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Equivalent Construction of Ordinary Differential Equations from Impulsive System

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Abstract

We construct an ordinary differential equation representation of an impulsive system by a bijective transformation that structurally maps the solutions of the initial value problem of the impulsive differential equations to the solutions of the initial value problems of the ordinary differential equations. Established in this work is the relationship between impulsive differential equations and ordinary differential equations which play a fundamental role in qualitative analysis of the former. It is also established that an n-dimensional impulsive differential equation can be represented in terms of a 2n-dimensional ordinary differential equation. Figures are used to demonstrate the practicability of the methodology developed.

Keywords: Stability; Impulsive; Ordinary differential equations; Bijective transformation.

1. Introduction

Various evolutionary processes from fields such as population dynamics, aeronautics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moment of times between intervals of continuous evaluation. Since the duration of this changes are often very small compared to the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or impulses. These processes tend to be more suitably modelled by impulsive differential equations, which allow for discontinuities in evaluation of the state. Impulsive differential

equations are usually defined by a pair of equations, an ordinary differential equation to be satisfied during the continuous portion of evolution and a difference equation defining the discrete impulsive actions.

Impulsive differential equations seem to have received very little attention not until in 1980s when interest in the area began to gather momentum. Among the earliest articles on impulsive differential equations was a seminar paper by Milman and Myshkis [1] where they considered differential equations with impulses occurring when certain spatio-temporal relations were satisfied [2-4].

Research into impulsive differential equations had culminated in the publishing of several monographs and articles [5-8]. These authors consider an impulsive differential equation to be an ordinary differential equation coupled with a difference equation to be satisfied at certain fixed or variable impulse times. The resulting solutions are thereby piecewise continuous with discontinuities occurring at these impulse times. This approach enabled them to apply many well established results for ordinary differential equations to these systems in order to develop the qualitative theory of impulsive differential equations which is still at its infancy. A few recent results in this new area can be found in [9-14].

Due to the nature of impulsive processes which are momentarily exposed to harsh impacts, their qualitative analysis is more complicated than that of ordinary differential equations. This work seeks to fill this gap by formulating a differential system that is equivalent to the impulsive system, thereby simplifying the analysis of the later.

To help in our investigation, we will define some important concepts.

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2. Definitions of Basic Terms/Concepts

2.1. Ordinary Differential Equation

Let a process evolve in a period of time T in an open set $\Omega\subset T\times R^n$, where $T:=(a,b)\subset R$. Let $f:\Omega\to R^n$ be an at least a continuous mapping fulfilling local Lipchitzian condition in $x\in R^n,\forall(t,x)\in\Omega$. Then an initial value problem of a differential equation is given by

$$\begin{cases} x'(t)=f(t,x(t)),\forall t\in T,(t,x(t))\in\Omega \\ x(t_0)=x_0, \quad t_0\in T\setminus S, \quad (t_0,x_0)\in\Omega \end{cases} \tag{1.1}$$

Definition 1.1. (Fixed Point): $x_0\in R^n$ is a fixed point of equation (1.1) if $\exists\,(a,b)\subset T$ such that $f(t,x_0)=0$ holds $\forall(t,x)\in(a,b)\times\{x_0\}\subset\Omega$ and (a, b) is maximal with this property. Hence or otherwise, x_0 is a fixed point of equation (1.1) if the constant function $x(t)\equiv x_0,\forall t\in(a,b)$ is a maximal solution of equation (1.1).

Definition 1.2. (Impulsive Differential Equation): The usual model for the simplest case of an impulsive differential equation
is as follows:

Let a process evolve in a period of time T in an open set $\Omega\subset T\times R^n$. Let $f:\Omega\to R^n$ be at least a continuous mapping fulfilling local Lipschitzian condition in $x\in R^n,\forall(t,x)\in\Omega$. Let the real time sequence $S=\{t_k\}_{k=1}^{\infty}\in T$ be increasing without finite accumulation points. Let $g:S\times R^n\to R^n$ be continuous/Lipschitzian function in its variable $x\ \forall\,(t_k,x)\in\Omega$. Then the controlling impulsive differentiable equation is given by

$$\begin{cases} x'(t)=f(t,x(t)),\forall t\in T\setminus S \\ \Delta x(t)=g(t_k,x(t_k)),\forall t_k\in S, \\ x(t_0)=x_0,t_0\in T\setminus S,(t_0,x_0)\in\Omega \end{cases} \tag{1.2}$$

Definition 1.3. (Motion): Let (X,d) be a metric space. Let $A\subset X$ and $T\subset R$. For any fixed $a\in A,\ t_0\in T$, a mapping
$p:T\times\{a\}\times\{t_0\}\to X$ is called a motion if

i) $p(t_0,a,t_0)=a$;

ii) $\forall t_1,t_2\in T,\ t_0\le t_1\le t_2,\ p(t_2,a,t_0)=p(t_2,p(t_1,a,t_0),t_1)$.

Definition 1.4. (Family of Motions): The family S of motions is defined as $S\subset P_{A,T}$, where

$P_{a,t_0}:=\{\,p:T\times\{a\}\times\{t_0\}\to X$ is a motion $\forall(a,t_0)\in A\times T\,\}$;

$$P_{A,T}:=\bigcup_{(a,t)\in A\times T}P_{a.t}$$

such that within S, requirements (i) and (ii) of Definition 1.3 are fulfilled.

Definition 1.5. (Dynamical System): The four-tuple $\{T,X,A,S\}$ is called a dynamical system.

Definition 1.6. (Bounded Motion): A motion $p\in S$ of a dynamical system $\{T,X,A,S\}$ is said to be bounded if there exist $x_0\in X$ and $\beta>0$ such that $d(p(t,a,t_0),x_0)<\beta$ for all $t\ge t_0$.

Definition 1.7. (Continuation of Motion): Let $\{T,X,A,S\}$ be a dynamical system and let $p,p^*\in P_{a,t_0},(a,t_0)\in A\times T$. Then
p^* is a continuation of p if for $t_0<b<c$, p is defined on $[t_0,b)$, p^* is defined on $[t_0,c)$ and $p(t,a,t_0)=p^*(t,a,t_0),\forall t\in[t_0,b)$.

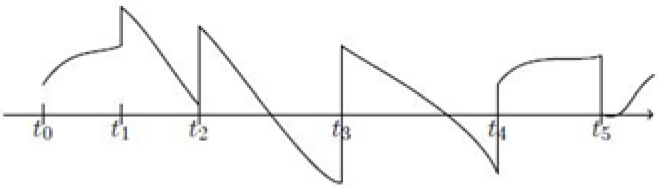
Definition 1.8. (Composite Dynamical System): A dynamical system $\{T,X,A,S\}$ is called a composite dynamical system if given the metric space $(X, d), \exists \{\{X_j, d_j\}: 1 \le j \le l\}$ such that $X = \prod_{j=1}^{l} X_j$ and $d = \max_{1 \le j \le l} d_j$ (or any alternative for product metric like sum, Euclidean distance etc. may be used).

Definition 1.9. (Diffeomorphism): Let $f : R^n \to R^m$, $n \le m$ be a bijective differentiable map. f is called a diffeomorphism
if f and f^{-1} are continuously differentiable (C^1-diffeomorphism). If f & f^{-1} are r-times continuously differentiable, then f is C^r-diffeomorphism and in the case of $r = 1$, f is C^1 diffeomorphism.

2.2. Dynamical System

A dynamical system is a concept in Mathematics where a fixed rule describes how a point in a geometrical space depend on time. Examples include the Mathematical model that describes the swinging of a clock pendulum, the flow of water in a pipe, etc. At any given time, a dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space. Small changes in the state of the system create small changes in the numbers. The evolution rule of the dynamical system is a fixed rule that describes what future state follows from the current state. The rule is deterministic; in other words, for a given time interval only one future state follows from the current state [15-17]. In what follows, we will denote a dynamical system as a four-tuple: $\{T, X, A, S\}$, where T denotes time set, X is the state space (a metric space with metric d), A is the set of initial states, and S denotes a family of motions. When $T = R^+ = [0,1)$, we speak of a continuous time dynamical system and when $T = N$ we speak of a discrete time dynamical system. For any motion $x(x_0, t_0) \in S$, we have $x(t_0, x_0, t_0) = x_0 \in A \subset X$ and $x(t, x_0, t_0) \in X, \forall t \in [t_0, t_1] \cap T, t_1 > t_0$, where t_1 may be finite or infinite. The set of motions S is obtained by varying (t_0, x_0) over $T \times A$. A dynamical system is said to be autonomous if every $x(., x_0, t_0) \in S$ is defined on $T \cap [t_0, \infty)$ such that for each $t_0 + \tau \in T$, there exist a motion $x(., x_0, t_0 + \tau) \in S$ such that $x(t_0 + \tau; x_0, t_0 + \tau) = x(t; x_0, t_0), \forall t$ and τ satisfying $t + \tau \in T$.

Figure-1. A trajectory of an impulsive differential equation



3. Statement of the Problem

Qualitative analysis has proved to be an important and useful tool to investigate the properties of solutions of differential equations, because it is able to analyze differential equations without solving analytically and numerically. The study of qualitative properties of differential equations has a long history, and qualitative theories have been developed for various equations such as ordinary differential equations, functional differential equations, integral equations, etc. Here we seek to find an equivalent representation of an impulsive differential equation by way of ordinary differential equations which makes qualitative analysis less cumbersome. This is done through some special dynamical system techniques. It is worthy to note that these aspects of investigations are at their infancy whereas their practical importance are highly rated.

4. Methodology

4.1. Transformation from Impulsive to Absolute Continuous Trajectory

Certain traditional qualitative analysis of ordinary differential equations is based on the properties of the absolute continuous solutions. These methods cannot be transferred to impulsive differential equations. Moreover, the solution of an impulsive differential equation is based on two independent ”forces”. One is the normal dynamics of the process while the other is a dynamical impact called impulses. The impulses are short term high power impacts which make their contributions to the dynamical behaviour of the system. This implies that the analysis of an impulsive system means to study the interaction of the ”two forces” (external impact and the normal dynamics). The first step to this study is the introduction of a transformation of the impulsive trajectories into absolute continuous trajectories and absolute continuous trajectories to impulsive trajectories. First we explain the underlying assumptions.

Assumption 4.1. Essentially, two assumptions will be used:

i) Bainov and his co-authors use the condition that the set of impulse points S does not have finite accumulation points. This means that in every compact subset of $K \subset T$, $K \cap S \subset T$ is a finite set.

ii) The condition in item (i) can be reformulated as follows: For any compact set $K \subset T$, the total variation of the solution

$x:[t_0,\alpha) \to R^N, \underset{K}{V}x < \infty$. The second condition implies the first when the discontinuity points form a finite set in all compact sets K.

Let $x:[t_0,\alpha) \to R^N$ be a solution of the initial value problem described in equation (1.2). By Assumption 4.1(i), we will have finite jump/impulse points in the interval $[t_0,\alpha) \subset [t_0,\alpha]$ provided that $[t_0,\alpha]$ is closed and bounded, hence compact, and we transform the impulsive trajectory onto the absolute continuous trajectory $\hat{x}:\left[t_0, \overset{\alpha}{\underset{s=t_0}{\vee}} x(s)\right] \to R^n$. This is stated in the following theorem:

Theorem 4.1. Defining the mapping in item (ii) above by induction:

Step 1. (a) Let $\hat{t}_0 := t_0$ *and* $\tau_0 := t_1$. We map $[t_0,t_1) \to [\hat{t}_0,\tau_0]$ using $s \in [t_0,t_1) \to \hat{t}_0 + s - t_0$ mapping and $\hat{x}(s - t_0 + \hat{t}_0) := x(s), \forall s \in [t_0,t_1)$.

(b) Let the jump of x *at* t_1 *be* $j_1 := x(t_1+0) - x(t_1-0)$. Then let $\hat{t}_1 := \tau_0 + j_1$ and let $\hat{x}(s) := 1 \times (s - \tau_0) + x(t_1 - 0)\ \forall\ s \in [\tau_0, \hat{t}_1]$. The linear function with gradient = 1 fulfils

$$\hat{x}(s) = \begin{cases} x(t_0-0) & s = \tau_0 \\ x(t_0+0) & s = \hat{t}_1 \end{cases}.$$

Step 2. Assume that $\hat{x}$ has been defined on the interval $[\hat{t}_0,\hat{t}_p)$ *and* $\{\tau_j\}_{j=0}^{p-1}$ are also defined.

(a) We define the image of the trajectory x on the interval $[t_p,t_{p+1})$ as follows:

Let $\tau_p := \hat{t}_p + t_{p+1} - t_p$. We map $[t_p,t_{p+1}) \to [\hat{t}_p,\tau_p]$ using the mapping $s \in [t_p,t_{p+1}) \to \hat{t}_p + s - t_p \in [\hat{t}_p,\tau_p]$ and $\hat{x}(s - t_p + \hat{t}_p) := x(s), \forall s \in [t_p,t_{p+1})$.

(b) Let the jump of x at t_{p+1} be $j_{p+1} = x(t_{p+1}+0) - x(t_{p+1}-0)$. Again, let $\hat{t}_{p+1} := \tau_p + j_{p+1}$ and let $\hat{x}(s) := 1 \times (s - \tau_p) + x(t_{p+1} - 0)\ \forall s \in [\tau_p, \hat{t}_{p+1}]$

This defines the absolute continuous trajectory for $[t_0,t_k)$, for any $t_k \in S$. Figure 2 shows the whole process. This construction will be very fundamental in our further discussions.
Let us first analyse and make some inferences about this:

Figure-2. The construction of the transformation of an impplusive trajectory to an absolute continuous one. $\hat{t}_i - \tau_{i-1} = j_i$.

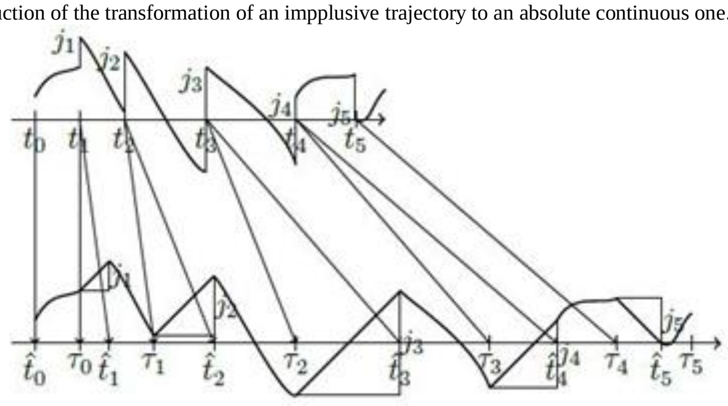


Figure-3. The figure shows the function with g defined in equation (3.15). We only presented the mapping on interval. This already shows that the segments of straight lines have infinite crossing points

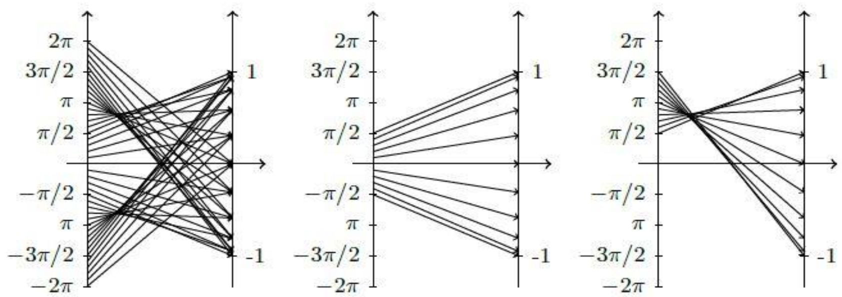


Figure-4. The Figure show three impulsive trajectories the jumps of which connect trajectories x and y into one trajectory while the segment of straight lines defined by jumps in trajectories y and zwill have a crossing point p hence these segment of straight lines cannot be obtained as solutionof adiffferential equation since the right side should have two different values

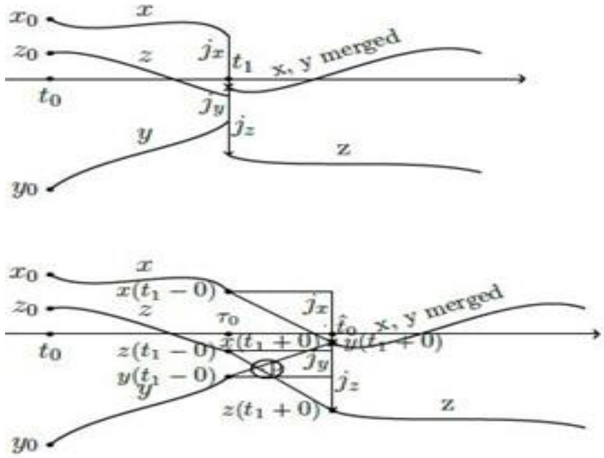


Figure-5. The common points {p}= $[x^-, x^+] \cap [y^-, y^+]$ of the segment of straight lines $[x^-, x^+]$ and $[y^-, y^+]$ can be separated by extruding one of the segment into an additional dimension with the help of flyover . the two curves one flyover connecting x^- to x^+ and the segment $[y^-, y^+]$ are obtainable as soultion of a differential equation

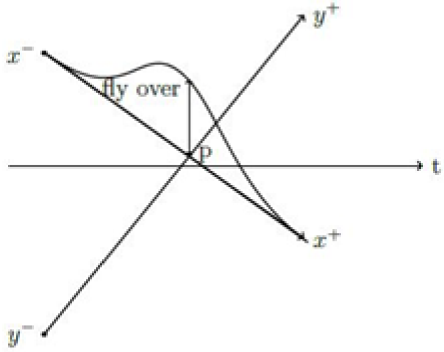
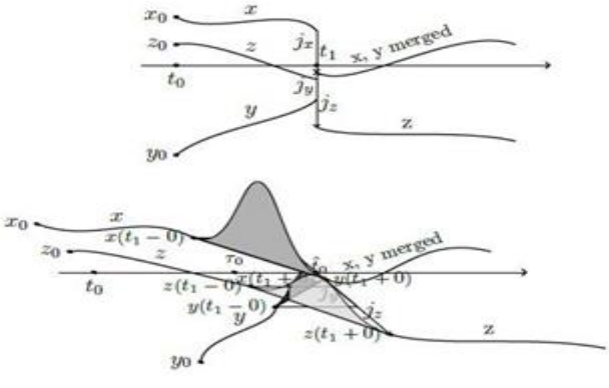


Figure-6. We creat a flyover for each curve in extra dimension so that the since the starting points of the segment are different for each different segment the flyover will be different for each point in the segment



In our result, we define the right side of the differential equation by using the gradients of the segments of straight lines connecting the left and right limits of the trajectory at impulse points. Let us consider the following case: This example shows that we cannot expect a unique gradient to any point of the region between τ_j and $\hat{t}_j + 1$. Let

$$g\left(t_j, y\right) := -y + \sin y,\ y \in R. \tag{4.1}$$

This jump function will produce infinite overlapping segments of straight lines as figure 3 shows and we cannot define a function by using the segments of straight lines $\left\{\left[\left(\tau_j, y\right), \left(\hat{t}_{j+1}, \sin y\right)\right]\right\}_{y \in R}$. The general form of the problem is shown on Figure 4. Also note on the cited figure that two solutions may be merged into one. It is worth to note that impulsive dynamics can merge two solutions into one and it is known that a differential equation with continuous right side may have infinite set of different solutions for some initial value problems. Merging trajectories define an equivalence relation which classifies the solutions. This is an interesting and important feature of impulsive differential equations. In the final analysis the asymptotic limits classify behavioural patterns. It is important to note that an impulse $y \in R^n \to y + g(t_k, y)$ is uniquely determined by $g(t_k, y), \forall (t_k, y) \in S \times R^n$, hence an impulse cannot create two trajectories from one. The way out of the problem of defining a suitable right hand side to an ordinary differential equation is shown on Figure 5. The idea is to use flyover curves to remove the overlapping segments of straight lines. The segments of straight lines $\left[x^-, x^+\right]$ and $\left[y^-, y^+\right]$ have a common point p. We have a separate connection by routing $\left[x^-, x^+\right]$ through a curve passing over $p \in \left[y^-, y^+\right]$. Figure 6 shows the idea that actually each straight line segment can be routed through an individual route, at a cost of doubling the dimension of the space.

The analysis here shows that the technique defined in Theorem 4.1 can be extended to an ordinary differential equation with right side measurable in t which we will construct in the next theorem. This will require the use of a ”flyover” technique and to make it easy and simple, we will work with a space with double dimension.

5. Results Obtained

5.1. Construction of an Associated Ordinary Differential Equation from an Impulsive Differential Equation

Let the impulsive differential equation be defined by equation (1.2). We will follow the steps of the construction of the absolute continuous trajectory of the Caratheodory type absolute continuous equations with a mapping connecting the two trajectories.

As a preparatory step, we will establish a relationship between a set

$$S := \left\{ t_j \mid t_j \in R,\ t_j < t_{j+1}, \forall j \in \mathbb{N} \right\}$$

of time point (impulse points) and sequences of intervals

$$S_c := \left\{ \left[\hat{t}_j, \tau_j\right) \mid \left[\hat{t}_j, \tau_j\right) \subset R,\ \hat{t}_j < \tau_j < \hat{t}_{j+1}, \forall\ j \in \mathbb{N} \right\};$$

$$S_i := \left\{ \left[\tau_j, \hat{t}_{j+1}\right) \mid \left[\tau_j, \hat{t}_{j+1}\right) \subset R, \hat{t}_j < \tau_j < \hat{t}_{j+1}, \forall j \in \mathbb{N} \right\}.$$

Notation 5.1. Let us denote by $\hat{S} := \left\{\hat{t}_j\right\}_{j=0}^{\infty}$, $S_U := \bigcup_{j=0}^{\infty} \left[\hat{t}_j, \tau_j\right)$ and $S_O := \bigcup_{j=0}^{\infty} \left[\hat{t}_j, \tau_j\right)$ the set of images of impulse points and unions of the intervals in S_c and its interior respectively.

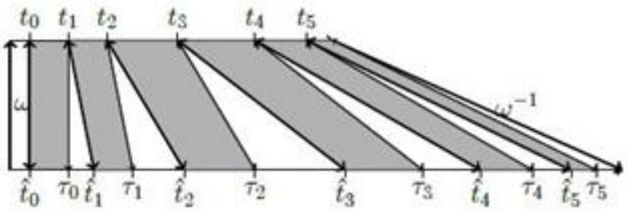
Definition 5.1. Let $\omega := S_U \to \left[t_0, \infty\right)$ be defined as follows:

$\omega\left(\hat{t}_k\right) := t_k$ and $\omega\left(\tau_k - 0\right) := t_{k+1}$ $\forall k \in \mathbb{N}$;

$$\omega(t) := t_k + \frac{\left(t - \hat{t}_k\right)\left(t_{k+1} - t_k\right)}{\tau_k - \hat{t}_k} \forall t \in \left[\tau_k, \hat{t}_k\right), \forall k \in N.$$

Moreover, let $\omega\left(\tau_k\right) := \omega(\tau_k - 0) = t_{k+1}$ $\forall k \in \mathbb{N}$.

Figure-7. the mappings ω *and* ω^{-1} are represented on the figure. The figure shows the idea of the creation of an absolute continuous continuation of an impulsive function. We map the impulse points in to intervals and there we define mixer connecting the source to the target for each impulse



Lemma 5.1. *The mapping* $\omega : S_U \to \left[t_0, \infty\right)$ *is a bijective continuous mapping if the restriction of the inverse* ω^{-1} *to* $[t_0, \infty) \setminus S$, $\omega^{-1} : [t_0, \infty) \setminus S \to S_O$ *is a* C^∞ *homeomorphism with*

$$\frac{d\omega(t)}{dt} = \frac{\tau_k - \hat{t}_k}{t_{k+1} - t_k}, \forall t \in \left(t_k, t_{k+1}\right), \forall k \in N.$$

Note that here we are going to use a uniform difference $\hat{t}_{j+1} - \tau_j = 1$ since we work with $g\left(t_{j+1}, y\right) \in R^n, \forall \left(t_k, y\right) \in S \times R^n$ and the values of g vary along R^n. However, g is continuous, hence the values are bounded on any compact subset of R^n. These investigations work with pre-compact neighbourhoods, hence their closure is compact. Therefore continuity of g, and hence its local boundedness grant conditions for gradients in our investigations similar to the result of the construction in Theorem 4.1.

Lemma 5.2. Let

$$d(s):=\begin{cases} e^{-1/s(1-s)} & s\in(0,1) \\ 0 & y\in R\backslash(0,1) \end{cases} \tag{4.2}$$

Then d is an infinitely differentiable function on R and positive in (0, 1).

Proof: (See [18-20])

Lemma 5.3. Let

$$m:y\in R^n\rightarrow\begin{cases} y & \forall\ y\in\bar{B}_1(0) \\ \left(2-\frac{1}{\|y\|}\right)\frac{y}{\|y\|} & y\in R^n\setminus\bar{B}_1(0) \end{cases} \tag{4.3}$$

Then m is a differentiable homeomorphism, hence a diffeomorphism of $R^n\rightarrow B_2(0)$. Moreover,

$$\left\|\frac{dm(y)}{dy}\right\|_{R^n\times R^n}\leq\begin{cases} 1 & if\ \|y\|\geq 1 \\ \frac{2}{\|y\|} & if\ \ \|y\|\geq 1 \end{cases} \tag{4.4}$$

Hence

$$\left\|\frac{dm}{dy}\right\|_C\leq 2\,.$$

Proof: Step 1: $m(y)=id_{\bar{B}_1(0)}(y)\ =y,\forall y\in\bar{B}_1(0)$ hence it is a C^∞–diffeomorphism in $\bar{B}_1(0)$.

Step 2: $m:y\in R^n\setminus\overline{B_1(0)}\rightarrow\left(2-\frac{1}{\mathrm{P}y\mathrm{P}}\right)\frac{y}{\mathrm{P}y\mathrm{P}}\in B_2(0)\setminus B_1(0)$ is a C^∞ mapping, such that this is a differentiable continuation of m as defined on $\overline{B_1(0)}$.

Step 3: $m:R^n\rightarrow B_2(0)$ is bijective: It is onto by the formula. If $y_1,y_2\in R^n\ \ and\ y_1\neq y_2$ then we have two cases:

(a) $\mathrm{P}y_1\mathrm{P}\neq\mathrm{P}y_2\mathrm{P}$: $\left\|\left(2-\frac{1}{\mathrm{P}y_1\mathrm{P}}\right)\frac{y_1}{\mathrm{P}y_1\mathrm{P}}\right\|=\left(2-\frac{1}{\mathrm{P}y_1\mathrm{P}}\right)\neq\ \left(2-\frac{1}{\mathrm{P}y_2\mathrm{P}}\right)=\left\|\left(2-\frac{1}{\mathrm{P}y_2\mathrm{P}}\right)\frac{y_2}{\mathrm{P}y_2\mathrm{P}}\right\|\Rightarrow m(y_1)\neq m(y_2)$.

(b) $\mathrm{P}y_1\mathrm{P}=\mathrm{P}y_2\mathrm{P}$: $\left(2-\frac{1}{\mathrm{P}y_1\mathrm{P}}\right)\frac{y_1}{\mathrm{P}y_1\mathrm{P}}=y_1\frac{2-\frac{1}{\mathrm{P}y_1\mathrm{P}}}{\mathrm{P}y_1\mathrm{P}}\neq\frac{2-\frac{1}{\mathrm{P}y_1\mathrm{P}}}{\mathrm{P}y_1\mathrm{P}}y_2=\frac{2-\frac{1}{\mathrm{P}y_2\mathrm{P}}}{\mathrm{P}y_2\mathrm{P}}y_2\Rightarrow m(y_1)\neq m(y_2)$ by the assumption that $y_1\neq y_2$.

The norm of $\left\|\frac{dm}{dy}\right\|_C$ comes from the maximum of the norms of the derivatives of m.

Verification: The mapping is defined in equation (4.3). We recall this definition.

$m(y)=y,\forall\ y\in\overline{B_1(0)}$ hence $\frac{dm(y)}{dy}=I:R^n\rightarrow R^n$, the unit matrix.

$m(y):=\left(2-\frac{1}{\mathrm{P}y\mathrm{P}}\right)\frac{y}{\mathrm{P}y\mathrm{P}},\forall y\in R^n\setminus\overline{B_1(0)}$.

We will use the fact that this function is defined in terms of polar co-ordinates: The radius is defined by

$$f(\rho):=\left(2-\frac{1}{\rho}\right)\underset{\rho=\mathrm{P}y\mathrm{P}}{=}\left(2-\frac{1}{\mathrm{P}y\mathrm{P}}\right)\quad y\in R^n\setminus\overline{B_1(0)};$$

$$e(y):=\frac{y}{\mathrm{P}y\mathrm{P}}\quad y\in R^n\setminus\overline{B_1(0)},$$

where $e(y)$ is a unit vector pointing to y. Note that ρ is a function of $y\in R^n,\rho(y):=\mathrm{P}y\mathrm{P}$ and $m(y)=f(\mathrm{P}y\mathrm{P})e(y)$.

Now we compute the derivative of m:

$$\frac{dm(y)}{dy}=\frac{df(\rho)}{d\rho}\frac{d\rho}{dy}e(y)+f(\rho)\frac{de(y)}{dy}\ \forall\ y\in R^n\setminus\overline{B_1(0)};$$

$$\frac{dm(y)}{dy}=\frac{1}{\mathrm{P}y\mathrm{P}^2}\frac{<y,y^T>}{\mathrm{P}y\mathrm{P}^2}+\left(2-\frac{1}{\mathrm{P}y\mathrm{P}}\right)\frac{1}{\mathrm{P}y\mathrm{P}}\left(I-\frac{<y,y^T>}{\mathrm{P}y\mathrm{P}^2}\right)=$$

$$\frac{1}{PyP}\left(\frac{1}{PyP}\frac{<y,y^T>}{PyP^2}+\left(2-\frac{1}{PyP}\right)\left(I-\frac{<y,y^T>}{PyP^2}\right)\right)$$

$\forall\ y\in R^n\setminus\overline{B_1(0)}.$

As a preparation for the main estimation, we group the big parenthesis on the right side:

$$\frac{1}{PyP}\frac{<y,y^T>}{PyP^2}+\left(2-\frac{1}{PyP}\right)\left(I-\frac{<y,y^T>}{PyP^2}\right)=$$

$$\left[\left(1-\frac{1}{PyP}\right)\left(I-\frac{<y,y^T>}{PyP^2}\right)+\frac{1}{PyP}\frac{<y,y^T>}{PyP^2}\right]+$$

$$\left(I-\frac{<y,y^T>}{PyP^2}\right).$$

We will now estimate the norms of the two terms on the right side. We use the following properties:

Remark 5.1.

(a) The operators $\frac{<y,y^T>}{PyP^2}$ and $I-\frac{<y,y^T>}{PyP^2}$ are orthogonal projectors hence their operator norms are 1.

(b) $0\le\frac{1}{PyP}\le 1$ by the selection $i\in R^n\setminus\overline{B_1(0)}$.

The estimates based on Remarks 3.2 are as follows:

$$\left\|\left(1-\frac{1}{PyP}\right)\left(I-\frac{<y,y^T>}{PyP^2}\right)+\frac{1}{PyP}\frac{<y,y^T>}{PyP^2}\right\|\le\left\|\left(1-\frac{1}{PyP}\right)\left(I-\frac{<y,y^T>}{PyP^2}\right)\right\|+\left\|\frac{1}{PyP}\frac{<y,y^T>}{PyP^2}\right\|\le\left(1-\frac{1}{PyP}\right)\times 1+\frac{1}{PyP}\times 1=1;$$

$$\left\|\left(I-\frac{<y,y^T>}{PyP^2}\right)\right\|\le 1.$$

From this, the statement of the lemma follows inevitably:

$$\left\|\frac{dm(y)}{dy}\right\|=\left\|\frac{1}{PyP}\left(\frac{1}{PyP}\frac{<y,y^T>}{PyP^2}+\left(2-\frac{1}{PyP}\right)\left(I-\frac{<y,y^T>}{PyP^2}\right)\right)\right\|\le\frac{2}{PyP}\ \forall\ y\in R^n\setminus\overline{B_1(0)}.$$

Corollary 5.1. The mapping m fulfils a global Lipschitz condition with Lipschitz constant $L=2$. Moreover it has a local Lipschitz property with Lipschitz constant $L_{loc,y}=\frac{2}{PyP},\ \forall\ y\in R^n\setminus\overline{B_1(0)}$.

Corollary 5.2. The mapping

$$h_\varepsilon(s,y):=\frac{\varepsilon d(s)m(y)}{2d(0.5)} \tag{4.5}$$

is a homeomorphism $R^n\to B_\varepsilon(0)$, $\forall$ fixed $t\in(0,1)$, $h_\varepsilon(0,y)=h_\varepsilon(1,y)=0$.

Proof:. $m:R^n\to B_2(0)$, and $d:[0,1]\to[0,e^{\frac{-1}{4}}]$. Hence $PmP_C=2$, $PdP_C=e^{\frac{-1}{4}}=d(\frac{1}{2})$. From this follows that

$$h_\varepsilon(s,y):=\frac{\varepsilon d(s)m(y)}{2d(0.5)}<\varepsilon,\forall(s,y)\in[0,1]\times R^n.$$

Lemma 5.4. $\forall\,\eta>0\,\exists\varepsilon(\eta)>0$ such that function h_1 fulfils the condition that

$$0\le h_1(y)<\varepsilon(\eta)PyP,\forall y\in B_\eta. \tag{4.6}$$

Lemma 5.5. Let

$$\lambda:(s,y^-,y^+)\in[0,1]\times R^n\times R^n\to s(y^+-y^-)+y^-\in R^n. \tag{4.7}$$

If $y_1^-\ne y_2^-$ and $\lambda(s_1,y_1^-,y_1^+)=\lambda(s_2,y_2^-,y_2^+),(s_1,y_1^-,y_1^+),(s_2,y_2^-,y_2^+)\in[0,1]\times R^n\times R^n$ then $s_1=s_2$.

Proof:. The situation is shown in figure 8. Let $y_1^- \neq y_2^-$. Let $p := \lambda(s_1, y_1^-, y_1^+) = \lambda(s_2, y_2^-, y_2^+), s_1, s_2 \in (0,1]$. Then the points $\{y_1^-, y_2^-, p, y_1^+, y_2^+\}$ are in one plane Pl and in this plane, the hyperplanes $\{\tau_j\} \times R^n \ \& \{\hat{t}_{j+1}\} \times R^n$ are represented by two parallel lines perpendicular to the time axes (see Figure 8). Let the coordinates of $p =: (s(p), u(p))$:

i) If $(s(p), u(p)) \in R \times R^n \setminus [0,1) \times R^n$ then no other $q \in [0,1) \times R^n$ can be common since the coincidence of two straight lines with two different common points contradicts the fact that $y_1^- \neq y_2^-$.

ii) We may assume that $y_1^+ \neq y_2^+$. In the plane Pl the vectors $(\tau_j, y_1^- - y_2^-)$ and $(\hat{t}_{j+1}, y_1^+ - y_2^+)$ are parallel hence the triangles $\Delta(y_1^-, y_2^- p)$ and $\Delta(y_1^+, y_2^+ p)$ are similar (see figure 8).

iii) Then the ratios fulfil:

$$\frac{\|\lambda(s_1, y_1^-, y_1^+) - \lambda(0, y_1^-, y_1^+)\|}{\|\lambda(1, y_1^-, y_1^+) - \lambda(s_1, y_1^-, y_1^+)\|} = \frac{\|\lambda(s_2, y_2^-, y_2^+) - \lambda(0, y_2^-, y_2^+)\|}{\|\lambda(1, y_2^-, y_2^+) - \lambda(s_2, y_2^-, y_2^+)\|} \tag{4.8}$$

and using formula (4.7), we have that

$$\begin{aligned}
&\lambda(s_1, y_1^-, y_1^+) - \lambda(0, y_1^-, y_1^+) = (s_1(y_1^+ - y_1^-) + y_1^-) - y_1^- = s_1(y_1^+ - y_1^-),\\
&\lambda(1, y_1^-, y_1^+) - \lambda(s_1, y_1^-, y_1^+) = y_1^+ - (s_1(y_1^+ - y_1^-) + y_1^-) = (1 - s_1)(y_1^+ - y_1^-),\\
&\lambda(s_2, y_2^-, y_2^+) - \lambda(0, y_2^-, y_2^+) = (s_2(y_2^+ - y_2^-) + y_2^-) - y_2^- = s_2(y_2^+ - y_2^-),\\
&\lambda(1, y_2^-, y_2^+) - \lambda(s_2, y_2^-, y_2^+) = y_2^+ - (s_2(y_2^+ - y_2^-) + y_2^-) = (1 - s_2)(y_2^+ - y_2^-).
\end{aligned} \tag{4.9}$$

Putting these back into equation (4.8), we get

$$\frac{s_1\|y_1^+ - y_1^-\|}{(1 - s_1)\|y_1^+ - y_1^-\|} = \frac{s_1}{1 - s_1} = \frac{s_2}{1 - s_2} = \frac{s_2\|y_2^+ - y_2^-\|}{(1 - s_2)\|y_2^+ - y_2^-\|}. \tag{4.10}$$

Equation (4.10) holds for $s_1 = s_2, s_1, s_2 \in (0,1)$ only. This completes the proof.

Figure-8. THE figure shows the plane determined by two intersecting jump segments $[y_1^-, y_1^+]$ and $[y_2^-, y_2^+]$. The common points is $\{p\} := [y_1^-, y_1^+] \cap [y_2^-, y_2^+]$.

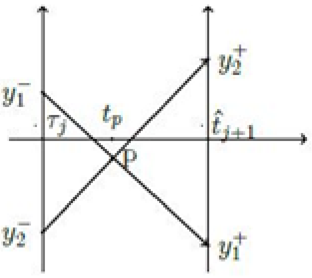
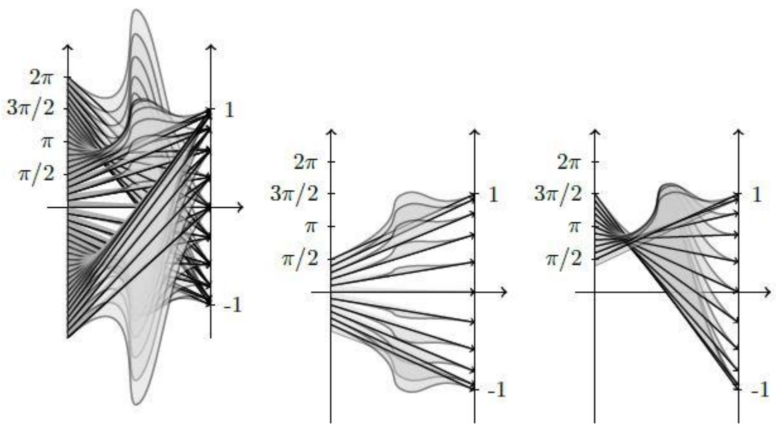


Figure-9. The figure represents the flyover system placed on figure 3. These are the representations of a sinusoidal jump function g as defined in equation (3.15). The infinite set of crossing segments are separated by flyovers being different by the different starting points of the segments involved



Notation 5.2. The definition of λ does not point to the jump function *g*. We need to express this fact for easy use so we introduce a notation to show this. The following relations hold: $y^- = y(t_k - 0), y^+ = y(t_k - 0) + g(t_{t_k}, y(t_k - 0))$.

Taking this into consideration: The definition (4.7) of λ will be rewritten as

$$\lambda_{g,\tau_k,\hat{t}_{k+1}}(s,y^-) := \frac{s-\tau_k}{\hat{t}_{k+1}-\tau_k} g(\omega_-(\tau_k), y^-) + y^-, \quad (s,y^-) \in [\tau_k, \hat{t}_{k+1}) \times R^n, \forall k \in N .$$

Figure 9 shows the separation of crossing points of the segments of intervals.

Now we will define the mappings/differential equations needed for the main representation theorem.

Let $\varphi : R^n \to R^n$ be a continuous mapping and let $J := [\tau_k, \hat{t}_{k+1}) \subset R$ be a non-empty bounded interval.

Definition 5.2. Let

$$\psi_{\varepsilon,g,\tau_k,\hat{t}_{k+1}} : (s,y^-) \in J \times R^n \to \left(s, \lambda_{g,\tau_k,\hat{t}_{k+1}}(s,y^-), h_\varepsilon\left(\frac{s-\tau_k}{\hat{t}_{k+1}-\tau_k}, y^- \right) \right) \in$$

$$\in J \times R^n \times R^n, (s,y^-) \in J \times R^n, k \in \phi_{\varepsilon,g,\tau_k,\hat{t}_{k+1}}(s,y^-,z) =$$

$$\begin{cases} \dfrac{\partial \psi_{\varepsilon,g,\tau_k,\hat{t}_{k+1}}(s,v)}{\partial s} & (s,y^-,z) = \psi_{\varepsilon,g,\tau_k,\hat{t}_{k+1}}(s,v), (s,y^-) \in J \times R^n \\ 0 & otherwise \end{cases} \quad (4.11)$$

Lemma 5.6. The mapping

$$\psi_{\varepsilon,g,\tau_k,\hat{t}_{k+1}} : (t,y^-) \in J \times R^n \to \left(t, \lambda_{g,\tau_k,\hat{t}_{k+1}}(t,y^-), h_\varepsilon\left(\frac{s-\tau_k}{\hat{t}_{k+1}-\tau_k}, y^- \right) \right)$$

$$\in J \times R^n \times R^n, (t,y) \in J \times R^n, \forall k \in N \quad (4.12)$$

is one to one from $J \times R^n \to J \times R^n \times R^n$.

Proof: We will use $a = \tau_k$ *and* $b = \hat{t}_{k+1}$ within this proof. Let $(t_1,y_1),(t_2,y_2) \in J \times R^n, (t_1,y_1) \neq (t_2,y_2)$. There are two cases:

Case 1: $\lambda_{g,a,b}(t_1,y_1) = \lambda_{g,a,b}(t_2,y_2)$. This case was discussed in Lemma 5.5 and it was proved that $t_1 = t_2 := \tau$ must hold. Then $h_\varepsilon(\frac{\tau-a}{b-a}, y_1) \neq h_\varepsilon(\frac{\tau-a}{b-a}, y_2)$ must hold since h_ε is defined in equation (4.5) as the product of a constant $c \times d(t) \times m(y)$ and m is one to one.

Case 2: $\lambda_{g,a,b}(t_1,y_1) \neq \lambda_{g,a,b}(t_2,y_2)$. Then the image of the two points are different.

Figure-10. The figure shows the construction of the ordinary differential equation with right side measurable in t to the impulsive differential equation defined in equation (1.2). The mapping plays a fundamental role in the sections of the base set split up by the impulse time-points/impulse points. The jump surfaces connected by a differential equation defined on the interval to connect the sources and targets of the jumps

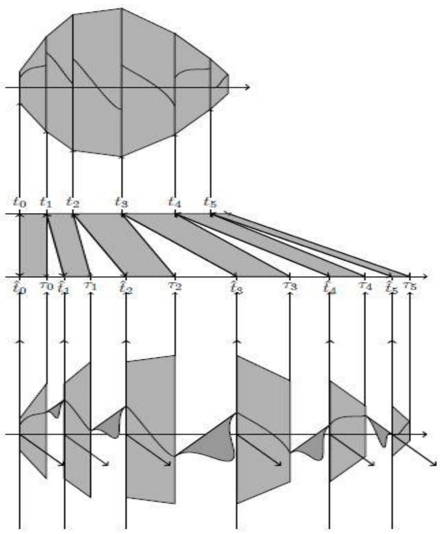
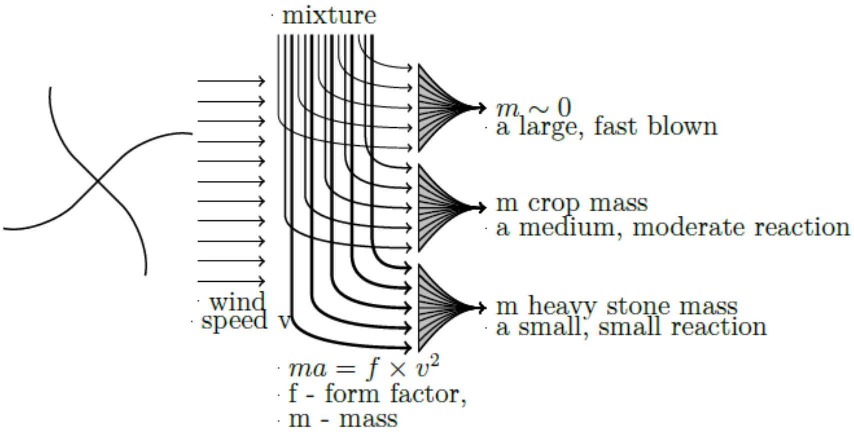


Figure-11. (The Mass Spectrometer): The figure represents an important application of merging solutions. Solutions are merged subject to the mass parameter. Since this property is valid for ODE-s with right side measurable in t, by our representation theorem (Theorem 3.6), impulsive differential equations have the same property



Lemma 5.7. The ordinary differential equation

$$(1,x',y')(t)=\phi_{\varepsilon,g,\tau_k,\hat{t}_{k+1}}(t,x(t),y(t)),$$

$$(x(t_0),y(t_0))=(x_0,y_0),\forall(t_0,x_0,y_0)\in[\tau_k,\hat{t}_{k+1})\times R^n\times R^n,k\in N \quad (4.13)$$

has a unique solution.

Proof:. The proof follows from Lemma 5.5. If $(t_0,x_0,y_0)=\psi_\varepsilon(t_0,v_0),v_0\in R^n$ then the unique solution is $\psi_\varepsilon(t,v_0),t\in a\le t_0<b$. Otherwise $\phi(t_0,x_0,y_0)=0$ hence the constant solution is unique.

Now we are ready with the tools needed to formulate our basic representation theorem.

Let the impulsive differential equation (1.2) be defined. Let $\omega:[\hat{t}_0,\infty)\to[t_0,\infty)$ be defined as constructed in Definition 5.1 together with $\omega^{-1}:S_U\to[t_0,\infty)$.

Definition 5.3. We define an ordinary differential equation with right side measurable in t, continuous or Lipschitzian in (x,y) for each fixed t, to the impulsive differential equation (1.2) as follows:

$$\Phi(t,x,y):=\begin{cases}(f(\omega(t),x),0), & \forall(\omega(t),x,0)\in\Omega\times\{0\}\\ 0 & (\omega(t),x,y)\notin\Omega\times\{0\}\\ \phi_{\varepsilon_k,g,\tau_k,\hat{t}_{k+1}}(t-\tau_k,x,y) & (t,x,y)\in[\tau_k,\hat{t}_{k+1})\times R^n\times R^n,k\in N,\varepsilon_k:=\frac{1}{k}\end{cases} \quad (4.14)$$

Theorem 5.1. The solution of the initial value problem of the differential equation with right side $\Phi(t,x,y),(t,x,y)\in[t_0,\infty)\times R^n\times R^n$ given as

$$(x'(t),y'(t))=\Phi(t,x(t),y(t));$$

$$(x(\omega^{-1}(s_0)),y(\omega^{-1}(s_0)))=(x_0,y_0),(\omega^{-1}(s_0),x_0,y_0)\in\Omega\times R^n \quad (4.15)$$

exists and is unique provided that the solution of the initial value problem $x(s_0)=x_0$ of the impulsive differentiable equation (1.2) exists and is unique. Moreover, the trajectory of solution (x,y) of initial value problems $s_0\notin S,(s_0,x_0,0)\in\Omega\times 0$ satisfies the condition that $x(\omega^{-1}(t)),t\in[s_0,\infty)$ is a solution of the impulsive differential equation (1.2).

Proof: The statement of the theorem follows from the construction, Lemma 5.7 and Definition 5.3.

Again, let us consider a solution x of equation (1.1) with an initial condition $x(t_0)=u$ and another solution y, with an initial value problem $y(t_0)=v$. We are interested in the behaviour of the difference: $h(t):=y(t)-x(t),\forall t\in[t_0,\alpha)$, where both solutions exist and are defined on the interval $[t_0,\alpha)$. Using equation (1.1), we have

$$h'(t):=y'(t)-x'(t)=f(t,y(t))-f(t,x(t))=$$

$$f\left(t,x(t)+h(t)\right)-f\left(t,x(t)\right)=\frac{\partial f\left(t,x(t)\right)}{\partial x}h(t)+r\left(t,h(t)\right), \tag{4.16}$$

where

$$\lim_{h\to 0}\frac{\|r(t,h)\|}{\|h\|}=0,\forall t\in[t_0,\alpha).$$

Lemma 3.8. Let an ordinary differentiable equation be given as equation (4.16). Let $S_c:=\{[\hat{t}_j,\tau_j)\,|\,[\hat{t}_j,\tau_j)\subset R,\hat{t}_j<\tau_j<\hat{t}_{j+1},\forall j\in\}$ be given. Then

$$S:=\{t_j\,|\,t_j:=\hat{t}_0+\sum_{s=0}^{j-1}(\tau_s-\hat{t}_s),j\in\}$$

fulfils the conditions of Definition 5.1. Hence, ω and the corresponding impulsive differentiable equation can be constructed.

5.2. Interpretation of Equivalent Formulation

An ordinary differential equation with right side measurable in t has been defined in definition (3.11) to the impulsive differential equation (1.2) and it is proved in theorem (5.1) that the constructed ordinary differential equation inherits its qualitative properties from the impulsive system and vice versa.

The construction of the ordinary differential equation which is shown on Figure 10 has the following interpretation: The right sides of the impulsive differential equations are used in the sections $[t_j,\tau_j),j\in\square$. A mixer/connector between the separated trajectories is defined in the sections $[\tau_j,t_{j+1}],\forall j\in\square$. These mixers are defined to correctly connect the trajectories back, and it is guaranteed that the solutions are unique in each $[\tau_j,t_{j+1}]$. Hence, the mixer acts as a switch-box. The solution offered however results in an increased dimensional differential equation. The mixer is constructed freely up to a large extent of the impulse function $g(t_k,y),(t_k,y),(t_k,g(t_k,y))\in\Omega\cap\{t_k\}\times R^n,\forall k\in\square$. Note that the mixer is a Lipschitian function in y for each fixed t if g is Lipschitian in y and continuous if g is continuous in y for any fixed t. Our construction is fully determined in the subspace where the impulsive process moves and the additional dimensions are needed to uniquely well define a right side to the trajectories.

6. Conclusion

To establish a structural relationship between ordinary differential equations having special mathematical mixture of dynamical systems and the impulsive differential equations. The qualitative properties of the constructed ordinary differential equation is the same as that of the impulsive system under consideration. The differential equations obtained here have locally unique solutions which may not possess global uniqueness. The qualitative properties of impulsive systems are inherited by the associated ordinary differential equations and vice versa. The practical importance of this is shown in the example of the mass spectrometer (figure 11) for the processing of rice, wheat, etc.

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