}{26460} & -\frac{21672}{26460} & \frac{47488}{26460} & -\frac{38052}{26460} & \frac{14976}{26460} & -\frac{2387}{26460}\\
\frac{571550}{677376} & -\frac{483000}{677376} & \frac{1336160}{677376} & -\frac{1002750}{677376} & \frac{391200}{677376} & -\frac{62125}{677376}\\
\frac{6622}{7840} & -\frac{5712}{7840} & \frac{16576}{7840} & -\frac{10542}{7840} & \frac{4416}{7840} & -\frac{707}{7840}\\
\frac{58310}{69120} & -\frac{49392}{69120} & \frac{142688}{69120} & -\frac{80262}{69120} & \frac{46368}{69120} & -\frac{6518}{69120}\\
\frac{5600}{6615} & -\frac{4956}{6615} & \frac{14336}{6615} & -\frac{8736}{6615} & \frac{6144}{6615} & -\frac{91}{6615}
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{5}{4}}\\ f_{n+\frac{3}{2}}\\ f_{n+\frac{7}{4}}\\ f_{n+2} \end{bmatrix}
$$

$$
+h\begin{bmatrix}
0&0&0&0&0&\frac{30585}{211680}\\
0&0&0&0&0&\frac{3735}{26460}\\
0&0&0&0&0&\frac{95685}{677376}\\
0&0&0&0&0&\frac{1107}{7840}\\
0&0&0&0&0&\frac{9765}{69120}\\
0&0&0&0&0&\frac{9333}{6615}
\end{bmatrix}
\begin{bmatrix} f_{n+2}\\ f_{n+\frac{7}{4}}\\ f_{n+\frac{3}{2}}\\ f_{n+\frac{5}{4}}\\ f_{n+1}\\ f_{n} \end{bmatrix}
\tag{3.1}
$$

The characteristic of polynomial of the hybrid block methods (2.6) and (3.1) is given as

$\rho(R) = \det(RA^0 - A^1), \quad where$

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad and \quad A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$therefore, \quad \det\,(RA^0 - A^1) = \det \begin{bmatrix} R & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & R-1 \end{bmatrix} = 0$$

$$= R(R(R(R(R(R-1))))) = 0$$

$$\Rightarrow R_1 = 0,\ R_2 = 0,\ R_3 = 0,\ R_4 = 0,\ R_5 = 0,\ R_6 = 1$$

Since $| R_j | \leq 1,\ j\varepsilon\{1, \cdots, 6\}$ hence the method as a block is zero stable on its own, and the hybrid block method is consistent as its order $P > 1$.

4. Convergence Analysis Order and Error Constants of the Block Hybrid Simpson's Methods

The block hybrid methods which are obtained in a block form with the help of a maple software have the following order and error constants.

Table-1. BHSM4

Evaluating point	order 'p'	Error constant
$y(x = x_{n+\frac{1}{2}})$	7	$\frac{965}{86704128}$
$y(x = x_{n+1})$	7	$\frac{29}{3010560}$
$y(x = x_{n+\frac{5}{4}})$	7	$\frac{107725}{11098128348}$
$y(x = x_{n+\frac{3}{2}})$	7	$\frac{31}{3211264}$
$y(x = x_{n+\frac{7}{4}})$	7	$\frac{245}{25165824}$
$y(x = x_{n+2})$	7	$\frac{1}{105840}$

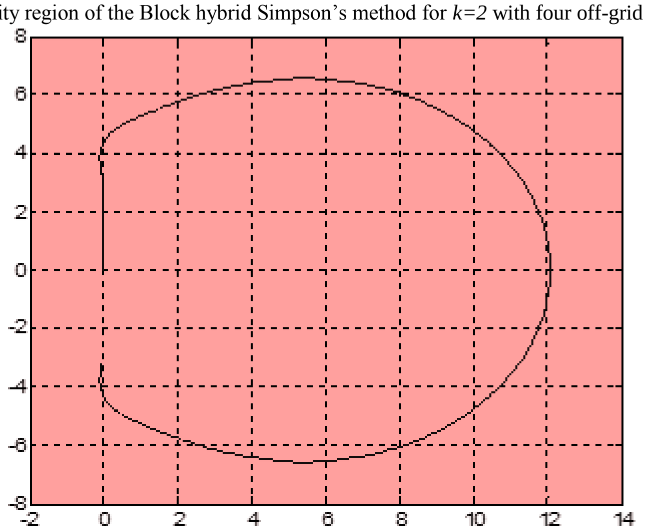
The method BHSM4 is of order 7 and has error constants

$$C_8 = [\frac{965}{86704128}, \frac{29}{3010560}, \frac{107725}{11098128348}, \frac{31}{3211264}, \frac{245}{25165824}, \frac{1}{105840}]^T$$

4.1. Region of Absolute Stability

Using the MATLAB package, we were able to plot the stability regions of the block method. This is done by reformulating the block method as general linear method to obtain the values of the matrices A, B, U and V. These matrices are substituted into the stability matrix and using MATLAB software, the absolute stability regions of the new methods are plotted as shown in fig(1)

Figure-1. Stability region of the Block hybrid Simpson's method for *k=2* with four off-grid points (BHSM4)



5. Numerical Experiments

The newly constructed method is demonstrated on some initial value problems and the results are displayed below

Example 1

$$y_1' = -8y_1 + 7y_2$$
$$y_2' = 42y_1 - 43y_2$$
$$where$$
$$h = \tfrac{1}{10}, \qquad y_1(0) = 1, \qquad y_2(0) = 8$$
$$Exact\ Solution$$
$$y_1(x) = 2e^{-x} - e^{-50x}, \qquad y_2(x) = 2e^{-x} + 6e^{-50x}$$
$$with\ stiff\ \ ratio\ \ 5.0 \times 10^1$$

Example 2

$$y_1' = 998y_1 + 1998y_2$$
$$y_2' = -999y_1 - 1999y_2$$
$$where$$
$$h = \tfrac{1}{10}, \qquad y_1(0) = 1, \qquad y_2(0) = 1$$
$$and\ exact\ solution$$
$$y_1(x) = 4e^{-x} - 3e^{-1000x}, \qquad y_2(x) = -2e^{-x} + 3e^{-1000x}$$
$$with\ stiff\ \ ratio\ \ 1.0 \times 10^3$$

Example 3

$$y_1' = -y_1 + 95y_2$$
$$y_2' = -y_1 - 97y_2$$

where

$$h = \tfrac{1}{10}, \qquad y_1(0) = 1, \qquad y_2(0) = 1$$

and exact solution

$$y_1(x) = \tfrac{95}{47}e^{-2x} - \tfrac{48}{47}e^{-96x}, \qquad y_2(x) = -\tfrac{48}{47}e^{-96x} - \tfrac{1}{47}e^{-2x}$$

with stiff ratio 4.8×10^1

Table-2. Absolute Stability Errors for Example 1

	BHSM with four off-grid points	
X	**Y_1**	**Y_2**
0.1	2.011 E^{-1}	1.815 E^{0}
0.2	4.699 E^{-1}	1.504 E^{0}
0.3	3.821 E^{-1}	1.484 E^{0}
0.4	4.668 E^{-1}	1.316 E^{0}
0.5	4.713 E^{-1}	1.219 E^{0}
0.6	4.951 E^{-1}	1.098 E^{0}
0.7	4.951 E^{-1}	9.981 E^{-1}
0.8	4.917 E^{0}	9.018 E^{-1}
0.9	4.791 E^{-1}	8.166 E^{-1}
1.0	4.624 E^{-1}	7.384 E^{-1}
1.1	4.420 E^{-1}	6.681 E^{-1}
1.2	4.191 E^{-1}	6.043 E^{-1}

Table-3. Absolute Stability Errors for Example 2

	BHSM with four off-grid points	
X	**Y_1**	**Y_2**
0.1	1.978 E^{0}	1.813 E^{0}
0.2	8.443 E^{-1}	1.728 E^{0}
0.3	1.853 E^{0}	1.494 E^{0}
0.4	1.779 E^{0}	1.345 E^{0}
0.5	1.683 E^{0}	1.220 E^{0}
0.6	1.586 E^{0}	1.104 E^{0}
0.7	1.492 E^{0}	9.984 E^{-1}
0.8	1.389 E^{0}	9.029 E^{-1}
0.9	1.292 E^{0}	8.096 E^{-1}
1.0	1.198 E^{0}	7.329 E^{-1}
1.1	1.108 E^{0}	6.681 E^{-1}
1.2	1.021 E^{0}	6.043 E^{-1}

Table-4. Absolute Stability Errors for Example 3

	BHSM with four off-grid points	
X	**Y_1**	**Y_2**
0.1	5.950 E^{-3}	5.950 E^{-3}
0.2	4.902 E^{-3}	4.902 E^{-3}
0.3	3.489 E^{-5}	2.889 E^{-5}
0.4	2.353 E^{-5}	2.353 E^{-5}
0.5	1.355 E^{-7}	6.049 E^{-4}
0.6	1.098 E^{-7}	7.294 E^{-8}
0.7	2.400 E^{-9}	7.060 E^{-10}
0.8	2.100 E^{-9}	5.460 E^{-10}
0.9	1.600 E^{-9}	7.999 E^{-7}
1.0	1.340 E^{-4}	2.999 E^{-4}
1.1	2.300 E^{-9}	1.200 E^{-10}
1.2	2.000 E^{-9}	2.900 E^{-11}

6. Conclusion and Recommendation

From the results of this finding the newly constructed hybrid block Simpson's method with four off-grid points was demonstrated on some stiff initial value problems (IVPs). From the result displayed on tables (2, 3, 4) it can be seen that the BHSM4 performs efficient and converges very well on example three and performs fairly on examples 1 and 2. Therefore, the newly constructed block hybrid Simpson's method is efficient, accurate and convergent on mildly stiff problems.

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