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Alphabetic Optimality Criteria for 2^K Central Composite Design

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Abstract

Designing an appropriate central composite design involves selection of the right type of central composite design. The various Central composite designs(CCDs); Spherical central composite design (SCCD), Rotatable central composite design (RCCD), Orthogonal central composite design (OCCD) and Face centered cube design (FCCD) were compared for factors k =2 to 6 using the D, G and A criteria. The α (axial) values were obtained for the various Central composite design (SCCD, RCCD, OCCD and FCD) and the values used to find relevant results for factors k=2 to 6. The efficiency values were obtained for the D, A and G optimality criteria in which larger values imply better design. The axial portion and cube portion were replicated with the center points increased one and three times. The results show that replicating the star points tends to reduce the D and G-optimality criteria of the CCDs (SCCD, RCCD, OCCD and FCCD) at some factor levels while it is different for the A optimality criterion. The results suggest replication affects the different criteria in very different ways because what improves one criterion may be detrimental to a different criterion. The overall results show that G optimality criteria performed better than D and A optimality in different factor levels and replications.

Keywords: SCCD; RCCD; OCCD; FCCD; Optimality criteria.

1. Introduction

The efficiency of an experiment is greatly influenced by the adoption of an appropriate experimental design capable of representing the response surface design. Selecting an appropriate experimental design, is based on finding the best optimality criterion in which larger efficiency values implies a better design [1]. Response surface methodology (RSM) is an important subject in the statistical design of experiments which is a collection of mathematical and statistical techniques useful for modeling and analysis of problems in which a response of interest is influenced by several variables and the objective is to optimize the response [2]. The goals of describing how the response varies as a function of the treatments as well as determining the treatments that give optimal responses perhaps maxima or minima. Response surface methodology could be used in three different ways: a) to show how a specific response is affected by a set of input variables over some particular region of interest b) to find out what settings of input variables will give simultaneously satisfying specifications for one or more response variables c) to define the optimum response and to determine the nature of this optimum. The Central composite design (CCD) emanated from the response surface designs and is the most popular and commonly used classes of experimental design for fitting a second-order response surface model is given as

$$y_{ij} = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j} \sum \beta_{ij} x_i x_j + \varepsilon_{ij} \qquad 1$$

Where y_{ij} is the measured response; $x_i, i = 1,...,k$ are the input variables; $\beta_0, \beta_i, \beta_{ij}$ are the unknown parameters and ε_{ij} is the random error with mean zero and variance σ^2.

Generally, the CCD consists of a number of 2^k factorial (or fractional factorial of resolution V) points with r_f (number of replication at each factorial point), 2k axial or star runs and n_c (number of replicated center point). There are two parameters in the CCD design that must be specified: the distance α of the axial runs from the design center and number of center points n_c. The choice of α value specifies the type of central composite design and these leads to the various classes of central composite designs; Spherical central composite design (SCCD), Rotatable central composite design (RCCD), Orthogonal central composite design (OCCD) and Face centered cube design (FCCD). These designs are selected based on the choice of the axial, cube and center points with the extent of replication.

There are situations in which selecting the right central composite design might be difficult, optimal designs have been suggested and are used frequently in practice. The optimal designs allow parameters to be estimated without bias and with minimum variance. However, in order to select the right central composite designs, certain optimality criteria are evaluated. Optimality criterion is often considered when deciding which response surface

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design to implement. There have been considerable studies in the problem of selecting efficient experimental designs and developing evaluation criteria for experimental designs.

Over the years many works have been done related to second order response surface design and design evaluation criteria.

In a study, Lucas [3] compared the performance of several types of second-order response surface designs (composite designs, Box-Behnken designs, uniform shell designs, Hoke designs, Box Draper designs) in symmetric experimental regions on the basis of D- and G- efficiency values and concluded that all the compared designs have high efficiencies.

Similarly, Crosier [4] compared response surface design by optimal design theory criteria, such as the D- and G-efficiencies and concluded that rotating a first- or second-order response surface design does not change its D- and G-efficiencies, but rotating a response surface design can change the number of levels of the factors and the ranges of the coded factors.

In a choice for axial values [5] discussed three basic choices for α: spherical, face centered cube and rotatable. The spherical choice of α is $\sqrt{k}$, which places all of the runs, other than the center runs, on the surface of the sphere defined by factorial points. When the experimental region is spherical, the choice of α makes the CCD the most D-efficient. The face centered cube (FCC), $\alpha = 1$, places all of the runs, other than the center runs on the surface of the cube defined by the factorial runs. The choice of α is the most D-efficient for cuboidal region and the rotatable choice of α is $f^{\frac{1}{4}}$ where f is the number of factorial runs in the CCD.

Comparing a class of central composite design on the basis of E-optimality and measuring the effect of missing observations on the variances of the estimates of the parameters and response [6] concluded that the missing of an axial point may create more problem than the missing of a factorial point when measuring the variance for $k \geq 4$.

Similarly, Chigbu, *et al.* [7] compared the prediction variances of some central composite designs in spherical regions and concluded that none of the designs are judged as superior in the three bases for comparison which are Central Composite Designs, Small Composite Designs and Minimum-run resolution (Min Res) V designs.

The partial replication of the central composite designs (CCDs) and the results showed that the optimum performance of the replicated variations of the CCD depends on the axial distance, α, and design region, cuboidal or spherical [8]. No particular replicated variation of the CCD is consistently optimum in both design regions and for all the available axial distances utilized in exploring the second-order response surfaces using the CCD. They concluded that replicating the star portion, in most cases, improves the designs.

An expository paper on optimal design showed that many standard designs are also optimal designs, so the use of design optimality for design construction in standard situations leads to the appropriate design choice [9]. They also concluded that it is important to evaluate designs carefully before conducting the experiment in order to answer questions regarding choice of optimality criterion, sample size, and choice of tentative model for the experiment.

Boonorm and Borkowski [1] compared the various response surface designs: Central composite designs (CCD), Box- Behnken designs (BBD), Small composite designs (SCD), Hybrid designs, and Uniform shell designs (USD) over sets of reduced models when the design is in a spherical region for 3 and 4 design variables. The two optimality criteria (*D* and *G)* were considered in which larger values implied a better design.

Oyejola and Nwanya [10] studied the performance of five varieties of Central Composite design: Spherical Central Composite Design (SCCD), Rotatable Central Composite Design (RCCD), Orthogonal Central Composite Design (OCCD), Slope Rotatable Central Composite Design (Slope -R) and Face Center Cube (FCC) and were evaluated using the D, A, G and IV-optimality criteria. The fraction of design space plot of these designs was also displayed. The results showed that replicating the star points tends to reduce the D and G-optimality criteria of the CCDs (SCCD, RCCD, OCCD, Slope-R, and FCC) in all the factors that were considered while it is not so for A-optimality criterion. In IV-optimality, the CCDs are relatively the same both when the center points and axial portion were increased.

The exploration of the optimality of CCDs that are augmented from one-half fractional factorial designs and investigating the rotatability and approximate orthogonality of this type of CCDs were studied by Capili [11]. In addition, the alphabetic criterion values based on the D, A, E, and V were calculated. The study revealed that rotatability and approximate orthogonality can be achieved using the five and six-factor designs.

Iwundu [12] studied the effects of addition of $n_c$center points on the optimality of Box-Benhken and Box-Wilson second-order designs. Relationships were seen to exist between optimal design properties and changing size of the designs by the addition of center points, the relationships between the Box-Benhken designs and the central composite design defined at $\alpha = \sqrt{k}$ and α=$f^{\frac{1}{4}}$ are very strong and variations seem to exist with central composite designs defined at α= 1.

Experimentation plays a vital role in many decision-making problems. Usually, one performs an experiment and hopes the outcome results in a near- optimal decision, but selecting the best possible experimental design for such a given situation has always been a serious issue. This is because there are a lot of criteria that ought to be taken into account when choosing one out of many alternative design options. There is also a problem with a large number of design variables and the experiments may be time consuming and also very expensive to carry out. In order to resolve this, experiment needs to be carried out with fewer experimental runs.

2. Methodology

In a 2^k factorial design, each control variable is measured at two levels coded -1, 1 corresponding to the low and high levels respectively. This design consists of all possible combinations of the levels of the k factors. This leads to what we call design matrix D. Each row of matrix D consists of all 1s, all -1s, or a combination of 1s and -1s which represents a particular treatment combination.

In Multiple Regression Analysis, the relationship between coded and actual values is

$$x_i = \frac{X_i - (X_{low} + X_{high})/2}{(X_{high} - X_{low})/2} \quad 2.1$$

where

x_i is the coded value and X is the actual value.

The three alphabetic optimality criteria to be employed in this work are A, D, and G optimality criteria. The values of these Alphabetic optimality criteria used was computed using the DESIGN EXPERT program.

The four varieties of central composite design that will be assessed and compared using the A, D and G optimality criteria are factors k = 2, 3, 4,5 and 6. The full factorial portion of the CCDs are employed for factors k = 2, 3 and 4 while full and half replicate of the factorial portion of the CCDs are employed for factors k = 5 and 6. The axial portions and cube portions are replicated and center points increased one and three times.

2.1. Central Composite Design (Ccd)

Central composite design also known as box Wilson design was developed by Box and Wilson [13] and is the most popular class of second order model design given as

$$y_{ij} = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j}\sum \beta_{ij} x_i x_j + \varepsilon_{ij} \quad 1$$

Where y_{ij} is the measured response; $x_i, i = 1,\ldots,k$ are the input variables, $\beta_0, \beta_i, \beta_{ii}$ and β_{ij} are the unknown parameters,

ε_{ij} is the random error with mean zero and variance σ^2.

A central composite design (CCD) consists of three different set of experimental runs.

a) A set of factorial design (cube portion) consisting of 2^{k-m} $(m \geq 0)$ units with each point replicated r_f times with the levels of factors coded $+1, -1$. The factorial portion allows estimation of all linear(β_0) and product (β_{ij}) term coefficient in the model.

For example, when k = 2 and m = 0, the CCD is made up of four cube portion of

$$\begin{pmatrix} +1 & +1 \\ -1 & +1 \\ +1 & -1 \\ -1 & -1 \end{pmatrix}$$

A set of axial points (star portions) consisting of $2k$ units where each point are replicated r_α times. The star portion allows the estimation of squared term coefficients (β_{ii}) in the model. For example when k = 2, and $r_\alpha = 1$, the CCD is made up of

$$\begin{pmatrix} +\alpha & 0 \\ -\alpha & 0 \\ 0 & +\alpha \\ 0 & -\alpha \end{pmatrix}$$

a) A set of center points replicated N_c times. This portion contains the number of center points at the center which provides an internal estimate of pure error used to test for lack of fit and also contribute towards estimation of the squared terms. For example if $N_c = 1$, the CCD is made up of $(0 \quad 0)$

The total number of experimental runs in a central composite design is given as

$$N = r_f 2^{k-m} + r_\alpha 2k + N_c \quad 2.1.2$$

Where k is the number of factors, f is the factorial part, r is the number of replications, m is the number of factors subtracted from k (where $m>0$ implies fractional factorial), α is the axial point and N_c is the number of replicated center points. See Nduka and Chigbu [14]

2.2. Rotatable Central Composite Design (RCCD)

The concept of rotatability was first introduced by Box and Hunter [15]. It is essential for a second order design to have a stable distribution given as

$$\text{Nvar}[\hat{y}(x)]/\sigma^2 \quad 2.2.1$$

Where N is the number of observations made in accordance with the experimental design, $var[\hat{y}(x)]$ is the variance of the estimated response at the point $x = (x_1, x_2, \ldots, x_k)$ where $\hat{y}(x)$ is the estimated response at the point $x = (x_1, x_2, \ldots, x_k)$.

A design is said to be rotatable if the prediction variance is constant at all points x which are the same distant from the center of the design. The advantage of this property is that under any rotation of the coordinate axes, the prediction variance remains the same.

Moments that affect rotatability in the case of a second order model are moment through order four. The necessary and sufficient condition for a design to be rotatable given by Box and Hunter [15] are that all odd moments through order four are zero and the ratio of moments

$$\frac{iiii}{iijj} = 3(i \neq j) \quad 2.2.2$$

Where

$$iiii = \frac{1}{N}\sum_{u=1}^{N} x_{iu}^{4} = f + 2\alpha^{4} \qquad 2.2.3$$

$$iijj = \frac{1}{N}\sum_{u=1}^{N} x_{iu}^{4} x_{ju}^{2} = f \qquad 2.2.4$$

The rotatability of a CCD depends on the number of factorial points. A design is rotatable if the condition below holds$[iiii]= 3[iijj]$ which leads to

$$\frac{iiii}{iijj} = \frac{n_f + 2\alpha^4}{n_f} = 3 \qquad 2.2.5$$

$$\alpha = \sqrt[4]{n_f} \qquad 2.2.6$$

Where n_f refers to the number of factorial points [2]. This also means that for a rotatable central composite design the value of α does not depend on the number of center points but if each axial point is observed r_α times, then the requirement for rotatability becomes

$$\alpha = \sqrt[4]{\frac{n_f}{r_\alpha}} \qquad 2.2.7$$

2.3. Orthogonal Central Composite Design

An orthogonal design is one in which the corresponding $X'X$ matrix is diagonal. However for a second order design a diagonal $X'X$ matrix is impossible to obtain, but this can be obtained if we consider the model with the pure quadratic terms corrected for their means, that is,

$$y_u = \beta_0' + \sum_{i=1}^{k} \beta_i x_{i\mu} + \sum \beta_{ii}\left(x_{i\mu}^2 - \overline{x}_i^2\right) + \sum_{i<j}\sum \beta_{ij} x_{i\mu} x_{j\mu} + \varepsilon_\mu, (\mu = 1, 2, ..., N) \qquad 2.3.1$$

Where

$$\beta_0' = \beta_0 + \sum_{i=1}^{k} \beta_{ii} \overline{x}_i^2 \qquad 2.3.2$$

and $$\overline{x}_i^2 = \sum_{i=1}^{N} \frac{x_i^2}{N} \qquad 2.3.3$$

Let b_0, b_i, b_{ii}, b_{ij} denote the least square estimators of $\beta_0, \beta_i, \beta_{ii}, \beta_{ij}$ respectively. In the CCD, all the covariance between estimated regression coefficient except $cov(b_{ii,} b_{ij})$ are zero. But if the inverse of the information matrix $(X'X)^{-1}$ is a diagonal matrix, then also $cov(b_{ii,} b_{ij})$ becomes zero. The condition for making a CCD to be orthogonal is by setting

$$\alpha = \left(\frac{\sqrt{FN} - F}{2}\right)^{\frac{1}{2}}$$ See [16]

2.3.4

Where $N = 2^{k-m} + r_\alpha 2k + N_c$

2.4. Spherical Central Composite Design (SCCD)

For a spherical region, the best choice of α from a prediction view point for the CCD is to set

$\alpha = \sqrt{k}$ [2] 2.4.1

This Spherical Central Composite design puts all the factorial and axial design point on the surface of a sphere of radius $\sqrt{k}$.

2.5. Face Centered Cube Design (FCCD)

In situations where the region of interest is cuboidal, a useful variation of the central composite design is the face centered cube in which $\alpha = 1$.This design locates the star or axial points on the centers of the faces of the cube. The face centered cube does not require as many center points as other CCDs design.

3. Design Optimality Criteria

Design optimality criteria are often used to evaluate a proposed experimental design in order to determine which experimental design should be run. The design optimality criteria are concerned with the optimal properties of the $X'X$ matrix for the design X. The choice of a design depends upon the choice of the evaluation criterion although a design maybe best among several designs by one optimality criterion but may perform poorly when evaluated by a different optimality criterion.

These design optimality criteria are characterized by the letters of the alphabet, hence are often called alphabetic optimality criteria. The four most frequently used alphabetic optimality criteria are the D, A, G, and IV criteria.

3.1. D-Optimality

According to Wald [17], the D-optimality criterion was the first alphabetical optimality criterion established. This is based on the determinant of $X'X$ which is inversely proportional to the square of the volume of the confidence region on the regression coefficients. It indicates how well the set of coefficients are estimated. Therefore, a smaller $|X'X|$or equivalently, a larger $|(X'X)^{-1}|$ implies poorer estimation of the regression coefficients in the model. The goal of D-optimality is to maximize $|X'X|$ or equivalently minimize $|(X'X)^{-1}|$ where X is the design matrix.

The D-efficiency is

$$100\frac{|X'X|^{\frac{1}{p}}}{N} \qquad 3.1.1$$

Where p is the number of model parameters and N is the number of runs.

3.2. G-Optimality

A design is considered to be good if it has low and stable values of scaled prediction variance throughout the experimental region. A G-optimality and the corresponding G-efficiency emphasize the use of designs for which the maximum Nvar $[\hat{y}(x)]/\sigma^2$in the region of the design is not too large. Hence G-optimality is based on

$$v(x) = Nf'(x)(X'X)^{-1}f(x) = N[\hat{y}(x)]/\sigma^2 \quad [15] \qquad 3.2.1$$

G-optimality is a minimax criterion; its aim is to minimize the maximum prediction variance in the design region.

$$\text{Hence, Minimize } \max_{x\in X}\{Nf'(x)(X'X)^{-1}f(x)\} \qquad 3.2.2$$

Where X is the design matrix, x is any point in the design region,$f(x) = [f_1(x), \dots, f_p(x)]'$ is a vector of p-real valued functions based on parameter model terms, and N is the design size.

The G-efficiency is

$$\frac{100p}{N\hat{\sigma}^2_{\max}} \qquad 3.2.3$$

3.3. A-Optimality

This is based on the individual variance of regression coefficients and the goal is to minimize the trace of the inverse of the $X'X$ matrix. Hence,

$$\text{Minimize trace}[(X'X)^{-1}] \qquad 3.3.1$$

Where X is the design matrix and trace is the sum of the scaled variances of the regression coefficients.

The A-efficiency is

$$\frac{100p}{trace\left[N(X'X)^{-1}\right]} \qquad 3.3.2$$

4. Illustration

The various types of central composite design (SCCD, RCCD, OCCD and FCCD) were compared using the D, A and G for factors k=2, 3, 4, 5, 6. Larger values imply a better design for the D, A and G optimality criteria.

4.1. CCD for 2^k Design

4.1.1. Two Factor Design

Table-4.1. The Optimality Criteria for k=2

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	9	1.414	62.85	66.67	30.48
	3	1	1	11	1.414	61.76	87.27	45.91
	1	2	1	13	1.414	59.82	46.15	22.84
	3	2	1	15	1.414	62.26	80.00	39.18
	1	1	2	13	1.414	59.82	46.15	23.57
	3	1	2	15	1.414	62.26	80.00	41.75
Mean						61.46	67.71	33.96
RCCD	1	1	1	9	1.414	62.85	66.67	30.48
	3	1	1	11	1.414	61.76	87.27	45.91
	1	2	1	13	1.414	59.82	46.15	22.84
	3	2	1	15	1.414	62.26	80.00	39.18
	1	1	2	13	1.682	77.25	61.95	33.42
	3	1	2	15	1.682	77.94	67.23	51.75

Mean						66.98	68.21	37.26
OCCD	1	1	1	9	1.000	46.22	82.71	31.17
	3	1	1	11	1.147	48.58	74.41	38.17
	1	2	1	13	0.896	39.37	60.89	28.51
	3	2	1	15	0.968	39.66	56.50	30.84
	1	1	2	13	1.048	48.54	78.76	26.60
	3	1	2	15	1.215	53.94	88.50	37.14
Mean						46.05	73.63	32.07
FCCD	1	1	1	9	1.000	46.22	82.71	31.17
	3	1	1	11	1.000	42.84	68.70	33.34
	1	2	1	13	1.000	42.56	64.64	28.09
	3	2	1	15	1.000	40.89	66.70	36.37
	1	1	2	13	1.000	47.35	87.25	26.57
	3	1	2	15	1.000	46.29	91.12	30.86
Mean						44.36	76.85	31.07
Overall mean						54.71	71.60	33.59

(Table of 2^2 factorial Design)

Table 4.1 above indicates that replicating axial points (increasing r_α) tends to reduce the D and G optimality criteria for OCCD and FCCD while replicating axial points tends to reduce A optimality criteria for SCCD, RCCD, OCCD and increases A optimality criteria for FCCD. Replicating factorial or cube points (increasing r_f) tends to increase the D optimality criterion for RCCD, OCCD and FCCD. Increasing the center points tends to reduce the D optimality criterion for SCCD, RCCD and FCCD except the OCCD while replicated star or cube point tends to increase the D optimality criterion of the CCDS except FCCD. Increasing the center points tends to increase the A optimality criterion for the CCDs whether or not star or cube points are replicated. Increasing the center points tends to increase the G optimality criterion for SCCD, RCCD and vice versa for OCCD and FCCD whether or not star or cube points are replicated. In all G optimality performed better than D and A optimality for the CCDs.

4.1.2. Three Factor Design

Table-4.2. The Optimality Criteria for k=3

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	15	1.732	71.13	66.67	32.39
	3	1	1	17	1.732	70.04	88.99	50.32
	1	2	1	21	1.732	67.31	47.62	24.66
	3	2	1	23	1.732	68.59	75.61	41.73
	1	1	2	23	1.732	68.43	43.48	24.38
	3	1	2	25	1.732	70.27	77.67	44.69
Mean						69.30	66.67	6.36
RCCD	1	1	1	15	1.682	68.70	67.48	32.07
	3	1	1	17	1.682	67.61	87.80	49.31
	1	2	1	21	1.682	64.72	48.44	24.48
	3	2	1	23	1.682	65.88	74.45	41.02
	1	1	2	23	2.000	83.08	60.89	33.57
	3	1	2	25	2.000	83.53	69.20	52.56
Mean						72.25	68.04	38.84
OCCD	1	1	1	15	1.215	52.03	87.03	37.66
	3	1	1	17	1.353	53.89	80.03	43.61
	1	2	1	21	1.114	44.82	65.59	35.88
	3	2	1	23	1.179	44.85	61.67	37.42
	1	1	2	23	1.261	53.95	84.59	31.26
	3	1	2	25	1.414	58.12	83.33	40.28
Mean						51.28	77.04	37.69
FCCD	1	1	1	15	1.00	44.72	83.65	31.28
	3	1	1	17	1.00	41.30	73.99	29.74
	1	2	1	21	1.00	40.94	63.74	33.09
	3	2	1	23	1.00	38.55	58.52	31.83
	1	1	2	23	1.00	45.85	85.08	24.65
	3	1	2	25	1.00	44.10	85.29	24.63
Mean						42.58	75.05	29.20
Overall mean						58.85	71.70	35.52
Overall mean						58.85	71.70	35.52

(Table of 2^3 factorial Design)

Table 4.2 above indicates that replicating axial points (increasing r_α) tends to reduce the D and G optimality criteria for the CCD (SCCD, RCCD, OCCD and FCCD) while replicating axial points tends to reduce A optimality criteria for SCCD, RCCD,OCCD and increases A optimality criteria for FCCD. Replicating factorial or cube points (increasing r_f) tends to increase the D optimality criterion for RCCD, OCCD and FCCD. Increasing the center points tends to reduce the D optimality criterion for SCCD, RCCD and FCCD except the OCCD while replicated star or cube point tends to increase the D optimality criterion of the CCDS except FCCD. Increasing the center points tends to increase the A optimality criterion for SCCD, RCCD and OCCD except the FCCD whether or not star or cube points are replicated. Increasing the center points tends to increase the G optimality criterion for SCCD, RCCD and vice versa for OCCD and FCCD whether or not star or cube points are replicated. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD. However, G optimality led in performance for k=3.

4.1.3. Four Factor Design

Table-4.3. The Optimality Criteria for k=4

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	25	2.000	76.73	60.00	31.65
	3	1	1	27	2.000	76.44	95.29	52.26
	1	2	1	33	2.000	73.49	45.45	25.18
	3	2	1	35	2.000	74.56	80.71	44.14
	1	1	2	41	2.000	73.19	36.59	22.35
	3	1	2	43	2.000	75.09	69.77	43.39
Mean						74.92	64.64	36.50
RCCD	1	1	1	25	2.000	76.73	60.00	31.65
	3	1	1	27	2.000	76.44	95.29	52.26
	1	2	1	33	2.000	73.49	45.45	25.18
	3	2	1	35	2.000	74.56	80.71	44.14
	1	1	2	41	2.378	90.78	62.33	38.11
	3	1	2	43	2.378	90.79	61.52	54.59
Mean						80.47	67.55	40.99
OCCD	1	1	1	25	1.414	58.17	94.49	41.81
	3	1	1	27	1.547	55.84	89.17	44.88
	1	2	1	33	1.321	52.10	74.52	42.60
	3	2	1	35	1.384	52.25	71.31	44.23
	1	1	2	41	1.453	58.32	71.18	32.61
	3	1	2	43	1.596	62.06	72.22	40.75
Mean						56.46	78.82	41.15
FCCD	1	1	1	25	1.000	44.52	91.04	25.49
	3	1	1	27	1.000	42.11	84.30	24.27
	1	2	1	33	1.000	41.47	71.36	30.67
	3	2	1	35	1.000	39.60	67.27	29.41
	1	1	2	41	1.000	44.44	72.16	17.87
	3	1	2	43	1.000	43.27	72.67	17.58
Mean						42.56	7647	24.22
Overall mean						63.60	71.87	35.72

(Table of 2^4 factorial Design)

Table 4.3 above indicates that replicating axial points (increasing r_α) tends to reduce the D and G optimality criteria for the CCD (SCCD, RCCD, OCCD and FCCD) while replicating axial points tends to reduce A optimality criteria for SCCD and RCCD and increases A optimality criteria for OCCD and FCCD. Replicating factorial or cube points (increasing r_f) tends to increase the D optimality criterion for RCCD and OCCD. Increasing the center points tends to reduce the D optimality criterion for SCCD, RCCD, OCCD and FCCD while replicated star or cube point tends to increase the D optimality criterion of the CCDS except FCCD. Increasing the center points tends to increase the A optimality criterion for SCCD, RCCD and OCCD except the FCCD whether or not star or cube points are replicated. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD. However, G optimality led in performance for k=4.

4.1.4. Five Factor Design (Full and Half Replicate)

Table 4.4 The optimality criteria for k = 5 (Full)

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	43	2.236	80.16	48.48	28.46
	3	1	1	45	2.236	80.71	85.94	50.95
	1	2	1	53	2.236	78.88	39.62	23.98
	3	2	1	55	2.236	80.10	89.21	44.81
	1	1	2	75	2.236	75.16	28.00	18.58
	3	1	2	77	2.236	77.14	56.70	38.58
Mean						78.69	57.99	34.23
RCCD	1	1	1	43	2.378	85.37	54.87	31.47
	3	1	1	45	2.378	85.64	83.04	53.33
	1	2	1	53	2.378	84.77	47.23	27.73
	3	2	1	55	2.378	85.61	90.48	47.25
	1	1	2	75	2.828	98.20	48.28	48.53
	3	1	2	77	2.828	97.69	48.53	59.29
Mean						89.55	62.07	44.60
OCCD	1	1	1	43	1.596	63.16	90.44	42.73
	3	1	1	45	1.724	65.36	90.09	49.12
	1	2	1	53	1.515	59.47	85.21	47.57
	3	2	1	55	1.577	59.91	82.82	49.65
	1	1	2	75	1.625	60.96	55.12	30.91
	3	1	2	77	1.761	64.36	56.94	38.04
Mean						62.20	76.77	43.00
FCCD	1	1	1	43	1.000	44.84	94.65	18.56
	3	1	1	45	1.000	43.33	93.71	17.97
	1	2	1	53	1.000	43.55	82.03	25.58
	3	2	1	55	1.000	42.05	79.22	24.83
	1	1	2	75	1.000	43.28	55.78	11.67
	3	1	2	77	1.000	42.64	56.47	11.54
Mean						43.28	76.98	18.36
Overall mean						68.43	68.45	35.05

(Table of 2^5 factorial Design full replicate)

In the above Table, replicating axial points (increasing r_α) tends to reduce the D optimality criterion for the CCDs (SCCD, RCCD, OCCD and FCCD) while replicating axial points tends to reduce A optimality criteria for SCCD, RCCD,OCCD and increases A optimality criteria for FCCD. Replicating factorial or cube points (increasing r_f) tend to increase the D optimality criterion for RCCD only. Increasing the center points tend to increase the D optimality criterion for SCCD, RCCD and OCCD except the FCCD while replicated star point or cube point tends to increase the D optimality criterion of the CCDS except FCCD. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD. However, D and G optimality performed equally for the CCDs when k=5 for full factorial design.

Table-4.5. The optimality criteria for k = 5 (half replicate)

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	27	2.236	80.02	77.78	36.94
	3	1	1	29	2.236	78.50	84.60	55.49
	1	2	1	37	2.236	73.13	56.76	28.25
	3	2	1	39	2.236	73.10	67.99	44.54
	1	1	2	43	2.236	80.16	48.83	28.46
	3	1	2	45	2.236	80.71	85.94	50.95
Mean						77.60	70.32	40.77
RCCD	1	1	1	27	2.000	72.46	88.18	40.43
	3	1	1	29	2.000	70.55	82.29	54.00
	1	2	1	37	2.000	65.76	68.80	33.11
	3	2	1	39	2.000	65.03	65.67	44.39
	1	1	2	43	2.378	85.37	54.87	31.47
	3	1	2	45	2.378	85.64	83.04	53.33
Mean						74.14	73.81	42.79
OCCD	1	1	1	27	1.547	59.70	83.99	48.67
	3	1	1	29	1.664	60.32	79.31	51.84
	1	2	1	37	1.443	50.70	63.70	44.69

	3	2	1	39	1.498	50.29	60.98	44.83
	1	1	2	43	1.596	63.06	90.44	42.73
	3	1	2	45	1.724	65.36	90.09	49.12
Mean						58.24	78.09	46.98
FCCD	1	1	1	27	1.000	42.69	80.60	25.20
	3	1	1	29	1.000	40.21	75.04	23.75
	1	2	1	37	1.000	37.55	60.57	28.61
	3	2	1	39	1.000	35.86	57.47	27.32
	1	1	2	43	1.000	44.84	94.65	18.56
	3	1	2	45	1.000	43.34	93.71	17.97
Mean						40.75	77.01	23.57
Overall mean						62.68	74.81	38.53

(Table of 2^5 factorial Design half replicate)

In the above table, replicating axial points (increasing r_α) tends to reduce the D optimality criterion for the CCDs (SCCD, RCCD, OCCD and FCCD) while replicating axial points tends to reduce A optimality criteria for SCCD, RCCD,OCCD and increases A optimality criteria for FCCD. Replicating factorial or cube points (increasing r_f) tends to increase the D and G optimality criteria for the CCDs (SCCD,RCCD OCCD and FCCD). Increasing the center points tend to reduce the D optimality criterion for SCCD, RCCD, OCCD and FCCD while replicated star point tends to reduce the D optimality criterion of the CCDS except FCCD and vice versa for cube point. Increasing the center points tends to increase the A optimality criterion for SCCD, RCCD, OCCD except the FCCD whether or not star or cube points are replicated. Increasing the center points tends to increase the G optimality criterion for SCCD, RCCD and vice versa for OCCD and FCCD whether or not star or cube points are replicated. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD. For k=5 (half replicate), G optimality performed better than others.

4.1.5. Six Factor Design (Full and Half Replicate)

Table-4.6. The optimality criteria for k = 6 (Full)

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	43	2.236	80.16	48.48	28.46
	3	1	1	45	2.236	80.71	85.94	50.95
	1	2	1	53	2.236	78.88	39.62	23.98
	3	2	1	55	2.236	80.10	89.21	44.81
	1	1	2	75	2.236	75.16	28.00	18.58
	3	1	2	77	2.236	77.14	56.70	38.58
Mean						71.00	51.65	30.53
RCCD	1	1	1	43	2.378	85.37	54.87	31.47
	3	1	1	45	2.378	85.64	83.04	53.33
	1	2	1	53	2.378	84.77	47.23	27.73
	3	2	1	55	2.378	85.61	90.48	47.25
	1	1	2	75	2.828	98.20	48.28	48.53
	3	1	2	77	2.828	97.69	48.53	59.29
Mean						98.19	62.84	53.52
OCCD	1	1	1	43	1.596	63.16	90.44	42.73
	3	1	1	45	1.724	65.36	90.09	49.12
	1	2	1	53	1.515	59.47	85.21	47.57
	3	2	1	55	1.577	59.91	82.82	49.65
	1	1	2	75	1.625	60.96	55.12	30.91
	3	1	2	77	1.761	64.36	56.94	38.04
Mean						65.43	68.39	41.25
FCCD	1	1	1	43	1.000	44.84	94.65	18.56
	3	1	1	45	1.000	43.33	93.71	17.97
	1	2	1	53	1.000	43.55	82.03	25.58
	3	2	1	55	1.000	42.05	79.22	24.83
	1	1	2	75	1.000	43.28	55.78	11.67
	3	1	2	77	1.000	42.64	56.47	11.54
Mean						43.72	67.99	12.76
Overall mean						69.59	62.72	34.52

(Table of 2^6 factorial Design full replicate)

Table 4.6 indicates that replicating axial points (increasing r_α) tends to increase the D optimality criterion for SCCD, RCCD and FCCD and reduces D optimality criterion for OCCD. Replicating axial points tends to reduce

Optimality criteria for SCCD and RCCD and increases Optimality criteria for OCCD and FCCD. Replicating factorial or cube points (increasing r_f) tends to reduce the D optimality criterion for SCCD, OCCD and FCCD. Increasing the center points tends to reduce the D optimality criterion for SCCD and OCCD while increasing center point increases the G optimality criterion for all the CCDs. Increasing the center points tends to increase the A optimality criterion for SCCD, RCCD and OCCD except the FCCD whether or not star or cube points are replicated. Increasing the center points tends to increase the G optimality criterion for all the CCDs. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD. G optimality still performed better for k=6 for full factorial.

Table-4.7. The Optimality Criteria for K=6 (Half Replicate)

Design	N_c	r_α	r_f	N	α	D-eff	G-eff	A-eff
SCCD	1	1	1	45	2.450	83.84	62.22	33.72
	3	1	1	47	2.450	83.48	94.86	55.83
	1	2	1	57	2.450	79.56	49.12	27.60
	3	2	1	59	2.450	79.94	79.89	47.36
	1	1	2	77	2.450	81.43	36.36	23.43
	3	1	2	79	2.450	82.54	69.63	45.79
Mean						81.85	65.35	38.96
RCCD	1	1	1	45	2.378	81.81	64.08	34.04
	3	1	1	47	2.378	81.40	94.41	55.37
	1	2	1	57	2.378	77.49	51.44	28.18
	3	2	1	59	2.378	77.78	79.36	47.17
	1	1	2	77	2.828	93.91	63.91	39.35
	3	1	2	79	2.828	93.83	64.09	55.64
Mean						84.37	69.54	43.29
OCCD	1	1	1	45	1.724	65.75	94.70	50.59
	3	1	1	47	1.841	67.01	91.23	55.01
	1	2	1	57	1.636	59.55	76.75	51.49
	3	2	1	59	1.692	59.50	74.62	52.38
	1	1	2	77	1.761	65.94	69.53	40.18
	3	1	2	79	1.885	68.40	70.60	46.51
Mean						64.36	79.57	49.36
FCCD	1	1	1	45	1.000	44.80	92.04	18.98
	3	1	1	47	1.000	43.19	88.13	18.30
	1	2	1	57	1.000	41.45	73.65	25.06
	3	2	1	59	1.000	40.21	71.15	24.30
	1	1	2	77	1.000	44.84	71.86	12.31
	3	1	2	79	1.000	44.02	71.89	12.09
Mean						43.09	78.12	18.51
Overall mean						68.42	73.15	37.53

(Table of 2^6 factorial Design half replicate)

Table 4.7 indicates that replicating axial points (increasing r_α) tends to reduce the D and G optimality criteria for the CCD (SCCD, RCCD, OCCD and FCCD) while replicating axial points tends to reduce A optimality criteria for SCCD and RCCD and increases A optimality criteria for OCCD and FCCD. Replicating factorial or cube points (increasing r_f) tends to increase the D optimality criterion for RCCD, OCCD and FCCD. Increasing the center points tends to reduce the D optimality criterion for SCCD, RCCD, and FCCD while replicated star point tends to increase the D optimality criterion for SCCD and RCCD. Increasing the center points for replicated cube point increases D optimality for SCCD and OCCD. Increasing the center points tends to increase the A optimality criterion for SCCD, RCCD and OCCD except the FCCD whether or not star or cube points are replicated. Increasing the center points tends to increase the G optimality criterion for SCCD and RCCD and vice versa for OCCD. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD. G optimality still performed better for k=6 for half replicate.

4.2. Comparisons for the Various Central Composite Design

Table-4.2.1. Ranking Table for various Central Composite Design

Factors	Design	D-eff	G-eff	A-eff
2	SCCD	61.46	67.71	33.96
	RCCD	66.98	68.21	37.26
	OCCD	46.05	73.63	32.07
	FCCD	44.36	76.85	31.07
3	SCCD	69.30	66.67	36.36

	RCCD	72.25	68.04	38.84
	OCCD	51.28	77.04	37.69
	FCCD	42.58	75.05	29.20
4	SCCD	74.92	64.64	36.50
	RCCD	80.47	67.55	40.99
	OCCD	56.46	78.82	41.15
	FCCD	42.56	76.47	24.22
5(Full)	SCCD	78.69	57.99	34.23
	RCCD	89.55	62.07	44.60
	OCCD	62.20	76.77	43.00
	FCCD	43.28	76.98	18.36
5 (Half)	SCCD	77.60	70.32	40.77
	RCCD	74.14	73.81	42.79
	OCCD	58.24	78.09	46.98
	FCCD	40.75	77.01	23.57
6 (Full)	SCCD	71.00	51.65	30.53
	RCCD	98.19	62.84	53.52
	OCCD	65.43	68.39	41.25
	FCCD	43.72	67.99	12.76
6(Half)	SCCD	81.85	65.35	38.96
	RCCD	84.37	69.54	43.29
	OCCD	64.36	79.57	49.36
	FCCD	43.09	78.12	18.51

Centered on the results in Table 4.2.1, RCCD is the superior design for all factors except factor k = 5(half replicate) where SCCD is the superior design based on D-criterion. OCCD is the superior design based on the G-criterion for all factors with the exception of factors k=2, 5(full replicate) where FCCD is the superior design. RCCD is the superior design for factors k = 2, 3, 5(full), 6(full) while OCCD is the superior design for factors k = 4, 5(half replicate), 6(half replicate) based on A-criterion.

4.3. Optimal Criteria Comparison

Table-4.2.2. Table of optimal criteria comparison

Number of factors	D-efficiency	G-efficiency	A-efficiency
k=2	54.71	71.60	33.59
k=3	58.85	71.70	35.52
k=4	63.60	71.87	35.72
k=5 (full)	68.43	68.45	35.05
k=5(half replicate)	62.68	74.81	38.53
k=6(full)	69.59	62.72	34.52
k=6(half replicate)	68.42	73.15	37.53
Mean	63.74	70.61	35.78
Rank	2	1	3

(Table of 2^k factorial Design comparisons)

From table 4.2.2, G-optimal criterion is most efficient.

4.4. Discussion of Findings

The results show that replicating the star points tends to reduce the D and G-optimality criteria of the CCDs (SCCD, RCCD, OCCD and FCCD) at some factor levels while it is different for the A optimality criterion. The results also show that replicating the cube point tends to increase D-optimality of some CCDs at some factor levels. D optimality criteria performed better than G and A optimality in SCCD and RCCD while G optimality performed better in OCCD and FCCD for all factors with the exception of factor k =2. However, G optimality criterion proved to be the best criteria among the three optimality criteria studied.

5. Conclusion

The effect of replicating the axial point, cube point or center point have been investigated and the results suggest replication affects the different criteria in very different ways because what improves one criterion may be detrimental to a different criterion due to the fact that some efficiency decreased when star points, cube points were replicated or numbers of center points were increased. In the case of a decrease in efficiency of the replicated star or cube portion, experimenters may be willing to sacrifice design efficiency to gain pure error degree of freedom for lack of fit test.

Moreover, none of these CCDs proved to be superior based on the entire criterion used for comparison (D, G and A optimality criteria). However, based on the most efficient optimal criterion (G) OCCD is the superior design for all factors with the exception of factors k=2, 5(full replicate).

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