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Existence and Uniqueness for the Controllability of a Dynamical System

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Abstract

In this paper, we consider the existence and uniqueness for the controllability of a dynamical system. Here, measure of non-compactness of set was employed to examine the conditions for darbo's fixed point theorem which is used to established the existence and uniqueness solution for nonlinear integro-differential equation with implicit derivatives.

Keywords: Nonlinear integro-differential equation; Darbo fixed point theorem; Controllability, Measure of non compactness of set.

1. Introduction

Controllability is one of the essential concepts in mathematical control theory. Controllability is a strong characteristic of dynamical control systems and it is of great importance in control theory. Klamka [1], Dacka [2], observed that many authors effectively applied Schauder's fixed-point theorem in solving the local and global controllability of nonlinear systems since the paper by Davison and Kunze [3]. It was also observed that there is a generalization of Schauder's theorem based on the notion of measure of non compactness of a set. The author then introduced a new method of analyzing for the controllability of nonlinear dynamical systems which consists of using the measure of non compactness of a set and the Darbo fixed-point theorem. This method was used in obtaining sufficient conditions for the global and local controllability of certain types of nonlinear time-varying systems with implicit derivative. See Aghajani, *et al.* [4] for some generalization of the Darbo fixed point theorem and its applications.

Dynamical systems theory deal with long-term qualitative behavior of dynamical systems. The focus is not finding precise solutions to the equations defining the dynamical system (which is often hopeless) but rather to some problem like "will the system settle down to a steady state in a long term? And if so, what ate the possible steady states or does the long term behavior of the system defend on its initial condition. The study of dynamical system is the focus of dynamical system theory, which has applications to wide variety of fields such as mathematics, physics, chemistry, biology, economics and medicine. Dynamical systems are a fundamental part chaos theory, logistic map dynamics, bifurcation theory, the self-assembly process, and the edge of chaos concept.

Real life example of dynamical systems that are found mostly in system surrounding us are in Mechatronic System, Temperature System, Biological Science, In Business System. Dynamical system as a system that evolves in time through the iterated application of an underlying dynamical rule. It is a mathematical model that one usually construct in order to investigate some physical phenomenon that evolves in time. This model usually involves mainly are ordinary differential equation, partial differential equations or functional differential equations which describe the evolution of the process under study in mathematical terms. Patrice [5] Davies and Jackreece [6], examined the controllability and null controllability of linear systems. The authors integrated the concept of null controllability into a generalized system with delay in the state and control. Sufficient conditions was obtained for the assumption of relative controllability for the null controllability with constraint. The result showed that if the uncontrolled system is asymptotically stable and the controlled system is relatively controllable then the system is null controllable with constraints. For a survey on controllability of dynamical systems, see Klamka [7].

Controllability of dynamical systems has been applied in various areas such as spacecraft, [8], mechanical systems [9], 2D linear systems [10, 11], chemical reactors in electric systems containing long lines and in the case of heat exchangers and acoustic systems; [12]. In this study, we consider existence and the uniqueness of the controllability of dynamical system. Balachandran and Somasundaram [13], studied the controllability of a class of nonlinear systems with distributed delays in control. The results obtained including the implicit derivatives of the systems was with the approach of Dacka [14].

Balachandran and Somasundaram [15], studied the relative controllability of nonlinear systems with time varying delays in control. The authors applied the method of Dacka [2] to time varying multiple delays in control and implicit derivatives.

Balachandran and Somasundaram [16], in their paper on controllability of nonlinear delay systems with delay depending on state variable showed that there exists a solution and controllability of nonlinear delay systems in which the delay depends on the state variable.

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A survey paper on the controllability of nonlinear systems including nonlinear delay systems via fixed-point theorems was done by Balachandran and Dauer [17].

Son [18], examined the controllability of the linear infinite-dimensional system defined in the Banach space. Necessary and sufficient conditions was obtained for the approximate controllability of the Banach space. The results obtained was applied to some controllability problems with constrained controls for the class of linear systems described by partial differential equations in the state space. See also the survey by Balachandran and Dauer [19], on the controllability of nonlinear systems in Banach spaces.

Klamka [1], studied the constrained controllability for abstract functional dynamical systems. The author formulated conditions for absolute and relative exact and approximate controllability with constraints posed on the control for linear time-invariant retarded dynamical systems with delays which are defined in infinite-dimensional Banach or Hilbert spaces. The obtained results were an extension of controllability conditions given by Son [18], for the case of constrained controls and delayed systems.

Park, *et al.* [20], studied the controllability of impulse neutral integrodifferential systems with infinite delay in Banach spaces. The authors obtained sufficient conditions for the controllability of the system using Schauder's fixed point theorem.

See Klamka and Niezabitowski [21] and Klamka, *et al.* [22] for a survey on the controllability of switched infinite-dimensional linear dynamical systems and some results in the exact controllability of second order infinite dimensional semilinear deterministic systems.

Sikora [23], studied the constrained controllability of dynamical systems with multiple delays in the state. In the paper, relative and approximate controllability properties with constrained controls were also examined and it was shown that approximate relative controllability is a weaker notion than relative controllability although it appears sufficient for many controllability tasks.

The system under consideration is nonlinear integro-differential equation defined as

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\,ds + B(t)u(t) + F\left(u(t), x\left(t - h(x(t),t)\right)\right)\dot{x}(t),t)) \qquad 1$$

Where the state x(t) is an n-vector and the control u(t) is an m-vector.

$$A : J \to R^{n2} B : J \to R^{nm}, K : D \to R^{n2}, f: JXR^{2n+m} \to R^{n},$$

Where A is a matrix, B is a matrix as well, K is the kernel of a matrix, and F is continuous vector function and using measure of noncompactness of set to formulate conditions for Darbo's fixed point theorem which is used to established existence and uniqueness of a solution.

2. Preliminaries

In this section, we present some basic definition which are useful for our discussion.

Definition 2.1: Condensing map: Let *X*be a subset of a Banach space. An operator $T : X \to X$ is called condensing if for any bounded subset E in X $\varsigma(E) \neq O$, we have $\varsigma(T(E)) \angle \varsigma(E)$ where $\varsigma(E)$ denotes the measure of non-compactness of the set E.

Definition 2.2: Lipschitz condition: suppose f is defined in a domain D of the (t,x) plane. If there exists a constant K>O such that for every (t,x_1) and (t,x_2) in D.

$$|f(t,x_1) - f(t,x_2)| \leq K|X_1 - X_2|$$

Then f is said to satisfy a lipschitz condition (with respect to x) [24].

Definition 2.3: A Mapping $\varsigma : M_E \to [0,\infty)$ is said to be measure of non compactness in E if it satisfies the following condition.

a. The family ker ς ={xϵM_E: ς (X)=0} is nonempty and ker $\varsigma \subseteq M_E$.

b. X $\subseteq$ y, this implies that ς (X)≤ ς (Y)

c. $\varsigma(\bar{X}) = \varsigma(X)$

d. ς (conv.x) = ς (x)

e. ς (λX+(1-λ)Y)≤λ ς (X)+(1-λ) ς (Y) forλϵ(0,1)

Condition e denote convex set

f. If (X_n) is a nested sequence of closed sets from M_E such that $\lim_{n\to\infty} \varsigma(X_n) = O$, then, the intersection set $X_\infty = n_n^\infty X_n$ is none empty

Where M_E is the set of all nonempty and bounded subsets of a metric space E, ς is a function called measure of non compactness, Y is a subset of E, conv (x) is the convex closure of x.

Definition 2.4. Open set: Let (x, p) be a metric space and Let E be an arbitrary subset of X, then the set

E is said to be an open set in X if for each given point, xϵEthere exists a positive real number r(i.e, r>o) such that

$B_r(x) \subseteq E$.

Definition 2.5. Matix: Matix can be define as an array of numbers in rows and columns.

Definition 2.6. Transpose of a Matrix: This is the process whereby the elements in the rows and columns interchange.

Definition 2.7. Nonsingular Matrix:This is a matrix in which its determinant is greater than zero.

Definition 2.8. Closure of set A: The closure $\bar{A}$ is the smallest closed set containing A.

Definition 2.9. Mathematically, let $G(t_{o,}t) = \int_{to}^{t} \varphi(t,s)B(s)B^1(s)\varphi^1(t,s)ds$ be defined as the controllability Matrix.

Where

The kernel φ (t,s) is a matrix, B(s) is a square matrix, $B^1(s)$ is a matrix transpose.

Definition 2.10. Boundedness: A linear functional on a normed linear space N is said to be bounded if there exist a constant M ≥0 such that |f(x)| ≤ M||x|| for all x ϵ N.

Lemma 2.1: Let X be a uniform space. A regular measure of noncompactness of X is an arbitrary function ϕ: $P(X) \to [o,\infty]$ which satisfies the following conditions.

1. $\phi(A) = \infty$ if an only if the set A is unbounded
2. $\phi(A) = \phi(\bar{A})$

Where $\bar{A}$ is the closure of A

3. $\phi(A) = O$ it follows that A is a totally bounded set
4. If X is a complete space, and if $\{B_n\}_{n\epsilon N}$ is a sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $_{n\epsilon N}$ and $\lim_{n\to\infty} \phi(B_n) = o,$ then $k = n_{n\epsilon N} B_n$ is a nonempty compact set.
5. From $A \subseteq B$ it follows that $\phi(A) \le \phi(B)$

Lemma 2.2 let $X \subset C_n(J),$ Let J= [0, 1] and

Let S ⊂ X be a bounded closed convex set. Let H: JXS→X be a continuous operator such that for any $\alpha\varepsilon J,$ the map H $(\alpha, \bullet):$ S→X is condensing. If X ≠ $H\ (\alpha, x)$ for any $\alpha\varepsilon J$ and any xϵδs(the boundary of S), then H (1.) has a fixed point.

Lemma 2.3 let X ⊂ $C_n(J)$ and Let β and γ be functions defined on $[O, t - t_o]$ such that

$\lim_{S\to O} \beta(s) = \lim_{S\to O} \gamma(s) = o.$

If a mapping $Q : X \to C_n(J)$ is given such that it map bounded sets into bounded sets and is such that

$\theta(Q(X), h) < w(\theta(x, \beta(h)) + \gamma(h)$

For all hϵ[O, t-t_0] and xϵXwith ωϵΩ, then Q is a condensing map.

2.1. Theorem

If f: K→R is continuous and K<R is compact, then F (K) is compact

2.2. Proof

We show that f (k) is sequentially compact.

Let (y_n) be a sequence in f (k). Then. y_n= f (X_n) for some X_nϵK. Since k is compact, the sequence (X_n) has a convergent subsequence (X_{ni}) such that $\lim_{i\to\infty} X_{ni=X}$

Where x ϵK-since f is continuous on k.

$\lim_{i\to\infty} f(X_{ni}) = f(x)$.

Putting y = f (x), we have y ϵf (K) and $\lim_{i\to\infty} y_{ni} = y$

Therefore every sequence (y_n) in f (K) has a convergent subsequence whose limit belongs to f (k), so f (k) is compact [25].

3. Existence Result

3.1. Theorem

If the function $f(u, x, y, t)$ satisfies the conditions below:

i. $|f(u, x, y, t)| \le M$ for $u\epsilon R^m, x, y\epsilon R^n, t\epsilon[t_0, t_1]$ and for each $y, y\epsilon R^n$ and $u\epsilon R^m, x\epsilon R^n, t\epsilon[t_0, t_1]$.

ii. $|f(u, x, y, t) - f(u, x, \bar{y}, t)| \le k|y - \bar{y}|$.

where M and K are positive constants such that $o \le k < 1$

with $h(x(t),t) \geq 0$, then

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\,ds + B(t)u(t) + F\left(u(t), x\left(t - h(x(t),t)\right)\right)\dot{x}(t),t)) \quad 1$$

For $t_0 \leq t \leq t_1$

$x(t) = \varphi(t,t_0)x_0 \quad a \leq t \leq t_0$

has at least one solution for any initial function

$\varphi_{\epsilon} D_n^1[a,t_o]$

3.2. Proof

If $u(t)$ is considered as fixed for instant then (1) is reduced to

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\,ds + F\left(x\left(t - h(x(t),t)\right)\right)\dot{x}(t),t)) \quad t_0 \leq t \leq t_1 \qquad 2$$

$$x(t) = \varphi(t,t_o)x_o \; a \leq t \leq t_o$$

where the initial function $\varphi(t)$ is continuous. Here the whole dependence on u is concealed in the dependence on t.

Now, if we consider the Banach space $D_n^1[t_o,t_1]$ and the subset

$H=[x{:}x\epsilon D_n^1[t_{o,}t_1, x(t_o) = \varphi(t_o)]$

again if A is fixed, then (2) becomes

$$\dot{x}(t) = \int_{t_0}^{t} K(t,s)x(s)\,ds + f\left(x\left(t - h(x(t),t)\right)\right)\dot{x}(t),t))$$

$$x(t) = \varphi(t)$$

Now, for any function $x\epsilon H$, we shall mean by $x(\alpha(t))$, the function defined in such a way that if for $t\epsilon[t_{0,}t_1]$ then $\alpha(t) < t_0$

$$x(\alpha(t)) = \varphi(\alpha(t))$$

Define the mapping Q from B into itself by

$$Q(x)(t) = \varphi(t_{1,}t_o)x_o + \int_{to}^{t} f(x(s-h(x(s),s)), x^1(s), s))ds + \int_{to}^{t} k(t,s)x(s)ds$$

Where f is a vector function and k is a kernel.

If we defined the bounded closed set B in H by

$B = [x\epsilon H{:} \|\dot{x}\| \leq r, \; \|\dot{x}\| \leq M$

Where M and r are positive constant such that

$$Q(x)(t) - \varphi(t_{1,}t_o)x_o = M(t - t_o)$$

Or

$$Q(x)(t) - \varphi(t_{1,}t_o)x_o = \left|\int_{to}^{t} f(x(s-h(x(s),s)), x^1(s), s)ds\right| + \left|\int_{to}^{t} k(t,s)x(s)ds\right|$$

Q is continuous, it maps B into itself since Q is continuous. Next, we find the modulus of continuity of $\Delta Q([u,x])(t)$ for $s\epsilon[t_{0,}t_1]$ t, as

$$\left|\Delta Q_1([u,x])(t) - \Delta Q_2([u,x])(s)\right| \leq k\left|x^1(t) - x^1(s)\right| + \beta(|t-s|)$$

hence from lemma 2.3 we have

$$\varphi_o(Q_1 E) = o \text{ and } \varphi_o(\Delta Q_2 E) \leq k\varphi_o(\Delta E_2)$$

Where E_2 is the natural projection of the set E on $D_n^1[t_{o,}t]$ Thus it follows that

$$\mu(QE) \leq k\mu(E)$$

Where μ is the measure of noncompactness. Hence by Darbo fixed point theorem. Q has a fixed point where k is the positive constant $o \leq k < 1$. Since Q has a fixed point then

$x(t) = Q(x)(t).$

Evidently, the extension of $x(t) = Q(x)(t)$. to the interval $[a,t_o]$ by means of the function φ is a solution of the form

$$x(t) = \varphi(t,t_0)x_0 + \int_{t_0}^{t} \left(\int_{t_0}^{t} K(t,s)x(s)\,ds + B(s)u(s) + \int_{t_0}^{t} \varphi(t,s)F\left(u(s), x\left(s - h(x(s),s)\right)\right)\dot{x}(s), s\right)$$

For $t \geq t_0$,
$x(t) = \varphi(t, t_0)x_0 \quad a \leq t \leq t_0$
Meaning the equation (1) has at least one solution

4. Uniqueness Theorem

To show uniqueness of equation (1)
Then equation 1 is unique if itsatisfiesLipschitz condition

4.1. Proof

From equation (1)

$$x^1(t) = A(t)x(t) + \int_{t_0}^{t} k(t,s)x(s)ds + B(t)u(t) + f(u(t), x(t - h(x(t)), x^1(t), t$$

Take

$$x(t_0) = x_0$$

and integrating (4.1), from t_o to t we have

$$x = \int_{t_o}^{t} \left[A(t)x(t) + \int_{t_0}^{t} k(t,s)x(s)ds + B(t)u(t) + f(u(t), x(t - h(x(t), t)), x(t), t \right] dx + x_0$$

Taking norm of both side, we obtain

$$\|x\| = \int_{t_o}^{t} \left[\| A(t)x_1(t) + \int_{t_0}^{t} k(t,s)\, x_1(s)\, ds \ + \ B(t)u(t) + f(u(t), x_1(t - h(x_1(t), t)), x_1^1(t), t) \| \right] -$$

$$\int_{t_o}^{t} \left[A(t)x_2(t) + \int_{t_0}^{t} k(t,s)\, x_2(s)ds + B(t)\, u(t) + f(u(t), x_2(t - h(x_2(t), t)), x_2^1(t), t) \| \right]$$

$$\leq \| A(t)x_1(t) - A(t)x_2(t) \| + \| \int_{t_0}^{t} k(t,s)x_1(s)ds - \int_{t_0}^{t} k(t,s)\, x_2(s)ds \| + \| B(t)u(t) - B(t)u(t) \| + \| f(u(t), x_1$$

$$(t - h(x_1(t), t)), x_1^1(t), t - f(u(t), x_2(t - h(x_2(t), t)), x_2^1(t), t \|$$

$$\leq \| x_1 - x_2 \| (K_1 + \beta K_2 + \mu K_3)$$

Where $k_1 + \beta k_2 + \mu k_3$ is the Lipchitz constant. Hence equation (1) is unique.

5. Conclusion

We considered a class of nonlinear integro-differential system with implicit derivatives of the form

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} k(t,s)x(s)ds + B(t)u(t) + f(u(t), x(t - h(x(t), t)), x^1(t), t \ for\ t \geq t_o$$

$x(t) = \varphi(t, t_0)x_0$ for $t \leq t_0$

By using measure of non compactness of set, we formulate conditions for existence and Uniqueness of a solution and the controllability of nonlinear integro-differential equation with implicit derivatives.

References

[1] Klamka, J., 2000. "Schauder's fixed-point theorem in nonlinear controllability problems." *Control and Cybernetics*, vol. 29, pp. 153-165.

[2] Dacka, 1980. "On the controllability of a class of nonlinear systems." *IEEE Transactions on Automatic Control*, vol. 25, pp. 263-266.

[3] Davison, E. J. and Kunze, E. G., 1970. "Some sufficient conditions for the global and local controllability of nonlinear time-varying systems." *SIAM Journal on Control*, vol. 8, pp. 489-497.

[4] Aghajani, A., Banas, J., and Sabzali, N., 2013. "Some generalizations of Darbo fixed point theorem and applications." *Bulletin of the Belgian Mathematical Society-Simon Stevin*, vol. 20, pp. 345-358.

[5] Patrice, N., 2016. *Contribution to the control theory of some partial functional integrodifferential equations in Banach Spaces*. A Ph.D thesis presented to the Department of Pure and Applied Mathematics (PAM) African University of Science and Technology, Abuja. Nigeria.

[6] Davies, I. and Jackreece, P., 2005. "Controllability and null controllability of linear systems." *Journal of Applied Sciences and Environmental Management*, vol. 9, pp. 31-36.

[7] Klamka, J., 2008. "Controllability of dynamical systems." *Matematyka Stosowana*, vol. 9, pp. 57-75.

[8] Liu, X. and Willms, A. R., 1996. "Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft." *Mathematical Problems in Engineering*, vol. 2, pp. 277-299.

[9] Klamka, J., 2005. "Approximate constrained controllability of mechanical system." *Journal of Theoretical and Applied Mechanics*, vol. 43, pp. 539-554.

[10] Klamka, J., 1999. "Controllability of 2d linear systems. Advances in control." *Highlights of ECC*, vol. 99, pp. 319-326.
[11] Kaczorek, T., 2000. "Reachability and controllability of 2D positive linear systems with state feedbacks." *IFAC Proceedings Volumes*, vol. 33, pp. 745-749.
[12] Luyben, W. L., 1990. *Process modeling, simulation, and control for chemical engineering*. McGraw-Hill.
[13] Balachandran, K. and Somasundaram, D., 1983. "Controllability of a class of nonlinear systems with distributed delays in control." *Kybernetika*, vol. 19, pp. 475-482.
[14] Dacka, 1983. "Relative controllability of perturbed nonlinear systems with delay in control." *IEEE Transactions on Automatic Control*, vol. 27, pp. 268-270.
[15] Balachandran, K. and Somasundaram, D., 1985. "Relative controllability of nonlinear systems with time varying delays in control." *Kybernetika*, vol. 21, pp. 65-72.
[16] Balachandran, K. and Somasundaram, D., 1986. "Controllability of nonlinear delay systems with delay depending on state variable." *Kybernetika*, vol. 22, pp. 439-444.
[17] Balachandran, K. and Dauer, J. P., 1987. "Controllability of nonlinear systems via fixed-point theorems." *Journal of Optimization Theory And Applications*, vol. 53, pp. 345-352.
[18] Son, N. K., 1990. "A unified approach to constrained approximate controllability for the heat equations and the retarded equations." *Journal of Mathematical Analysis and Applications*, vol. 150, pp. 1-19.
[19] Balachandran, K. and Dauer, J. P., 2002. "Controllability of nonlinear systems in Banach spaces: a survey." *Journal of Optimization Theory and Applications*, vol. 115, pp. 7-28.
[20] Park, J. Y., Balachandran, K., and Arthi, G., 2009. "Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces." *Nonlinear Analysis: Hybrid Systems*, vol. 3, pp. 184-194.
[21] Klamka, J. and Niezabitowski, M., 2014. "Controllability of switched infinite-dimensional linear dynamical systems." In *Methods and Models in Automation and Robotics (MMAR), 2014 19th International Conference on .IEEE*. pp. 171-175.
[22] Klamka, J., Zawiski, R., and Niezabitowski, M., 2017. "On some recent results in the exact controllability of second order infinite dimensional semilinear deterministic systems." In *AIP Conference Proceedings.AIP Publishing*. p. 080004.
[23] Sikora, B., 2003. "On the constrained controllability of dynamical systems with multiple delays in the state." *International Journal of Applied Mathematics and Computer Science*, vol. 13, pp. 469-479.
[24] Earl, A. C. and Norman, L., 1995. *Theory of ordinary differential equations*. New York, Toronto London: McGraw-Hill Book Company.
[25] Reza, A., 2015. "Some fixed point theorem in generalized Darbo fixed point theorem and the existence of solutions for systems of integral equation." *J. Korean Maths Soc.*, vol. 52, pp. 125-139.