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On the Pell Equations $x^2 - 104y^2 = 1$ and $y^2 - Dz^2 = 25$

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Abstract

If $D = 2^t p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_4} p_4^{\alpha_3}$, where $\alpha_s = 0$ or 1, $p_s\,(1 \le s \le 4)$ are distinct odd primes, the system of indefinite equations in title only has positive integer solution only when $D = 2^{t_1} \times 7 \times 743\,(t_1 = 1, 3)$ or $D = 2^{t_2} \times 3^2 \times 5 \times 7 \times 17^2 \times 743$ $(t = 1, 3, 5, 7)$.

Keywords: The system of indefinite equations; Pell equation; Integer solution; Common solution; Odd prime.

1. Introduction

In recent years, the common solution of pell equations

$$\begin{cases} x^2 - D_1 y^2 = k \\ y^2 - Dz^2 = m \end{cases} \quad (1)$$

is a hot field in indefinite equations. When k=1 and m=1, the research results of the system focus on the scope and estimation of the solution, and the main conclusions are shown in Ljunggrenn [1], Pan, *et al.* [2]. When k=1 and m=4, for the solution of the system, the main conclusion is shown in Chen [3], Hu and Han [4], Dong and Yang [5], Le Maohua [6], Chen [7], Cao zhenfu [8], Chen [9] when D_1=2, When D_1=6, it is shown in Du and Li [10], Du, *et al.* [11], Ran [12], when D_1=12, it is shown in the main conclusion [13-16]. When k=1 and m=25, the situation of the system is discussed in Zhao [17] when D_1=23.

In this paper, we deal with the solutions (x, y) of the system of the indefinite equations

$$\begin{cases} x^2 - 104y^2 = 1 \\ y^2 - Dz^2 = 25 \end{cases} \quad (1)$$

And the following conclusions are obtained:

Theorem If $D = 2^t p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_4} p_4^{\alpha_3}$, where $\alpha_s = 0$ or 1, $p_s\,(1 \le s \le 4)$ are distinct odd primes, t is a positive integer, and the solution of the indefinite system (1) is as follows:

(i) D=2×7×743, the system (1) has non-trivial solutions (x,y,z)=(±530451,±52020,±510);

(ii) D=2^3×7×743时, the system (1) has non-trivial solutions (x,y,z)=(±530451,±52020,±255).

(iii) D=2×54100801×108191201, the system (1) has non-trivial solutions
(x,y,z)=(±585550569867227751, ±58498526893288080,±2^3×3^2×5×7×172×743).

(iv) D=2^3×54100801×108191201, the system (1) has non-trivial solutions
(x,y,z)=(±585550569867227751,±58498526893288080,±2^2×3^2×5×7×172×743).

(v) D=2^5×54100801×108191201, the system (1) has non-trivial solutions
(x,y,z)=(±585550569867227751,±58498526893288080,±2^2×3^2×5×7×172×743).

(vi) D=2^7×54100801×108191201, the system (1) has non-trivial solutions
(x,y,z)=(±585550569867227751, ±58498526893288080, ±2^2×3^2×5×7×172×743).

(v) Otherwise, the system (1) only has trivial solutions (x,y,z)=(±51, ±5,0).

2. Preliminaries

Lemma 1 Zhao [17] If p is an odd prime number, then the diophantine equation x^4-py^2=1 has no other positive integer solution except p=5, x=3, y=4 and p=29, x=99, y=1820.

Lemma 2 Zhao [17] If a is a square number and a >1, the equation $ax^4 - by^2 = 1$ has only one positive integer solution.

Lemma 3 Zhao [17] If D is a non-square positive integer, then $x^4 - Dy^4 = 1$ has at most two positive integer solutions. And the sufficient and necessary condition for the equation to have two groups of solutions is that D=1785 or D=28560, or that $2x_0$ and $2y_0$ are squares, where (x_0, y_0) is the fundamental solution of the equation.

Lemma 4 If x_n,y_n is any integer solution of Pell equation $x^2-104y^2=1$, then x_n,y_n has the following properties:

(I) $x_n \equiv 1(\mathrm{mod}\,2), x_n \equiv 1(\mathrm{mod}\,5), x_{2n} \equiv \pm 1(\mathrm{mod}\,51),$

$x_{2n+1} \equiv 0(\mathrm{mod}\,51), x_{2n+1} \equiv 0(\mathrm{mod}\,102), x_{2n} \equiv \pm 1(\mathrm{mod}\,102)$

$y_{2n} \equiv 0(\mathrm{mod}\,2),\ y_{2n+1} \equiv 1(\mathrm{mod}\,2), y_n \equiv 0(\mathrm{mod}\,5),$

$y_{2n} \equiv 0(\mathrm{mod}\,51), y_{2n+1} \equiv \pm 5(\mathrm{mod}\,51);$

(II) $(x_n, y_n)=1, (x_n, x_{n+1})=1, (y_n, y_{n+1})=5;$

(III) $(x_{2n}, y_{2n+1})=1, (x_{2n+2}, y_{2n+1})=1, (x_{2n+1}, y_{2n})=(x_{2n+1}, y_{2n+2})=51;$

(IV) $x_{n+2}=102x_{n+1}-x_n, x_0=1, x_1=51, y_{n+2}=102y_{n+1}-y_n, y_0=0, y_1=5.$

Lemma 5 If (x_1,y_1) is the fundamental solution of Pell equation $x^2-104y^2=1$, and all integer solutions are (x_n,y_n),n∈Z. For any (x_n,y_n),it has the following properties:

i) x_n is square if and only if n=0;

ii) $\frac{x_n}{51}$ is square if and only if n=±1;

iii) $\frac{y_n}{5}$ is square if and only if n=0,1.

3. Proof of Theorem

proof： Since the fundamental solution of Pell equation $x^2-104y^2=1$ is $(x_1,y_1)=(51,5)$, all integer solutions of pell equation are $x_n+y_n\sqrt{104}=\left(51+5\sqrt{104}\right)^n, n\in Z.$ Thus:

If $(x,y,z)=(x_n,y_n,z)$ is the integer solution to (1), then $\forall n\in Z$,

$$y_n^2-25=y_n^2-25\left(x_n^2-104y_n^2\right)=2601y_n^2-25x_n^2=(51y_n+5x_n)(51y_n-5x_n)=y_{n+1}y_{n-1} \quad (2)$$

By (1) $Dz^2=y_n^2-25$

Then
$$Dz^2=y_{n+1}y_{n-1} \quad (3)$$

case1 Let n be odd, might as well $n=2m-1,(m\in Z)$, At this point, equation (3) becomes：

$$Dz^2=y_{n-1}y_{n+1}=y_{2m-2}y_{2m}=4x_{m-1}y_{m-1}x_my_m \quad (4)$$

case1. 1 Let m be odd, might as well $m=2r,(r\in N^*)$, At this point, equation (4) becomes：

$$Dz^2=4x_{2r-1}y_{2r-1}x_{2r}y_{2r}=8x_{2r-1}y_{2r-1}x_{2r}x_ry_r \quad (5)$$

case 1.1.1 Let r be odd, might as well $r=2u-1,(u\in Z)$, At this point, equation (5) becomes：

$$Dz^2=8x_{4u-3}y_{4u-3}x_{4u-2}x_{2u-1}y_{2u-1} \quad (6)$$

From lemma 5, $\frac{x_{2u-1}}{51},\frac{x_{4u-3}}{51},\frac{y_{2u-1}}{5},\frac{y_{4u-3}}{5},x_{4u-2}$ are two relatively prime，and $\frac{y_{2u-1}}{5},\frac{y_{4u-3}}{5}$ are odd，$x_{4u-2},\frac{x_{2u-1}}{51},\frac{x_{4u-3}}{51}$ are odd, namely $\frac{x_{2u-1}}{51},\frac{x_{4u-3}}{51},\frac{y_{2u-1}}{5},\frac{y_{4u-3}}{5},x_{4u-2}$ are two relatively odd prime.

From lemma 5, if and only if u=0,1, $\frac{x_{2u-1}}{51}$ is a square, and if and only if u=1, $\frac{x_{4u-3}}{51}$ is a square; For any u∈Z, x_{4u-2}, $\frac{y_{4u-1}}{5}$ are not squares. If and only if u=1, $\frac{y_{2u-1}}{5},\frac{y_{4u-3}}{5}$ all are square numbers. So if u ≠0,1, $\frac{x_{2u-1}}{51},\frac{x_{4u-3}}{51},\frac{y_{2u-1}}{5},\frac{y_{4u-3}}{5},x_{4u-2}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (6) is not true, so when u≠0,1, the system (1) has no solution.

When u=0, equation (6) is

$$Dz^2=2^3\cdot 5^2\cdot 51^2\cdot x_2\cdot\frac{x_3}{51}\cdot\frac{y_3}{5} \quad (7)$$

However, $x_2=5201=7\times 743, \frac{x_3}{51}=\frac{530451}{51}=3\times 3467, \frac{y_3}{5}=10404=2^2\times 3^2\times 17^2$

Therefore, the right hand side of (7) contains five different odd prime Numbers, so formula (7) does not hold, and the system (1) has no solution.

When u = 1, $Dz^2 = 8x_1y_1x_2x_1y_1 = 2^3\times5^2\times51^2\times5201 = 2\times7\times743\times(2\times5\times51)^2 = 2^3\times7\times743\times(5\times51)^2$.

So when D=2×7×743, the system (1) has anontrivial solutions (x, y, z) = (+ 530451, + 52020, + 510); D=2^3×7×743, (1) has a nontrivial solution (x, y, z) = (+ 530451, + 52020, + 255).

case 1.1.2 If r is even, let $r = 2v, (v \in Z)$, then equation (5) can be written into

$$Dz^2 = 8x_{4v-1}y_{4v-1}x_{4v}x_{2v}y_{2v} = 16x_{4v-1}y_{4v-1}x_{4v}x_{2v}x_vy_v \tag{8}$$

From lemma 5, when v is even $\frac{x_{4v-1}}{51}, \frac{y_{4v-1}}{5}, x_{4v}, x_{2v}, x_v, \frac{y_v}{255}$ are two relatively prime, when v is odd $\frac{x_{4v-1}}{51}, \frac{y_{4v-1}}{5}, x_{4v}, x_{2v}, \frac{x_v}{51}, \frac{y_v}{5}$ are two relatively prime. And when v is odd, $\frac{y_{4v-1}}{5}, \frac{x_{4v-1}}{51}, \frac{x_v}{51}, x_{4v}, x_{2v}, \frac{x_v}{51}$ all are odd; when v is even, $\frac{y_{4v-1}}{5}, \frac{x_{4v-1}}{51}, \frac{x_v}{51}, x_{4v}, x_{2v}$ all are odd;.

From lemma 5, if and only if v=0, $\frac{x_{4v-1}}{51}, x_{4v}, x_{2v}, x_v\ \frac{x_{2u-1}}{51}$ are squares, and if and only if v=±1, $\frac{x_v}{51}$ is a square; For any v∈Z, x_{4u-2}, $\frac{y_{4v-1}}{5}$ is not square. If and only if v=0, 1, $\frac{y_v}{5}$ is a square. So if v ≠0 and v is even $x_{4v}, x_{2v}, x_v, \frac{x_{4v-1}}{51}, \frac{y_{4v-1}}{5}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (8) is not true, so when u≠0,1, the system (1) has no solution.

when $v \neq \pm 1$ and v is odd, $x_{4v}, x_{2v}, \frac{x_v}{51}, \frac{x_{4v-1}}{51}, \frac{y_{4v-1}}{5}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (8) is not true, so when u≠0,1, the system (1) has no solution.

So when v≠0, v≠±1 and v is even, the system (1) has no solution.

when v=0, (8) can be written into $Dz^2 = 16\cdot x_0^3\cdot y_0\cdot x_{-1}\cdot y_{-1} = 0$, thus z=0, At this point, the system (1) only has ordinary solutions (x,y,z)= (±51,±5, 0).

when v=0, (8) can be written into

$$\begin{aligned} Dz^2 &= 16\cdot x_3\cdot y_3\cdot x_4\cdot x_2\cdot x_1\cdot y_1 = 16\times530451\times52020\times54100801\times5201\times51\times5 \\ &= 2^4\times3^2\times17\times3467\times17^2\times5\times3^2\times2^2\times54100801\times7\times743\times17\times3\times5 \\ &= 2^4\times3^5\times7\times17^4\times743\times3467\times54100801 \end{aligned},$$

The right hand side of the above equation contains 6 odd prime Numbers, so the above formula is impossible. Therefor when v=1, the system (1) has no common solution.

when v=-1,

$$\begin{aligned} Dz^2 &= 16x_{-5}y_{-5}x_{-4}x_{-2}x_{-1}y_{-1} = 16\cdot51^2\cdot5^2\cdot x_2\cdot x_4\cdot\frac{x_5}{51}\cdot\frac{y_5}{5} \\ &= 16\times51^2\times5^2\times5201\times54100801\times108191201\times108222408 \\ &= 2^4\times3^2\times17^2\times5^2\times7\times743\times54100801\times108191201\times743\times17^2\times7\times3^2\times2^3 \\ &= 2^7\times3^4\times5^2\times7^2\times17^4\times743^2\times54100801\times108191201 \\ &= 2^7\times54100801\times108191201\times(3^2\times5\times7\times17^2\times743)^2 \\ &= 2^5\times54100801\times108191201\times(2\times3^2\times5\times7\times17^2\times743)^2 \\ &= 2^3\times54100801\times108191201\times(2^2\times3^2\times5\times7\times17^2\times743)^2 \\ &= 2\times54100801\times108191201\times(2^3\times3^2\times5\times7\times17^2\times743)^2 \end{aligned}$$

Therefore,

When D=2×54100801×108191201,The system (1) has non-trivial solutions (x,y,z)=(±585550569867227751, ±58498526893288080, $\pm2^3\times3^2\times5\times7\times17^2\times743$);

When D=2^3×54100801×108191201,The system (1) has non-trivial solutions (x,y,z)=(±585550569867227751, ±58498526893288080, $\pm2^2\times3^2\times5\times7\times17^2\times743$);

When D=2^5×54100801×108191201,The system (1) has non-trivial solutions (x,y,z)=(±585550569867227751, ±58498526893288080, $\pm2\times3^2\times5\times7\times17^2\times743$);

When D=$2^7 \times 54100801 \times 108191201$,The system (1) has non-trivial solutions (x,y,z)=($\pm$585550569867227751, $\pm$58498526893288080, $\pm 3^2 \times 5 \times 7 \times 17^2 \times 743$);

case 1.2 If m is odd, Modelled on the case 1.1, it can be proved that the equation (1) is only the common solution (x,y,z)= ($\pm$51,$\pm$5,0).

case2 If n is even, by lemma4, $y_{n-1} \equiv y_{n+1} \equiv 1 \pmod 2$, the right-hand side of equation (3) is odd, while the left-hand side is even in the form of D, so the system (1) has no common solution.

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