Coefficient Estimates for Generalized Analytic Bi-Univalent Functions

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Abstract
For the normalized analytic functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \]

We obtain the initial coefficient estimates for the subclass \( I_{a,q}^s(p,q;\mu) \). The relationship with some coefficient estimates in the literature with that of the subclass above was also considered.

Keywords: Analytic; Bi-univalent; Linear transformation; Coefficient estimates.

1. Introduction
Let \( A \) be the class of normalized analytic functions \( f \) in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( f(0) = f'(0) - 1 = 0 \) and of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \]

and \( S \) the class of all functions in \( A \) that are univalent in \( U \). This class of functions is defined on the entire complex plane, thus, one-to-one and onto. Hence, the inverse exists but the inverse may not be defined in the entire unit disc. It was established by Koebe in his one quarter theorem, that every univalent function maps the unit disc to a disc of radius \( \frac{1}{2} \). The inverse function for every univalent function \( S \in A \) can be defined by

\[ f^{-1}(f(z)) = z, \quad |z| < 1 \]

and

\[ f^{-1}(f(w)) = w, \quad (|w| < r_0(f); \quad r_0(f) \geq \frac{1}{2}). \]

Let

\[ f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \]

And \( f(z) = w \), using (3), we write

\[ f^{-1}(f(z)) = \sum_{n=2}^{\infty} c_n(f(z))^n, \quad a_1 = 1 \]

which is equivalent to

\[ f^{-1}(w) = \sum_{n=1}^{\infty} c_n w^n, \quad a_1 = 1, \]

equating coefficients in (2) and (4) with some calculations, we have

\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \]

Let \( f^{-1}(w) = g(w) \), we write

\[ g(w) = w + \sum_{n=2}^{\infty} A_n w^n \]

A function of the form (1) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} = g \) are univalent in \( U \). We denote the class of bi-univalent functions by \( \Sigma \).

The study of the coefficient of bi-univalent functions began with the work of Jahangiri and Hamidi [1], Lewin [2], while the coefficient bounds of bi-univalent functions started with the work of Brannan and Taha [3]. This aspect of research has gained the attention of several researchers in the recent years and a lot of work has been done in this regard see [1, 4-8]. The author in Makinde, et al. [9] defined a certain linear operator by:

Definition: Let \( s, \beta, \gamma \geq 0, \alpha \) a real number such that \( s + \beta + \gamma > 0 \).

Then for a subclass \( f \) of \( A \), of the form:

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]

(6)
and \( i(1 \leq i \leq k) \), the linear operator \( I_{a, \beta, \gamma}^{s}f \) is given by:

\[
I_{a, \beta, \gamma}^{s}f(z) = I_{a, \beta, \gamma}(I_{a, \beta, \gamma}^{s-1}f(z))
\]

and of the form

\[
I_{a, \beta, \gamma}^{s}f(z) = z + \sum_{n=2}^{\infty} \left( \frac{\alpha + n\beta + n^{2}\gamma}{\alpha + \beta + \gamma} \right)^{s} a_{n}z^{n}
\]  

The coefficients \( a_{n} \) in (1) play a prominent role in analytic functions. The convergence of the analytic functions depend largely on \( a_{n} \), for \( |z| < 1 \). Thus, the importance of the coefficient estimates of the analytic functions cannot be overemphasized. Initial coefficient estimates is used in the study of Fekete-Szego functional and the Hankel determinant among others. Also, the author in Makinde [10] applied the generalized linear function introduced in Makinde, et al. [9] to a disease control measure using the coefficient estimate of the class of function. Motivated by the work of Mustafa [11], using the linear operator in (10) we define a new subclass of bi-univalent functions and obtained the initial coefficient estimates for the subclass. For the linear operator of the form (10), we obtain with simple calculations, the inverse function given by:

\[
I_{a, \beta, \gamma}^{s}f^{-1}(w) = \frac{t_{2}a_{2}w^{2} + (t_{2}^{2}a_{2}^{2} - t_{3}a_{3})w^{3} - (5t_{2}^{3}a_{2}^{3} - 5t_{3}^{3}a_{2}a_{3} + t_{3}^{3}a_{4})w^{4} + ...}{2(1 + \mu)}
\]

where \( t_{2} = \frac{(\alpha + 2\beta + 4\gamma)}{\alpha + \beta + \gamma} \), \( t_{3} = \frac{(\alpha + 3\beta + 9\gamma)}{\alpha + \beta + \gamma} \), \( t_{4} = \frac{(\alpha + 4\beta + 16\gamma)}{\alpha + \beta + \gamma} \).

Let \( f^{-1}(w) = g(w) \), we write

\[
g(w) = z + \sum_{n=2}^{\infty} A_{n}w^{n}
\]

In what follows, we define our subclass of analytic functions and obtain the estimates of the initial coefficients.

2. Coefficient Estimates for the Subclass \( \Sigma I_{a, \beta, \gamma}^{s}(p, q; \mu) \)

Definition 2 Let \( p,q \) be analytic functions such that \( p,q: U \rightarrow C, p(0) = q(0) = 1 \) and

\[
\min\{\text{Re}(p(z)), \text{Re}(q(z))\} > 0, z \in U.
\]

Also, let \( f \) be as in (1), then \( f \) is said to be in \( \Sigma I_{a, \beta, \gamma}^{s}(p, q; \mu) \) if:

\[
f \in \Sigma
\]

and

\[
\left( I_{a, \beta, \gamma}^{s}f(z) \right)^{'} + \mu z \left( I_{a, \beta, \gamma}^{s}f(z) \right)^{''} \in p(U), \ z \in U, g \in \Sigma
\]

and

\[
\left( I_{a, \beta, \gamma}^{s}g(w) \right)^{'} + \mu w \left( I_{a, \beta, \gamma}^{s}g(w) \right)^{''} \in p(U), \ w \in U \in p(U), \ w \in U,
\]

where \( \mu > 0 \) and \( I_{a, \beta, \gamma}^{s}f \) is as given in (10).

We now give the statement and the proof of the results in this study.

Theorem 1 Let \( f \in \Sigma I_{a, \beta, \gamma}^{s}(p, q; \mu) \) where \( a, \beta, \gamma \geq 0, a \) a real number such that \( a, \beta, \gamma > 0; s = 0; 1; 2; ::; p; q \) satisfy the condition in definition 1. Then

\[
|a_{2}| \leq \min \left\{ t_{2}^{-8} \frac{1}{2(1 + \mu)} \left( |p'(0)|^{2} + |q'(0)|^{2} \right) , t_{2}^{-8} \frac{1}{12(1 + 2\mu)} \left( |p''(0)| + |q''(0)| \right) \right\}
\]

and

\[
|a_{3}| \leq \min \left\{ t_{3}^{-8} \frac{(|p'(0)|^{2})}{8(1 + \mu)^{2}} + t_{3}^{-8} \frac{(|p''(0)| + |q''(0)|)}{12(1 + 2\mu)}, t_{3}^{-8} \frac{|p''(0)|}{6(1 + 2\mu)} \right\}
\]

Proof 1 Let \( f \in \Sigma I_{a, \beta, \gamma}^{s}(p, q; \mu) \) and \( I_{a, \beta, \gamma}^{s}g = (I_{a, \beta, \gamma}^{s}f)^{-1} \), thus by definition, we have respectively

\[
\left( I_{a, \beta, \gamma}^{s}f(z) \right)^{'} + \mu z \left( I_{a, \beta, \gamma}^{s}f(z) \right)^{''} = p(z), \ z \in u
\]

And

\[
\left( I_{a, \beta, \gamma}^{s}g(w) \right)^{'} + \mu w \left( I_{a, \beta, \gamma}^{s}g(w) \right)^{''} = q(w), \ w \in u
\]

satisfies the condition of the definition 1 and

\[
p(z) = 1 + b_{1}z + b_{2}z^{2} + b_{3}z^{3} + ...
\]

\[
q(w) = 1 + c_{1}w + c_{2}w^{2} + c_{3}w^{3} + ...
\]

Using (10) in (14) and (11) in (15) respectively give

\[
1 + 2t_{2}^{2}a_{2}z + 3t_{3}^{3}a_{3}z^{2} + \cdots + 2\mu t_{2}^{2}a_{2}z + \cdots + 1 + b_{1}z + b_{2}z^{2} + b_{3}z^{3} + ...
\]

and

\[
1 - 2t_{2}^{2}a_{2}w + (6t_{2}^{2}a_{2}^{2} - 3t_{3}^{3}a_{3})w^{2} - (20t_{3}^{3}a_{2}^{3} - 20a_{2}a_{3} + 4a_{4})w^{3} + ...
\]
equating coefficients in (18) and (19), we obtain
\[ 2t_2^3(1 + \mu)a_2 = b_1 \]  
\[ 3t_2^3(1 + 2\mu)a_3 = b_2 \] respectively.
\[ -2t_2^3(1 + \mu)a_2 = c_1 \]  
\[ 6t_2^3(1 + 2\mu)a_2 - 3t_2^3((1 + 2\mu)a_3 = c_2 \] Comparing (20) and (22), we obtain
\[ c_1 = -b_1 \] and
\[ 4t_2^2(1 + \mu)^2a_2 = b_2^2 \]  
\[ 4t_2^2(1 + \mu)^2a_2 = c_2^2 \] From (24) and (25), we have
\[ 8t_2^3(1 + \mu)\ \ \ \ a_2^2 = b_2^2 + c_2^2 \] Also, from (21) and (23), we have
\[ 6t_2^3((1 + 2\mu)a_2^2 = b_2 + c_2 \] Now from (26) and (27), we have
\[ a_2^2 = t_2^{-2s}(b_2^2 + c_2^2) \] \[ 8(1 + \mu)^2 \] Respectively,
\[ a_2^2 = t_2^{-2s}(b_2 + c_2) \] \[ 6(1 + 2\mu) \] Thus, from (28) and (29), using (16) and (17) we have
\[ |a_2|^2 \leq t_2^{-2s}\frac{|q''(0)| + |q''(0)|}{12(1 + 2\mu)} \] (31) And from (30) and (31), we obtain our desired result for the coefficient |a_2|.
Furthermore, from (21) and (23), we have
\[ -6t_2^{-2s}(1 + 2\mu)a_2^2 + 6t_3^{-3}(1 + 2\mu)a_3 = b_2 - c_2 \] (32) Using (28) and (29) in (32), we obtain
\[ |a_3| = t_3^{-3}(b_2 - c_2) \] \[ 6(1 + 2\mu) \] \[ + t_3^{-3}(b_2^2 - c_2^2) \] \[ 8(1 + \mu)^2 \] And
\[ |a_3| = t_3^{-s}\frac{b_2}{3(1 + 2\mu)} \] (34) And from (33) and (34), we obtain our desired result for the Coefficient |a_3|.
Hence, we conclude the proof of the Theorem 1.
Remark 1
1. It is noted that the generalize multiplier transform in Makinde, et al. [9] is an extension of Swamy’s Swamy [12]. Thus, Theorem 2.1 will give the result for Swamy linear operator in place of the generalized multiplier transform by the author in Makinde, et al. [9] when \( \gamma = 0 \).
2. Also, when \( \alpha = 1, \gamma = 0 \), the result for \( N_{\beta}^{\gamma} \) Swamy [12] is obtained in the Theorem. This is given in some of the corollaries below.
Corollary 1 Let \( f \in I_{s,\alpha,\beta}(p, q; 1) \) where \( s, \beta, \gamma \geq 0, \alpha \) a real number such that \( \alpha + \beta + \gamma > 0 \); \( s = 0, 1, 2, \ldots \) and \( p; q \) satisfy the condition in definition 1. Then
\[ |a_2| \leq \min\left\{ t_2^{-s}\frac{1}{4}\sqrt{\frac{|p''(0)|^2 + |q''(0)|^2}{2}}, 6t_2^{-s}\sqrt{|p''(0)| + |q''(0)|}\right\} \]
And
\[ |a_3| \leq \min\left\{ t_3^{-s}\frac{|p''(0)|^2 + |q''(0)|^2}{32}, t_3^{-s}\frac{|p''(0)| + |q''(0)|}{36}, t_3^{-s}\frac{|p''(0)|}{18}\right\} \]
Corollary 2 Let \( f \in T_s^{\gamma}(p, q; \mu) \) where \( s, \beta, \gamma \geq 0, \alpha \) a real number \( \alpha + \beta > 0; s = 0,1,2, \ldots \) and \( p; q \) such that satisfy the condition in definition 1. Then

\[
|a_2| \leq \min \left\{ t_{2\alpha \beta}^{-s} \frac{1}{2(1+\mu)}, t_{2\alpha \beta}^{-\frac{s}{2}} \sqrt{\frac{|p''(0)|^2 + |q''(0)|^2}{2}}, t_{2\alpha \beta}^{-s} \frac{|p''(0) + |q''(0)|}{12(1+\mu)} \right\},
\]

and

\[
|a_3| \leq \min \left\{ t_{3\alpha \beta}^{-s} \frac{|p'(0)|^2 + |q'(0)|^2}{8(1+\mu)^2} + t_{3\alpha \beta}^{-s} \frac{|p''(0)| + |q''(0)|}{12(1+2\mu)} + t_{3\alpha \beta}^{-s} \frac{|p''(0)|}{6(1+2\mu)} \right\},
\]

where

\[
t_{3\alpha \beta} = \frac{\alpha + \beta}{\alpha + 2\beta} \text{ and } t_{3\alpha \beta} = \frac{\alpha + \beta}{\alpha + 3\beta}.
\]

Remark 2 The corollary 2 gives the result for the Swamy's linear operator for the subclass of this study.

Corollary 3 Let \( f \in T_s^{\gamma}(p, q; 1) \) where \( s, \beta, \gamma \geq 0, \alpha \) a real number \( \alpha + \beta > 0; s = 0,1,2, \ldots \) and \( p; q \) such that satisfy the condition in definition 1. Then

\[
|a_2| \leq \min \left\{ t_{2\alpha \beta}^{-s} \frac{1}{2(1+\mu)}, t_{2\alpha \beta}^{-\frac{s}{2}} \sqrt{\frac{|p''(0)|^2 + |q''(0)|^2}{2}}, 6t_{2\alpha \beta}^{-s} \sqrt{|p''(0)| + |q''(0)|} \right\},
\]

and

\[
|a_3| \leq \min \left\{ t_{3\alpha \beta}^{-s} \frac{|p'(0)|^2 + |q'(0)|^2}{32} + t_{3\alpha \beta}^{-s} \frac{|p''(0)| + |q''(0)|}{36} + t_{3\alpha \beta}^{-s} \frac{|p''(0)|}{18} \right\},
\]

where

\[
t_{3\alpha \beta} = \frac{\alpha + \beta}{\alpha + 2\beta} \text{ and } t_{3\alpha \beta} = \frac{\alpha + \beta}{\alpha + 3\beta}.
\]

Corollary 4 Let \( f \in T_s^{\gamma}(p, q; \mu) \) where \( \beta \geq 0; s = 0,1,2, \ldots \) \( \text{and } p, q \) satisfy the condition in definition 1. Then

\[
|a_2| \leq \min \left\{ t_{\alpha \beta}^{-s} \frac{1}{2(1+\mu)}, t_{\alpha \beta}^{-\frac{s}{2}} \sqrt{\frac{|p''(0)|^2 + |q''(0)|^2}{2}}, t_{\alpha \beta}^{-s} \sqrt{\frac{|p''(0)| + |q''(0)|}{12(1+\mu)}} \right\},
\]

And

\[
|a_3| \leq \min \left\{ t_{\alpha \beta}^{-s} \frac{|p'(0)|^2 + |q'(0)|^2}{8(1+\mu)^2} + t_{\alpha \beta}^{-s} \sqrt{\frac{|p''(0)| + |q''(0)|}{12(1+2\mu)}}, t_{\alpha \beta}^{-s} \frac{|p''(0)|}{6(1+2\mu)} \right\}.
\]

Remark 3 The corollary 2 gives the result the operator \( N_\beta^2(z) \) for the subclass of this study.

Corollary 5 Let \( f \in T_s^{\gamma}(p, q; 1) \) where \( \beta \geq 0; s = 0,1,2, \ldots \) and \( p, q \) satisfy the condition in definition 1. Then

\[
|a_2| \leq \min \left\{ t_{\alpha \beta}^{-s} \frac{1}{2(1+\mu)}, t_{\alpha \beta}^{-\frac{s}{2}} \sqrt{\frac{|p''(0)|^2 + |q''(0)|^2}{2}}, 6t_{\alpha \beta}^{-s} \sqrt{|p''(0)| + |q''(0)|} \right\},
\]

and

\[
|a_3| \leq \min \left\{ t_{\alpha \beta}^{-s} \frac{|p'(0)|^2 + |q'(0)|^2}{32} + t_{\alpha \beta}^{-s} \frac{|p''(0)| + |q''(0)|}{36} + t_{\alpha \beta}^{-s} \frac{|p''(0)|}{18} \right\}.
\]

3. Conclusion

The generalized multiplier transform in Makinde, et al. [9] which is an extension of the Swamy's linear operator in Swamy [12] was applied to the method of Mustafa in Mustafa [11]. The results in this study extends that of its form in the literature.

References


