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An Implicit Two-Step One Off-Grid Point Third Derivative Hybrid Block Method for the Direct Solution of Second-Order Ordinary Differential Equations

D. Raymond
Department of Mathematics and Statistics, Federal University, Wukari-Nigeria

Y. Skwamw
Department of Mathematics, Adamawa State University, Mubi-Nigeria

J. Sunday*
Department of Mathematics, Adamawa State University, Mubi-Nigeria

Abstract

In this paper, we propose an implicit two-step third derivative hybrid block method with one off-step point for the direct solution of second order Ordinary Differential Equations. We adopted the method of interpolation and collocation of power series approximate solution to generate the continuous hybrid linear multistep method, which was evaluated at non-interpolated step points to give a continuous block method. The discrete block method was recovered when the continuous block method was evaluated at all step points. The basic properties of the method were investigated and the method was found to be zero-stable, consistent and convergent. The efficiency of the method was tested on some stiff equations and was found to give better approximation than the existing methods with which we compared our results.

Keywords: Two-step; Hybrid; Third derivative; Off-step; Collocation; and interpolation.
AMS Subject Classification: 65L05; 65L06; 65D30.

1. Introduction

In this paper, an implicit two-step one off-grid third derivative hybrid block method is derived for the integration of second-order differential equation of the form;

$$y'' = f(x, y(x), y'(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{1}$$

Where f is continuous within the interval of integration. Different method have been proposed for the solution of (1) ranging from predictor-corrector method to hybrid methods. Despite the success recorded by the predictor-corrector methods, its major setback is that the predictor are in reducing order of accuracy especially when the value of the step-length is high and moreover the result are at overlapping interval. Direct method of solving (1), which we shall employ has been discussed by many authors and they concluded that it is more convenient and accurate. Among the authors that proposed direct methods were [1-6]. Block methods have the advantage of incorporating function evaluation at off-step points which afford the opportunity of circumventing the Dahlquist zero-stability barrier and it is actually possible to obtain convergent k-step methods of order 2k+1. Hence, hybrid block method is less expensive in terms of number of function evaluation compare to predictor-corrector methods; it also possesses the properties of Runge-Kutta for being self-starting and does not require starting values. Other authors who proposed block methods are [7-11].

In this paper, we shall develop a two-step third derivative hybrid block method for direct solution of second order ordinary differential equations of the form (1), which is implemented in block method mode. The method developed evaluates less function per step and circumventing the Dahlquist barrier's by introducing a hybrid point.
The paper is organized as follows: In section 2, we discuss the methods and the materials for the development of the method. Section 3 considers analysis of the basis properties of the method, which include convergence and stability region. The numerical experiments where the efficiency of the derived method is tested on some stiff numerical examples shall be discussed in section 4. Lastly, the conclusion shall be drawn in section 5.

2. Derivation of the Method

We consider a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{2s+r-1} a_i \left(\frac{x - x_n}{h} \right)^j \tag{2}$$

*Corresponding Author

where $r = 2$ and $s = 4$ are the numbers of interpolation and collocation points respectively, is considered to be a solution to (1).

The second and third derivative of (2) gives

$$\left.\begin{aligned} y''(x) &= \sum_{j=2}^{2s+r-1} \frac{a_j\, j!}{h^2 (j-2)} \left(\frac{x - x_n}{h}\right)^{j-2} = f(x, y, y') \\ y'''(x) &= \sum_{j=3}^{2s+r-1} \frac{a_j\, j!}{h^3 (j-3)} \left(\frac{x - x_n}{h}\right)^{j-3} = g(x, y, y') \end{aligned}\right\} \qquad (3)$$

Substituting (3) into (1) gives

$$f(x, y, y'') = \sum_{j=2}^{2s+r-1} \frac{a_j\, j!}{h^2 (j-2)} \left(\frac{x - x_n}{h}\right)^{j-2} + \sum_{j=3}^{2s+r-1} \frac{a_j\, j!}{h^3 (j-3)} \left(\frac{x - x_n}{h}\right)^{j-3} \qquad (4)$$

Collocating (4) at all points $x_{n+s}, s = 0, \frac{1}{2}, 1, 2$ and interpolating equation (2) at $x_{n+r}, r = 0, \frac{1}{4}$, gives a system of nonlinear equation of the form

$$AX = U \qquad (5)$$

Where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]^T$$

$$U = \left[y_n, y_{n+\frac{1}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+2}, g_n, g_{n+\frac{1}{2}}, g_{n+1}, g_{n+2} \right]^T$$

And

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \frac{1}{256} & \frac{1}{512} \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{3}{h^2} & \frac{5}{2h^2} & \frac{15}{8h^2} & \frac{21}{16h^2} & \frac{7}{8h^2} & \frac{9}{16h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{24}{h^2} & \frac{40}{h^2} & \frac{60}{h^2} & \frac{82}{h^2} & \frac{112}{h^2} & \frac{144}{h^2} \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{12}{h^3} & \frac{15}{h^3} & \frac{15}{h^3} & \frac{105}{8h^3} & \frac{21}{2h^3} & \frac{63}{8h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} & \frac{336}{h^3} & \frac{504}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{120}{h^3} & \frac{240}{h^3} & \frac{420}{h^3} & \frac{672}{h^3} & \frac{1008}{h^3} \end{bmatrix}$$

Solving (5) for $a_i{}'s$ using Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$p(x) = \sum_{j=0,\frac{1}{2}} \alpha_j y_{n+j} + h^2 \left[\sum_{j=\frac{1}{2}} \beta_j f_{n+j} + \sum_{j=0}^{2} \beta_i f_{n+i} \right] + h^3 \left[\sum_{j=\frac{1}{2}} \gamma_j g_{n+j} + \sum_{i=0}^{2} \gamma_j g_{n+j} \right] \qquad (6)$$

The coefficients of $y_{n+j},\ j=0,\frac{1}{2}$ and $f_{n+j},\ j=0,\frac{1}{2},1,2$ are given by;

$$\alpha_0 = 1 - 2\,t\,\alpha_{\frac{1}{2}} = 2\,t$$

$$\beta_0 = -\frac{94457}{645120}\,t\,h^2 + \frac{1}{2}\,t^2\,h^2 - \frac{119}{48}\,t^4\,h^2 + \frac{433}{80}\,t^5\,h^2 - \frac{133}{24}\,t^6\,h^2 + \frac{73}{24}\,t^7\,h^2 - \frac{6}{7}\,t^8\,h^2 + \frac{7}{72}\,t^9\,h^2$$

$$\beta_{\frac{1}{2}} = -\frac{10267}{136080}\,t\,h^2 + \frac{64}{81}\,t^4\,h^2 + \frac{64}{135}\,t^5\,h^2 - \frac{224}{81}\,t^6\,h^2 + \frac{1472}{567}\,t^7\,h^2 - \frac{184}{189}\,t^8\,h^2 + \frac{32}{243}\,t^9\,h^2$$

$$\beta_1 = -\frac{2243}{80640}\,t\,h^2 + \frac{5}{3}\,t^4\,h^2 - \frac{29}{5}\,t^5\,h^2 + \frac{49}{6}\,t^6\,h^2 - \frac{116}{21}\,t^7\,h^2 + \frac{25}{14}\,t^8\,h^2 - \frac{2}{9}\,t^9\,h^2$$

$$\beta_2 = -\frac{5557}{17418240}\,t\,h^2 + \frac{29}{1296}\,t^4\,h^2 - \frac{187}{2160}\,t^5\,h^2 + \frac{91}{648}\,t^6\,h^2 - \frac{517}{4536}\,t^7\,h^2 + \frac{17}{378}\,t^8\,h^2 - \frac{13}{1944}\,t^9\,h^2$$

$$\gamma_0 = -\frac{139}{18432}\,t\,h^3 + \frac{1}{6}\,t^3\,h^3 - \frac{7}{12}\,t^4\,h^3 + \frac{77}{80}\,t^5\,h^3 - \frac{53}{60}\,t^6\,h^3 + \frac{11}{24}\,t^7\,h^3 - \frac{1}{8}\,t^8\,h^3 + \frac{1}{72}\,t^9\,h^3$$

$$\gamma_{\frac{1}{2}} = \frac{173}{6480}\,t\,h^3 - \frac{32}{27}\,t^4\,h^3 + \frac{32}{9}\,t^5\,h^3 - \frac{592}{135}\,t^6\,h^3 + \frac{512}{189}\,t^7\,h^3 - \frac{52}{63}\,t^8\,h^3 + \frac{8}{81}\,t^9\,h^3$$

$$\gamma_1 = \frac{853}{161280}\,t\,h^3 - \frac{1}{3}\,t^4\,h^3 + \frac{6}{5}\,t^5\,h^3 - \frac{53}{30}\,t^6\,h^3 + \frac{53}{42}\,t^7\,h^3 - \frac{3}{7}\,t^8\,h^3 + \frac{1}{18}\,t^9\,h^3$$

$$\gamma_2 = \frac{379}{5806080}\,t\,h^3 - \frac{1}{216}\,t^4\,h^3 + \frac{13}{720}\,t^5\,h^3 - \frac{4}{135}\,t^6\,h^3 + \frac{37}{1512}\,t^7\,h^3 - \frac{5}{504}\,t^8\,h^3 + \frac{1}{648}\,t^9\,h^3$$

where $t = \frac{x - x_n}{h}$, $y_{n+j} = y(x_n + jh)$ and $f_{n+j} = f((x_n + jh), y(x_n + jh), y'(x_n + jh))$

Differentiating (6) once yields

$$p'(x) = \frac{1}{h}\sum_{j=0,\frac{1}{2}} \alpha_j y_{n+j} + h\left[\sum_{j=\frac{1}{2}} \beta_j f_{n+j} + \sum_{j=0}^{2} \beta_j f_{n+j}\right] + h^2\left[\sum_{j=\frac{1}{2}} \gamma_j g_{n+j} + \sum_{j=0}^{2} \gamma_j g_{n+j}\right] \quad (7)$$

The coefficients of f_{n+j}, and f_{n+k} give

$$\sigma_0 = -\frac{94457}{645120}\,h^2 + t\,h^2 - \frac{119}{12}\,t^3\,h^2 + \frac{433}{16}\,t^4\,h^2 - \frac{133}{4}\,t^5\,h^2 + \frac{511}{24}\,t^6\,h^2 - \frac{48}{7}\,t^7\,h^2 + \frac{7}{8}\,t^8\,h^2$$

$$\sigma_{\frac{1}{2}} = -\frac{10267}{136080}\,h^2 + \frac{256}{81}\,t^3\,h^2 + \frac{64}{27}\,t^4\,h^2 - \frac{448}{27}\,t^5\,h^2 + \frac{1472}{81}\,t^6\,h^2 - \frac{1472}{189}\,t^7\,h^2 + \frac{32}{27}\,t^8\,h^2$$

$$\sigma_1 = -\frac{2243}{80640}\,h^2 + \frac{20}{3}\,t^3\,h^2 - 29\,t^4\,h^2 + 49\,t^5\,h^2 - \frac{116}{3}\,t^6\,h^2 + \frac{100}{7}\,t^7\,h^2 - 2\,t^8\,h^2$$

$$\sigma_2 = -\frac{5557}{17418240}\,h^2 + \frac{29}{324}\,t^3\,h^2 - \frac{187}{432}\,t^4\,h^2 + \frac{91}{108}\,t^5\,h^2 - \frac{517}{648}\,t^6\,h^2 + \frac{68}{189}\,t^7\,h^2 - \frac{13}{216}\,t^8\,h^2$$

$$\Upsilon_0 = -\frac{139}{18432}\,h^3 + \frac{1}{2}\,t^2\,h^3 - \frac{7}{3}\,t^3\,h^3 + \frac{77}{16}\,t^4\,h^3 - \frac{53}{10}\,t^5\,h^3 + \frac{77}{24}\,t^6\,h^3 - t^7\,h^3 + \frac{1}{8}\,t^8\,h^3$$

$$\Upsilon_{\frac{1}{2}} = \frac{173}{6480}\,h^3 - \frac{128}{27}\,t^3\,h^3 + \frac{160}{9}\,t^4\,h^3 - \frac{1184}{45}\,t^5\,h^3 + \frac{512}{27}\,t^6\,h^3 - \frac{416}{63}\,t^7\,h^3 + \frac{8}{9}\,t^8\,h^3$$

$$\Upsilon_1 = \frac{853}{161280}\,h^3 - \frac{4}{3}\,t^3\,h^3 + 6\,t^4\,h^3 - \frac{53}{5}\,t^5\,h^3 + \frac{53}{6}\,t^6\,h^3 - \frac{24}{7}\,t^7\,h^3 + \frac{1}{2}\,t^8\,h^3$$

$$\Upsilon_2 = \frac{379}{5806080}\,h^3 - \frac{1}{54}\,t^3\,h^3 + \frac{13}{144}\,t^4\,h^3 - \frac{8}{45}\,t^5\,h^3 + \frac{37}{216}\,t^6\,h^3 - \frac{5}{63}\,t^7\,h^3 + \frac{1}{72}\,t^8\,h^3$$

Evaluating (7) at all points gives a discrete block formula of the form

$$A^{(0)}Y_m^{(i)} = \sum_{i=0}^{1} h^i e_i y_n^{(i)} + h^2 b_i f(y_n) + h^2 d_i f(Y_m) + h^3 c_i g(y_n) + h^3 r_i g(Y_m) \quad (8)$$

where

$$Y_m=\left[y_{n+\frac{1}{2}},\,y_{n+1},y_{n+2}\right]^T\quad,\quad f(y_m)=\left[f_{n+\frac{1}{2}},\,f_{n+1},f_{n+2}\right]^T\;,\;g(y_m)=\left[g_{n+\frac{1}{2}},\,g_{n+1},g_{n+2}\right]^T$$

$$y_n^{(i)}=\left[y_{n-1}^{(i)},\,y_{n-2}^{(i)},y_n^{(i)}\right]^T\quad,\quad f(y_n)=\left[f_{n-1},\,f_{n-2},f_n\right]^T\;,\;g(y_n)=\left[g_{n-1},\,g_{n-2},g_n\right]^T$$

and $A^{(0)}=3\times 3$ identity matrix.

When $i=0$

$$e_0=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\;,\;e_1=\begin{bmatrix}0&0&\frac{1}{2}\\0&0&1\\0&0&2\end{bmatrix}\;,\;b_0=\begin{bmatrix}0&0&\frac{94457}{1290240}\\0&0&\frac{437}{2520}\\0&0&\frac{173}{315}\end{bmatrix}\;,\;d_0=\begin{bmatrix}\frac{10267}{272160}&\frac{2243}{161280}&\frac{5557}{34836480}\\\frac{2152}{8505}&\frac{23}{315}&\frac{37}{68040}\\\frac{11264}{8505}&\frac{16}{315}&\frac{643}{8505}\end{bmatrix}$$

$$c_0=\begin{bmatrix}0&0&\frac{139}{36864}\\0&0&\frac{7}{720}\\0&0&\frac{2}{45}\end{bmatrix}\;,\;r_0=\begin{bmatrix}-\frac{173}{12960}&-\frac{853}{322560}&-\frac{379}{11612160}\\-\frac{92}{2835}&-\frac{1}{90}&-\frac{1}{9072}\\\frac{512}{2835}&\frac{16}{63}&-\frac{4}{405}\end{bmatrix}$$

When $i=1$

$$e_0=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\;,\;b_0=\begin{bmatrix}0&0&\frac{8399}{43008}\\0&0&\frac{23}{112}\\0&0&\frac{13}{21}\end{bmatrix}\;,\;d_0=\begin{bmatrix}\frac{1145}{4536}&\frac{139}{2688}&\frac{667}{1161216}\\\frac{32}{63}&\frac{2}{7}&\frac{1}{1008}\\\frac{1024}{567}&-\frac{16}{21}&\frac{191}{567}\end{bmatrix}$$

$$c_0=\begin{bmatrix}0&0&\frac{347}{30720}\\0&0&\frac{1}{80}\\0&0&\frac{1}{15}\end{bmatrix}\;,\;r_0=\begin{bmatrix}-\frac{1679}{30240}&-\frac{523}{53760}&-\frac{227}{1935360}\\-\frac{8}{315}&-\frac{1}{35}&-\frac{1}{5040}\\\frac{512}{945}&\frac{64}{105}&-\frac{31}{945}\end{bmatrix}$$

3. Analysis of Basic Properties of the Method

3.1. Order of the Block

According to Fatunla [12], the order of the new method in equation (8) is obtained by using the Taylor series and it is found that the developed method has a uniformly order nine, with an error constants vector of:

$$C_{10}=\left[\frac{457}{34681651200},\frac{17}{406425600},\frac{1}{1058400},\frac{173}{3715891200},\frac{1}{14515200},\frac{1}{453600}\right]^T$$

3.2. Consistency

The hybrid block method (8) is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent.

3.3. Zero Stability of Our Method

Definition: A block method is said to be zero-stable if as $h \to 0$, the root $z_i, i = 1(1)k$ of the first characteristic polynomial $\rho(z) = 0$ that is $\rho(z) = \det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right] = 0$ satisfies $|z_i| \le 1$ and for those roots with $|z_i| = 1$, the multiplicity must not exceed two. The block method for k=2, with one off-grid collocation point is expressed in the form

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \frac{h}{2} \\ 0 & 0 & 1 & 0 & 0 & h \\ 0 & 0 & 1 & 0 & 0 & 2h \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = z^4 (z-1)^2$$

$$\rho(z) = z^4 (z-1)^2 = 0, \quad z = 0, 0, 0, 0, 1, 1$$

Hence, our method is zero-stable.

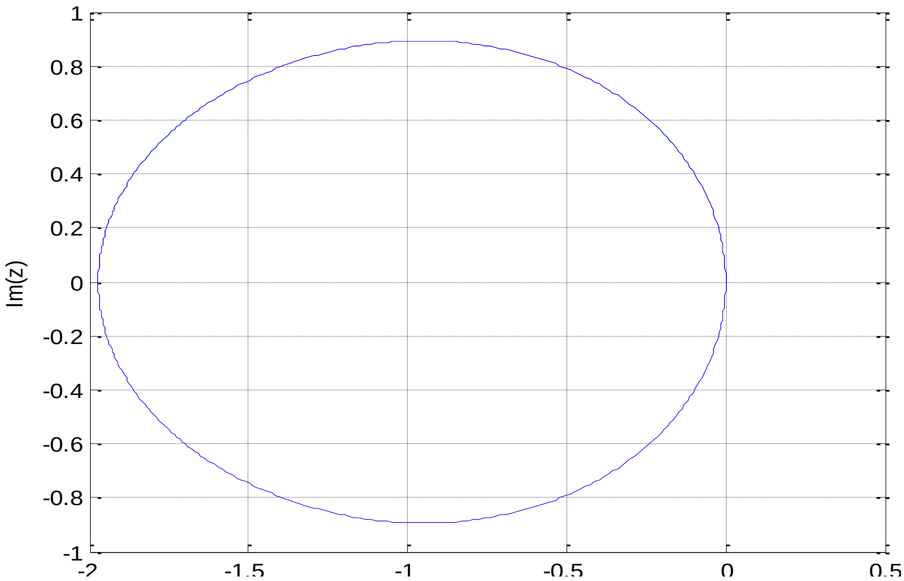
3.4. Region of Absolute Stability of the Two-Step One Off-Grid Point

We shall adopt the boundary locus method to determine the region of absolute stability of the implicit two-step third derivative hybrid block method. This gives stability polynomial below

$$\bar{h}(w) = h^9\left(\left(\frac{1}{1360800}\right)w^3 - \left(\frac{289}{38102400}\right)w^2\right) + h^8\left(\left(\frac{13711571}{72013536000}\right)w^2 + \left(\frac{229}{32659200}\right)w^3\right)$$
$$+ h^7\left(\left(\frac{1}{136080}\right)w^3 + \left(\frac{21724447}{5401015200}\right)w^2\right) + h^6\left(\left(\frac{2033}{5080320}\right)w^3 + \left(\frac{14473757}{2880541440}\right)w^2\right)$$
$$- h^5\left(\left(\frac{121}{36288}\right)w^3 + \left(\frac{1026013}{11430720}\right)w^2\right) + h^4\left(\left(\frac{191}{25920}\right)w^3 + \left(\frac{1729481}{57153600}\right)w^2\right)$$
$$- h^3\left(\left(\frac{145}{288}\right)w^2 + \left(\frac{89}{2592}\right)w^3\right) - h^2\left(\left(\frac{161}{864}\right)w^3 + \left(\frac{1567}{864}\right)w^2\right) + w^3 - w^2$$

Thus, the absolute stability region of the new method is plotted and shown below

Figure-3.1. Stability region of the method



4. Numerical Experiments

The method derived shall be employed in solving some problems so as to test how computationally reliable the method is.

Problem 4.1.

Consider a highly stiff problem

$$y''+1001y'+1000y, \quad y(0)=1, y'(0)=-1 \tag{9}$$

with the exact solution,

$$y(x)=\exp(-x) \tag{10}$$

Table-4.1. Showing the result for Problem 4.1

x-values	Exact Solution	Computed Solution	Error in our method	Error in [8]
0.100	0.90483741803595957316	0.90483741803595957253	6.300E(-19)	1.054712E(-14)
0.200	0.81873075307798185867	0.81873075307798180886	4.981E(-17)	1.776357E(-14)
0.300	0.74081822068171786607	0.74081822068171781705	4.9020E(-19)	2.342571E(-14)
0.400	0.67032004603563930074	0.67032004603563918790	1.1284E(-16)	2.797762E(-14)
0.500	0.60653065971263342360	0.60653065971263314623	2.7737E(-16)	3.130829E(-14)
0.600	0.54881163609402643263	0.54881163609402593546	4.9717E(-16)	3.397282E(-14)
0.700	0.49658530379140951470	0.49658530379140871125	8.0345E(-16)	3.563816E(-14)
0.800	0.44932896411722159143	0.44932896411722043065	1.16078E(-15)	3.674838E(-14)
0.900	0.40656965974059911188	0.40656965974059751543	1.59645E(-15)	3.730349E(-14)
1.00	0.36787944117144232160	0.36787944117144023830	2.0833E(-15)	3.741452E(-14)

Problem 4.2.

Consider the problem

$$f(x,y,y')=100y', \quad y(0)=1, \ y'(0)=-10 \tag{11}$$

with the exact solution,

$$y(x)=\exp(-10x) \tag{12}$$

Table-4.2. Showing the result for Problem 4.2

x-values	Exact Solution	Computed Solution	Error in our method	Error in [10]
0.01	0.90483741803595957316	0.90483741803595956538	7.780E(-18)	0.0000(+00)
0.02	0.81873075307798185867	0.81873075307798177085	8.782E(-17)	2.431388(-14)
0.03	0.74081822068171786607	0.74081822068171757330	2.9277E(-16)	7.105427(-14)
0.04	0.67032004603563930074	0.67032004603563873584	5.649E(-16)	1.384448(-13)
0.05	0.60653065971263342360	0.60653065971263248017	9.4343E(-16)	2.257083(-13)
0.06	0.54881163609402643263	0.54881163609402504957	1.38306E(-15)	3.316236(-13)
0.07	0.49658530379140951470	0.49658530379140759480	1.9199E(-15)	4.555800(-13)
0.08	0.44932896411722159143	0.44932896411721907405	2.51738E(-15)	5.974665(-13)
0.09	0.40656965974059911188	0.40656965974059590272	3.20916E(-15)	7.575052(-13)
0.10	0.36787944117144232160	0.36787944117143835538	3.96622E(-15)	9.361956(-13)
0.11	0.33287108369807955329	0.33287108369807473419	4.81910E(-15)	1.134093(-12)
0.12	0.30119421191220209664	0.30119421191219634900	5.74764E(-15)	1.352474(-12)

5. Conclusions

It is evident from the results obtained that our proposed method is indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is A-stable. Comparing the new method with the existing methods [8, 10]; the results presented in the Tables 4.1 and 4.2 show that the new method performs better than the existing methods [8, 10]. In this article, a two-step block method with one off-step point has been derived via the interpolation and collocation approach. The developed method is also consistent, convergent and zero-stable.

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