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Coefficient Estimates for Generalized Analytic Bi-Univalent Functions

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Abstract

For the normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C},$$

We obtain the initial coefficient estimates for the subclass $I^{s}_{\alpha\beta\gamma}(p, q; \mu)$. The relationship with some coefficient estimates in the literature with that of the subclass above was also considered.

Keywords: Analytic; Bi-univalent; Linear transformation; Coefficient estimates.

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1. Introduction

Let A be the class of normalized analytic functions f in the open unit disc $U = \{z \in C: |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \tag{1}$$

and S the class of all functions in A that are univalent in U. This class of functions is defined on the entire complex plane, thus, one- to-one and onto. Hence, the inverse exists but the inverse may not be defined in the entire unit disc. It was established by Koebe in his one quarter theorem, that every univalent function maps the unit disc to a disc of radius $\frac{1}{4}$. The inverse function for every univalent function $S \in A$ can be defined by

$$f^{-1}(f(z)) = z, \ |z| < 1 \tag{2}$$

and

$$f^{-1}(f(w)) = w, \ \left(|w| < r_0(f); \ r_0(f) \geq \tfrac{1}{4}\right).$$

Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \tag{3}$$

And$f(z) = w$, using (3), we write

$$f^{-1}(f(z)) = \sum_{n=2}^{\infty} c_n (f(z))^n, \quad a_1 = 1$$

which is equivalent to

$$f^{-1}(w) = \sum_{n=1}^{\infty} c_n w^n, \quad a_1 = 1, \tag{4}$$

equating coefficients in (2) and (4) with some calculations, we have

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{5}$$

Let$f^{-1}(w) = g(w)$, we write

$$g(w) = w + \sum_{n=2}^{\infty} A_n w^n$$

A function of the form (1) is said to be bi-univalent in U if both f and $f^{-1} = g$ are univalent in U. We denote the class of bi-univalent functions by Σ.

The study of the coefficient of bi-univalent functions began with the work of Jahangiri1 and Hamidi [1], Lewin [2], while the coefficient bounds of bi-univalent functions started with the work of Brannan and Taha [3]. This aspect of research has gained the attention of several researchers in the recent years and a lot of work has been done in this regard see [1, 4-8]. The author in Makinde, *et al.* [9] defined a certain linear operator by:

Definition: *Let* $s, \beta, \gamma \geq 0, \alpha$ *a real number such that* $s + \beta + \gamma > 0$.

Then for a subclass f *of* A, *of the form*:

$$f(z) = z + \sum_{n=2}^{\infty} a_n^i z^n, \tag{6}$$

and $i(1 \leq i \leq k$, the linear operator $I^s_{\alpha,\beta,\gamma}f$ is given by:

$$:I_{\alpha,\beta,\gamma}f(z) = \frac{\alpha f(z)+\beta zf'(z)+\gamma z(zf'(z))'}{\alpha+\beta+\gamma} \quad (7)$$

$$I^2_{\alpha,\beta,\gamma}f(z) = I_{\alpha,\beta,\gamma}\left(I_{\alpha,\beta,\gamma}f(z)\right) \quad (8)$$

$$I^s_{\alpha\beta\gamma}f(z) = I_{\alpha\beta\gamma}(I^{s-1}_{\alpha\beta\gamma}f(z)) \quad (9)$$

and of the form

$$I^s_{\alpha,\beta,\gamma}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^s a^i_n z^n \quad (10)$$

The coefficients a_n in (1) play a prominent role in analytic functions. The convergence of the analytic functions depend largely on a_n, for $|z| < 1$. Thus, the importance of the coefficient estimates of the analytic functions cannot be overemphasized. Initial coefficient estimates is used in the study of Fekete-Szego functional and the Hankel determinant among others. Also, the author in Makinde [10] applied the generalized linear function introduced in Makinde, *et al.* [9] to a disease control measure using the coefficient estimate of the class of function. Motivated by the work of Mustafa [11], using the linear operator in (10) we define a new subclass of bi-univalent functions and obtained the initial coefficient estimates for the subclass. For the linear operator of the form (10), we obtain with simple calculations, the inverse function given by:

$$I^s_{\alpha,\beta,\gamma}f^{-1}(w) = w - t^s_2 a_2 w^2 + (t^{2s}_2 a^2_2 - t^s_3 a_3)w^3 - (5t^{3s}_2 a^3_2 - 5t^s_2 t^s_3 a_2 a_3 + t^s_4 a_4)w^4 + \cdots \quad (11)$$

Where $t_2 = \left(\frac{\alpha+2\beta+4\gamma}{\alpha+\beta+\gamma}\right)$, $t_3 = \left(\frac{\alpha+3\beta+9\gamma}{\alpha+\beta+\gamma}\right)$, $t_4 = \left(\frac{\alpha+4\beta+16\gamma}{\alpha+\beta+\gamma}\right)$

Let $f^{-1}(w) = g(w)$, we write

$$g(w) = z + \sum_{n=2}^{\infty} A_n W^n$$

In what follows, we define our subclass of analytic functions and obtain the estimates of the initial coefficients.

2. Coefficient Estimates for the Subclass $\Sigma I^s_{\alpha,\beta,\gamma}(p,q;\mu)$

Definition 2 *Let p, q be analytic functions such that $p, q: U \longrightarrow C$, $p(0) = q(0) = 1$ and*
$min\{Re(p(z)), Re(p(z))\} > 0,\ z \in U.$
Also, let f be as in (1), then f is said to be in $\Sigma I^s_{\alpha,\beta,\gamma}(p,q;\mu)$ if:

$f \in \Sigma$ and

$$\left(I^s_{\alpha,\beta,\gamma}f(z)\right)' + \mu z\left(I^s_{\alpha,\beta,\gamma}f(z)\right)'' \in p(U),\ \ z \in U, g \in \Sigma \quad (12)$$

and

$$\left(I^s_{\alpha,\beta,\gamma}g(w)\right)' + \mu w\left(I^s_{\alpha,\beta,\gamma}g(w)\right)'' \in p(U),\ \ w \in U \in p(U), w \in U, \quad (13)$$

where $\mu > 0$ and $I^s_{\alpha,\beta,\gamma}f$ is as given in (10).

We now give the statement and the proof of the results in this study.

Theorem 1 *Let $f \in \Sigma I^s_{(\alpha,\beta,\gamma)}(p,q;\mu)$ where $\alpha,\beta,\gamma \geq 0, \alpha$ a real number such that $\alpha,\beta,\gamma > 0;\ s\ =\ 0;\ 1;\ 2; ::: and\ p;\ q$ satisfy the condition in definition 1. Then*

$$|a_2| \leq min\left\{t_2^{-8}\frac{1}{2(1+\mu}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{2}}. t_2^{-8}\sqrt{\frac{|p''(0)|+|q''(0)|}{12(1+2\mu)}}\right\}$$

and

$$|a_3 \leq min\left\{t_3^{-8}\frac{(|p'(0)|^2}{8(1+\mu)^2} + t_3^{-8}\frac{(|p''(0)|+|q''(0)|)}{12(1+2\mu)}, t_3^{-8}\frac{|p''\ (0)|}{6(1+2\mu)}\right\}$$

Proof 1 *Let $f \in \Sigma I^s_{\alpha,\beta,\gamma}(p,q;\mu)$ and $I^s_{\alpha,\beta,\gamma}g = (I^s_{\alpha,\beta,\gamma}f)^{-1}$, thus by definition, we have respectively*

$$(I^s_{\alpha,\beta,\gamma}f(z))' + \mu z(I^s_{\alpha,\beta,\gamma}f(z))'' = p(z), \qquad z \in u \quad (14)$$

And

$$(I^s_{\alpha,\beta,\gamma}g(w))' + \mu w(I^s_{\alpha,\beta,\gamma}g(w))'' = q(w), \qquad w \in u \quad (15)$$

satisfies the condition of the definition 1 and

$$p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \quad (16)$$

$$q(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots \quad (17)$$

Using (10) in (14) and (11) in (15) respectively give

$$1 + 2t^s_2 a_2 z + 3t^3_3 a_3 z^2 + \cdots + 2\mu t^3_2 a_2 z + \cdots 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \quad (18)$$

and

$$1 - 2t^s_2 a_2 w + (6t^{2s}_2 a^2_2 - 3t^3_3 a_3)w^2 - (20t^{3s}_2 a^3_2 - 20a_2 a_3 + 4a_4)w^3 + \cdots$$

$$-2\mu t_2^s a_2 w + \mu(12t^{2s}_2 a_2^2 - 6t_3^s a_3)w^2 - \mu(60t_2^{3s} a_2^3 - 60a_2 a_3 + 12a_4)w^3 + \cdots$$
$$= 1 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots \quad (19$$

equating coefficients in (18) and (19), we obtain

$$2t_2^s(1+\mu)a_2 = b_1 \quad (20)$$

$$3t_3^s(1+2\mu)a_3 = b_2 \quad (21)$$

respectively

$$-2t_2^s(1+\mu)a_2 = c_1 \quad (22)$$

$$6t_2^{2s}(1+2\mu)a_2^2 - 3t_3^s((1+2\mu)a_3 = c_2 \quad (23)$$

Comparing (20) and (22), we obtain

$$c_1 = -b_1$$

and

$$4t_2^{2s}(1+\mu)^2 a_2^2 = b_1^2 \quad (24)$$

$$4t_2^{2s}(1+\mu)^2 a_2^2 = c_1^2 \quad (25)$$

From (24) and (25), we have

$$8t_2^{2s}(1+\mu)^2 a_2^2 = b_1^2 + c_1^2 \quad (26)$$

Also, from (21) and (23), we have

$$6t_2^{2s}((1+2\mu)^2 a_2^2 = b_2 + c_2 \quad (27)$$

Now from (26) and (27), we have

$$a_2^2 = t_2^{-s}\frac{(b_1^2+c_1^2)}{8(1+\mu)^2} \quad (28)$$

Respectively,

$$a_2^2 = t_2^{-2s}\frac{(b_2+c_2)}{6(1+2\mu)} \quad (29)$$

Thus, from (28) and (29), using (16) and (17) we have

$$|a_2|^2 \le t_2^{-2s}\frac{(|p'(0)|^2+|q'(0)|^2}{8(1+\mu)^2} \quad (30)$$

Respectively,

$$|a_2|^2 \le t2^{-2s}\frac{(|p''(0)|+|q''(0)|)}{12(1+2\mu)} \quad (31)$$

And from (30) and (31), we obtain our desired result for the coefficient $|a_2|$.

Furthermore, from (21) and (23), we have

$$-6t_2^{-2s}(1+2\mu)a_2^2 + 6_3^{t-3}(1+2\mu)a_3 = b_2 - c_2 \quad (32)$$

Using (28) and (29) in (32), we obtain

$$|a_3| = t_3^{-3}\frac{(b_2-c_2)}{6(1+2\mu)} + t_3^{-3}\frac{(b_1^2-c_1^2)}{8(1+\mu)^2} \quad (33)$$

And

$$|a_3| = t_3^{-s}\frac{b_2}{3(1+2\mu)} \quad (34)$$

And from (33) and (34), we obtain our desired result for the
Coefficient $|a_3|$
Hence, we conclude the proof of the Theorem 1.

Remark 1

1. It is noted that the generalize multiplier transform in Makinde, et al. [9] is an extension of Swammy's Swamy [12]. Thus, Theorem 2.1 will give the result for Swammy linear operator in place of the generalized multiplier transform by the author in Makinde, et al. [9] when $\gamma = 0$

2. Also, when $\alpha = 1. \gamma = 0$, *the result for* N_β^n *Swamy [12] is obtained in the Theorem. This is given in some of the corollaries below.*

Corollary 1 Let $f \in I^s_{\alpha,\beta,\gamma}(p,q;1)$ *where* $s, \beta, \gamma \ge 0, \alpha$ *a real number such that* $\alpha+\beta+\gamma > 0; s = 0,1,2,\ldots$ *and* $p;\ q$ *satisfy the condition in definition 1. Then*

$$|a_2| \le min\left\{t_2^{-s}\frac{1}{4}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{2}}, 6t_2^{-s}\sqrt{|p''(0)|+|q''(0)|}\right\},$$

And

$$|a_3| \le min\left\{t_3^{-s}\frac{(|p'(0)|^2+|q'(0)|^2)}{32} + t_3^{-s}\frac{(|p''(0)|+|q''(0)|)}{36}, t_3^{-s}\frac{|p'(0)|}{18}\right\}.$$

Corollary 2 Let $f \in I^{s}_{\alpha,\beta,0}(p,q;\mu)$ where $s,\beta,\gamma \geq 0, \alpha$ a real number$\alpha+\beta>0; s=0,1,2,\ldots$ and $p;\ q$ such that satisfy the condition in definition 1. Then

$$|a_2| \leq min\left\{t_{2\alpha\beta}^{-s}\frac{1}{2(1+\mu)}\sqrt{\frac{|p\prime(0)|^2+|q\prime\,(0)|^2}{2}}, t_{2\alpha\beta}^{-s}\sqrt{\frac{|p''(0)|+|q''\,(0)|}{12(1+\mu)}}\right\},$$

and

$$|a_3| \leq min\left\{t_{3\alpha\beta}^{-s}\frac{|p'(0)|^2+|q'(0)|^2}{8(1+\mu)^2}+t_{3\alpha\beta}^{-s}\frac{|p''(0)|+|q''(0)|}{12(1+2\mu)}, t_{3\alpha\beta}^{-s}\frac{|p''(0)|}{6(1+2\mu)}\right\}$$

where

$$t_{3\alpha\beta}=\frac{\alpha+\beta}{\alpha+2\beta}\ and\ t_{3\alpha\beta}=\frac{\alpha+\beta}{\alpha+3\beta}$$

Remark 2 The corollary 2 gives the result for the Swammy's linear operator for the subclass of this study.

Corollary 3 Let $f \in I^{s}_{\alpha,\beta,0}(p,q;1)$ where $s,\beta,\gamma \geq 0, \alpha$ a real number$\alpha+\beta>0; s=0,1,2,\ldots$ and $p;\ q$ such that satisfy the condition in definition 1. Then

$$|a_2| \leq min\left\{t_{2\alpha\beta}^{-s}\frac{1}{4}\sqrt{\frac{|p\prime(0)|^2+|q\prime\,(0)|^2}{2}}, 6t_{2\alpha\beta}^{-s}\sqrt{|p''(0)|+|q''\,(0)|}\right\},$$

and

$$|a_3| \leq min\left\{t_{3\alpha\beta}^{-s}\frac{|p'(0)|^2+|q'(0)|^2}{32}+t_{3\alpha\beta}^{-s}\frac{|p''(0)|+|q''(0)|}{36}, t_{3\alpha\beta}^{-s}\frac{|p''(0)|}{18}\right\}$$

where

$$t_{3\alpha\beta}=\frac{\alpha+\beta}{\alpha+2\beta}\ and\ t_{3\alpha\beta}=\frac{\alpha+\beta}{\alpha+3\beta}$$

Corollary 4 *Let $f \in I^{s}_{1,\beta,0}(p,q;\mu)$ where$\beta \geq 0; s-0,1,2\ldots$ and p,q satisfy the condition in definition 1. Then*

$$|a^2| \leq min\left\{t_{2\beta}^{-s}\frac{1}{2(1+\mu}\sqrt{\frac{|p\prime(0)|^2+|q\prime(0)|^2}{2}}, t_{2\beta}^{-s}\sqrt{\frac{|p''(0)|+|q\prime\prime(0)|}{12(1+2\mu)}}\right\},$$

And

$$|a_3| \leq min\left\{t_{3\beta}^{-s}\frac{|p\prime(0)|^2+|q\prime(0)|^2}{8(1+\mu)^2}+t_{3\beta}^{-s}\frac{|p''(0)|+|q\prime\prime(0)|}{12(1+2\mu)}, t_{3beta}^{-s}\frac{|p\prime\prime(0)|}{6(1+2\mu)}\right\}.$$

Remark 3 The corollary 2 gives the result the operator $N_{\beta}^{n}(z)$ for the subclass of this study.

Corollary 5 *Let $f \in I^{s}_{1,\beta,0}(p,q;1)$ where $\beta \geq 0, s=0,1,2,\ldots$ and p,q satisfy the condition in definition 1. Then*

$$|a_2| \leq min\left\{t_{2\beta}^{-s}\frac{1}{4}\sqrt{\frac{|p\prime(0)|^2+|q\prime(0)|^2}{2}}, 6t_{2\beta}^{-8}\sqrt{|p'(0)|+|q'(0)|}\right\},$$

and

$$|a_3| \leq min\left\{t_{3\beta}^{-s}\sqrt{\frac{|p'(0)|^2+|q'(0)|^2}{32}}+t_{3\beta}^{-s}\frac{(|p'(0)|+|q'(0)|)}{36}, t_{3\beta}^{-s}\frac{|p'(0)|}{18}\right\}$$

3. Conclusion

The generalized multiplier transform in *Makinde, et al. [9]* which is an extension of the Swammy's linear operator in Swamy [12] was applied to the method of Mustafa in Mustafa [11]. The results in this study extends that of its form in the literature.

References

[1] Jahangiri1, J. M. and Hamidi, S. G., 2013. "Coefficient estimates for certain classes of bi-univalent functions." *International Journal of Mathematics and Mathematical Sciences*, vol. 2013, pp. 1-4.

[2] Lewin, M., 1967. "On a coefficient problem for bi-univalent functions." *Proceedings of the American Mathematical Society*, vol. 18, pp. 63-68.

[3] Brannan, D. A. and Taha, T. S., 1986. "On some classes of bi- univalent functions, studia universitatis babes-bolyai." *Mathematica*, vol. 31, pp. 70-77.

[4] Ali, R. M., Lee, S. K., Ravichandran, V., and Supramaniam, S., 2012. "Coefficient estimates for bi-univalent ma-minda starlike and convex functions." *Applied Mathematics Letters*, vol. 25, p. 344?351.

[5] Frasin, B. A. and Aouf, M. K., 2011. "New subclasses of bi- univalent functions." *AppliedMathematics Letters*, vol. 24, pp. 1569-1573.

[6] Hamidi, S. G., Halim, S. A., and Jahangiri, J. M., 2013. "Faber polynomial coefficient estimates for meromorphic bi-starlike functions." *International Journal of Mathematics and Mathematical Sciences*, vol. 2013, pp. 1-4.

[7] Srivastava, H. M., Mishra, A. K., and Gochhayat, P., 2010. "Certain subclasses of analytic and bi-univalent functions?" *Applied Mathematics Letters,* vol. 23, pp. 1188-1192.

[8] Vijaya, R., Sudharsan, T. V., and Sivasubramanian, S., 2016. "Coefficient estimates for certain Subclasses of biunivalent functions defined by convolution." *International Journal of Analysis,* vol. 2016, pp. 1-5.

[9] Makinde, D. O., Hamzat, J. O., and Gbolagade, A. M., 2019. "A generalized multiplier transform on a univalent integral operator." *Journal of Contemporary Applied Mathematics,* vol. 9, pp. 31-38.

[10] Makinde, D. O., 2019. "A generalized multiplier transform on a p-valent integral operator with application." *American Journal of Applied Mathematics and Statistics,* vol. 7, pp. 115-119.

[11] Mustafa, N. "The various properties of certain subclasses of analytic functions of complex order." pp. 1-9. Available: https://arxiv.org/ftp/arxiv/papers/1704/1704.04980.pdf

[12] Swamy, S. R., 2012. "Inclusion Properties of certain subclasses of analytic functions." *International Mathematical Forum,* vol. 7, pp. 1751-1760.