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Original Research

Semi-Magic Permutation: A Composition Study on the Structure ω_i

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Abstract

In this paper we propose a definition for a semi-magic permutation and study the composition function behavior between the magic, semi-magic and non-magic permutation using the structure ω_i defined as:

$$\omega_i = \begin{pmatrix} 1 & 2 & \ldots\ldots\ldots. & p \\ 1 & (1+i)_{modp} & \ldots\ldots\ldots.. & (1+(p-1)i)_{modp} \end{pmatrix}$$

Where p is a prime greater than or less than five. We equally observed that no permutations $\lambda_i \in D_4$ for $i = 1, \ldots 6.$ is magic or semi-magic.

Keywords: Magic square; Permutation; Semi-magic permutation; D_4-permutations and ω_i-pattern.

1. Introduction

The discovery of magic square an ancient mathematical structure lead to an interesting area of mathematics called recreational mathematics. Recreational mathematics was developed from cultural, religion, and philosophical symbols.

According to Stephens [1] the oldest magic square of order 3 by 3 appeared in an ancient Chinese literature and later in India a magic square of order 4 by 4. Durer in 1514 also constructed a magic square of order 4 by 4. Since then, the structure had attract the attention of great mathematicians and the construction of magic square of different order has been ongoing. In fact, [2] gave a generalization of magic square of order 4, [3] develop an algorithm for all magic squares of order four and Dawood, *et al.* [4] uses folded magic square to generalize the construction of cubes (magic cubes). Magic matrices and magic stars are resulting structures from magic square. Fanja [5] study the magic squares relative to their permutation matrix and define a magic permutation. Below are ancient magic squares, for more historical development and recent work on magic squares see Andrew [6], Nordgren [7], Ms. Rupali and Sabharwal [8], Rungratgasame, *et al.* [9] and Neeradha, *et al.* [10].

4	9	2
3	5	7
8	1	6

Loh-shu (China, 2858-2738 B.C)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Jaina (India, 12th Century)

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Durer (Germany, 1514)

A *magic square* is an $n \times n$ array of positive integers such that each raw, column and diagonals (i.e. main and cross) entries all sum to the same number called the magic constant or magic sum. Ahmed [11], report that, the magic constant for nth order general magic square starting with an integer A and with entries in an increasing arithmetic series with difference D between terms is $\frac{1}{2}n[2A + D(n^2 - 1)]$.

A square matrix with a magic sum δ is called a δ-magic square. Of course, magic squares are square matrices with the properties as defined. However, if the square matrix is arranged such that the entries in each diagonal (main and cross diagonal) sum up to distinct numbers, then, the square matrix is called a *semi-magic square*. Magic square and semi-magic square enumeration has a long history date back at least to MacMahon, Anand et al., and Stanley [12]. According to Fanja [5], Hertzpring defined the number of magic permutations as well as the number of permutations without fixed points and without reflected point. Fanja [5], was able to propose definition for a magic permutation by considering the bijection via the use of the matrix representation of permutation. He equally show that the inverse and reflected permutation of a magic permutation are magic, and state that, there exist not a magic permutation of length n for $n = 2,3$

In this paper, we propose a semi-magic definition for permutations using the fixed point feature of permutations, this was possible via the matrix representation of permutations an approach employed by Fanja [5], in defining magic permutation. Also the composition behavior of the magic and semi-magic permutations of some permutations were studied.

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2. Preliminaries

Definition: A *permutation* φ is a bijecion from a non-empty set of positive integers to itself. That is, for any set X, such that $\varphi: X \to X$ is a bijection.

Example. Let $X = (123)$. Then, $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is a permutation on X.

Permutation derangement are permutation without fixed point while non-derangement permutations are permutation with a fixed point.

Example: $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is a deranged permutation.

$\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ is a non-deranged permutation.

Definition: Let A, B and C be non-empty sets and $f: A \to B$ and $f: B \to C$ the composition $*$ of f and g written $g * f$ is the relation from A into C defined as

$g * f = \{(x, z) | x \in A \,\&\, z \in C, \exists\, y \in B \text{ such that } f(x) = y \text{ and } g(y) = z\}$.

We note that the operation $*$ of a group $(G,*)$ is a composition function.

Example: let I_n be a permutation group of length n and permutations $\mu, \sigma \in I_n$ then the composition $\mu * \sigma$ is defined as $(\mu * \sigma)(i) = \mu(\sigma(i))\ \forall\, i \in I_n$.

Definition: Let $\xi \subset S_n$, the *permutation matrix* M_ξ is the $n \times n$ matrix obtained by putting $M_\xi = (e_{\xi(1)}, e_{\xi(2)} \ldots \ldots . . e_{\xi(n)})$, where $e_{\xi(i)}$ is the standard basis vector whose $\xi(i)^{th}$ component is 1.

Definition: An integer i is said to be a *fixed point* of a permutation ξ. If $\xi(i) = i$. We denote it with $Fix(\xi)$.

Example: $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ then $Fix(\varphi) = 3$.

Definition: An integer i is called a *reflected point* of a permutation ξ. If $\xi(i) = n - i + 1$. We denote it as $Rlf(\xi)$.

Example: $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ has $Rlf(1) = 3$. Which is true by φ, thus $Rlf(\varphi)$ is 1.

Definition: A *magic permutation* is a permutation π whose matrix representation (i.e. permutation matrix) is a magic square of magic sum 1.

We said that an integer i is a pivot point if i is a fixed point as well as a reflected point. The *reflected permutation* ξ' of a permutation ξ is defined by $\xi'(i) = n - \xi(i) + 1$.

The *dihedral structure* D_n a permutation group whose elements are generated via rotation (R) and the reflection lines (R') of a regular n-polygon was described by Conrad [13] as a rigid motion of a regular n-gon. Consider D_4, we see in one-line notation that:

$R = \{(1234), (2341), (3412), (4123)\}$ and $R' = \{(1432), (2143), (3214), (4321)\}$ which gives the cardinality $|D_4| = 8$.

3. Results

3.1. Proposition

A permutation φ is said to be semi-magic if for the permutation matrix M_φ the main diagonal (upper left to lower right) sum is 1.

Proof:

Let $\varphi = (\varphi(i) \ldots \ldots \ldots \varphi(n))$ such that there exist $i \to \varphi(i) = i$ and M_φ be the permutation matrix of φ then,

$\sum a_{ij} = 1$ for $a_{ij} \in M_\varphi$ where $i = j = 1,2, \ldots . n$.

Definition: A permutation φ is said to be semi-magic if there exist a point i such that $(i, \varphi(i))$ where $\varphi(i) = i$.

Explicitly, a semi-magic permutation is any permutation with a fixed point. Thus every non-deranged permutation is semi-magic.

Example: the permutation $\omega_3 = (14253)$ is semi-magic and $\varphi = (1432)$ is not.

Remark: Every magic permutation is semi-magic but not every semi-magic is magic.

Proposition:

There exist a semi-magic permutation of length for n=3.

Theorem:

Let $\omega_i = \begin{pmatrix} 1 & 2 & \ldots \ldots \ldots . & p \\ 1 & (1+i)_{modp} & \ldots \ldots \ldots .. & (1+(p-1)i)_{modp} \end{pmatrix}$ and G_p the permutation group such that $\omega_i \in G_p$ for any prime $p \geq 5$. Then, there exist $p - 3$ numbers of $\omega_i \in G_p$ that are magic.

Proof:

Let Γ denote the number of $\omega_i \in G_p$ that are magic. By definition, we observe that, there exist some i for which ω_i has reflected point. Since for any ω_i the reflected point is preserved in the corresponding ω_i^{-1}. therefore;

$$\Gamma = |G_p| = p - 1 - 2 = p - 3.$$

Remark: Observe that ω_i is a non-deranged permutation, $|G_p| = p - 1$ and $\Gamma = p - 3$, then, there exist $p - 4$ of ω_i that are semi-magic since G_p is a group.

Proposition:

For any prime $p \geq 5$ and $\omega_i \in G_p$ the permutation ω_{p-1} is semi-magic.

Proof: Prove is trivial by definition.

Remark: For any permutation set $G_p \subset S_n$ of prime $p \geq 5$, there exist at least two $\omega_i \subset G_p$ that are magic.

Proposition:

Let D_n be the dihedral structure with $|D_n| = 2n$ then, $D_4 \subset D_n$ contain $2n - 3$ number of permutations with no reflected point.

Proof:

Let the permutations $\lambda_i \subset D_4$ for $i = 1, \dots 2n$. by definition, we observe that there exist three $\lambda_i \in D_4$ that have points defined by Rlf. Thus,

$$|D_4| - 3$$

is the number of λ_i that has no reflected point.

Lemma

There exist no permutation $\lambda \in D_4$ that is magic or semi-magic.

Proof: The proof is trivial from definitions above.

Proposition:

Let $\omega_i \in G_p$ for all prime $p \geq 5$. then for any magic, non-magic and semi-magic permutations of ω_i the following holds:

I Composition of distinct magic equals non-magic.

II Self-composition of a magic equals semi-magic.

III Composition of a magic and a semi-magic equals magic.

IV Self-composition of a semi-magic equals non-magic.

V Self-composition of a non-magic equals non-magic.

VI Composition of distinct non-magic equals non-magic.

Proof:

The proof is trivially seen from the composition table for $p = 7$ of $\omega_i \in G_p$ below.

Table-1. $p = 7$

$\circ$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
ω_1	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
ω_2	ω_2	ω_4	ω_6	ω_1	ω_3	ω_5
ω_3	ω_3	ω_6	ω_2	ω_5	ω_1	ω_4
ω_4	ω_4	ω_1	ω_5	ω_2	ω_6	ω_3
ω_5	ω_5	ω_3	ω_1	ω_6	ω_4	ω_2
ω_6	ω_6	ω_5	ω_4	ω_3	ω_2	ω_1

4. Conclusion

The author established his findings on a permutation of prime length $p \geq 5$ of the structure ω_i which is true for permutations of similar length. The dihedral structure D_4 with 8-element permutation set has (2n-3) permutations with no reflected points and has no valid magic and semi-magic composition function since, there exist no permutation with one point fixed in D_4. This can be checked for all D_n where n is even.

With this, we recommend for study, possible existence of magic and semi-magic permutations on permutations of even length n.

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