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The Real Representation of Canonical Hyperbolic Quaternion Matrices and Its Applications

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Abstract

In this paper, we construct the real representation matrix of canonical hyperbolic quaternion matrices and give some properties in detail. Then, by means of the real representation, we study linear equations, the inverse and the generalized inverse of the canonical hyperbolic quaternion matrix and get some interesting results.

Keywords: T_1-MP inverse; Canonical hyperbolic quaternion matrix; Real representation.

1. Introduction

In 1843, Hamilton introduced the quaternion, which has the form of

$$a = a_1 + a_2 i + a_3 j + a_4 k,$$

where

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$$

and a_1, a_2, a_3, a_4 are real numbers. Since quaternions are non-commutative, they differ from complex numbers and real numbers. Quaternions and quaternion matrices play an important role in quaternionic quantum mechanics and field theory [1]. In 1849, the split quaternion(or coquaternion), which was found by James Cockle, is in the form of

$$a = a_1 + a_2 i + a_3 j + a_4 k,$$

where

$$i^2 = -1, j^2 = k^2 = 1, ij = -ji = k, jk = -kj = -i, ki = -ik = j$$

and a_1, a_2, a_3, a_4 are real numbers. Split quaternions are noncommutative, too. But split quaternion set contains zero-divisors, nilpotent elements and nontrivial idempotents [2, 3].

In 1892, Segre proposed modified quaternions so that commutative property in multiplication is possible [4]. In Catoni, *et al.* [5], the authors studied three three types of commutative quaternions: Elliptic quaternions, Parabolic quaternions and Hyperbolic quaternions. They are 4-dimensional like the set of quaternions, but contain zero-divisor and isotropic elements. Although commutative quaternion algebra theory is becoming more and more important in recent years and has many important applications in the areas of mathematics and physics [5-10], the current focus is mainly on canonical elliptic quaternions [11-14]. In these papers, H. Kösal and M. Tosun gave some properties of canonical elliptic quaternions and their fundamental matrices. After that, they investigated canonical elliptic quaternion matrices using properties of complex matrices. Then they defined the complex adjoint matrix(complex representation matrix) of canonical elliptic quaternion matrices and gave some of their properties. Recently, they proposed real matrix representations of canonical elliptic quaternions and their matrices and derived their algebraic properties and fundamental equations.

As has been noticed, there is no paper that studied the theory on canonical hyperbolic quaternion matrices. In this paper, we will discuss canonical hyperbolic quaternion matrices.

Let $\mathbf{R}$ denote the real number field and $\mathbf{Q}_{\mathbf{hc}} = \mathbf{R} + \mathbf{R}i + \mathbf{R}j + \mathbf{R}k$ denote the canonical hyperbolic quaternion set, where

$$i^2 = j^2 = k^2 = 1, ij = ji = k, jk = kj = i, ki = ik = j.$$

For $a = a_1 + a_2 i + a_3 j + a_4 k, b = b_1 + b_2 i + b_3 j + b_4 k \in \mathbf{Q}_{\mathbf{hc}}$, it is clear that

$$ab = ba = (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4) + (a_2 b_1 + a_1 b_2 + a_4 b_3 + a_3 b_4) i$$

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$$+(a_1b_3+a_3b_1+a_2b_4+a_4b_2)j+(a_4b_1+a_1b_4+a_2b_3+a_3b_2)k.$$

This paper is organized as follows. In Section 2, we construct the real representation of canonical hyperbolic quaternion matrices and systematically study its properties. In Section 3, we discuss the canonical hyperbolic quaternion linear equations and study the judgment and construction of solutions. Next, we give the necessary and sufficient condition for canonical hyperbolic quaternion matrix invertibility. Finally, we define a generalized inverse and initially discuss its existence and uniqueness. Some results are interesting. In Section 4, we give some conclusions.

2. Real Representation of Canonical Hyperbolic Quaternion Matrices

In this section, we define the real representation of canonical hyperbolic quaternion matrices and systematically study its properties. It is worth mentioning that, unlike other quaternions, the canonical hyperbolic quaternion is not the natural generalization of complex number. It is hard for its matrices to construct the complex representation and we only discuss the real representation. For real representations of quaternion matrices, split quaternion matrices and elliptic quaternion matrices, many results have been obtained [2, 3, 14-17] and their references for details). Inspired by them, we define the real representation of canonical hyperbolic quaternion matrices as follows.

For any $A=A_1+A_2i+A_3j+A_4k\in\mathbf{Q_{hc}}^{m\times n}, A_l\in\mathbf{R}^{m\times n}(l=1,2,,3,4),$ we define its real representation matrix or real representation A^R as follows.

$$A^R\equiv\begin{pmatrix}A_1&A_2&A_3&A_4\\A_2&A_1&A_4&A_3\\A_3&A_4&A_1&A_2\\A_4&A_3&A_2&A_1\end{pmatrix}\in\mathbf{R}^{4m\times 4n}. \tag{2.1}$$

The set of all matrices shaped like (2.1) is denoted by $\mathbf{Rr}^{4m\times 4n}$.

Let

$$Q_t=\begin{pmatrix}0&I_t&0&0\\I_t&0&0&0\\0&0&0&I_t\\0&0&I_t&0\end{pmatrix},S_t=\begin{pmatrix}0&0&0&I_t\\0&0&I_t&0\\0&I_t&0&0\\I_t&0&0&0\end{pmatrix},R_t=\begin{pmatrix}0&0&I_t&0\\0&0&0&I_t\\I_t&0&0&0\\0&I_t&0&0\end{pmatrix}.$$

By simple computation, we can obtain the following properties.

2.1. Theorem

Let $A,B\in\mathbf{Q_{hc}}^{m\times n},C\in\mathbf{Q_{hc}}^{n\times s},\alpha\in\mathbf{R}$. Then

(1). $(A+B)^R=A^R+B^R,(\alpha A)^R=\alpha A^R,(AC)^R=A^RC^R;$

(2). $Q_m^2=S_m^2=R_m^2=I_{4m},Q_m^T=Q_m,R_m^T=R_m,S_m^T=S_m;$

(3). $R_mQ_m=Q_mR_m=S_m,Q_mS_m=S_mQ_m=R_m,S_mR_m=R_mS_m=Q_m;$

(4). $Q_mA^RQ_n=A^R,R_mA^RR_n=A^R,S_mA^RS_n=A^R.$

It is easy to verify that the following results are right.

2.2. Theorem

For any $V\in\mathbf{R}^{4m\times 4n}$,

$$V+Q_mVQ_n+R_mVR_n+S_mVS_n\in\mathbf{Rr}^{4m\times 4n},$$

and it is the real representation matrix of the canonical hyperbolic quaternion matrix

$$\tilde V=\frac14(I_m,I_mi,I_mj,I_mk)(V+Q_mVQ_n+R_mVR_n+S_mVS_n)\begin{pmatrix}I_n\\I_ni\\I_nj\\I_nk\end{pmatrix}.$$

Proof. Partitioning V into

$$V=\begin{pmatrix}V_{11}&V_{12}&V_{13}&V_{14}\\V_{21}&V_{22}&V_{23}&V_{24}\\V_{31}&V_{32}&V_{33}&V_{34}\\V_{41}&V_{42}&V_{43}&V_{44}\end{pmatrix}$$

and taking

$$\hat V=V+Q_mVQ_n+R_mVR_n+S_mVS_n,$$

we can verify $\hat V\in\mathbf{Rr}^{4m\times 4n}$ with

$$\hat{V}_{11} = V_{11} + V_{22} + V_{33} + V_{44}, \hat{V}_{21} = V_{21} + V_{12} + V_{43} + V_{34},$$
$$\hat{V}_{31} = V_{31} + V_{42} + V_{13} + V_{24}, \hat{V}_{41} = V_{41} + V_{32} + V_{23} + V_{14},$$

and $\hat{V}$ is the real representation matrix of the canonical hyperbolic quaternion matrix

$$\tilde{V} = \hat{V}_{11} + \hat{V}_{21} i + \hat{V}_{31} j + \hat{V}_{11} k = \frac{1}{4}(I_m, I_m i, I_m j, I_m k)\hat{V}\begin{pmatrix} I_n \\ I_n i \\ I_n j \\ I_n k \end{pmatrix}.$$

W

Further, we can also get the following construction method.

2.3. Theorem

For any $V \in \mathbf{R}^{4m\times n}$,

$$(V, Q_m V, R_m V, S_m V) \in \mathbf{Rr}^{4m\times 4n}.$$

2.4. Theorem.

For any $V \in \mathbf{R}^{4m\times 4n}$, $V \in \mathbf{Rr}^{4m\times 4n}$ if and only if

$$V = Q_m V Q_n = R_m V R_n = S_m V S_n.$$

A square matrix $A \in \mathbf{Q}_{\mathbf{hc}}{}^{n\times n}$ is said to be orthogonal matrix , if $AA^T = I$ and invertible matrix, if $AB = BA = I$ for some $B \in \mathbf{Q}_{\mathbf{hc}}{}^{n\times n}$, where A^T is the transpose of A. For the above concepts, the following results can be easily verified.

2.5. Theorem

Let $A \in \mathbf{Q}_{\mathbf{hc}}{}^{m\times n}, B \in \mathbf{Q}_{\mathbf{hc}}{}^{n\times p}, U \in \mathbf{Q}_{\mathbf{hc}}{}^{n\times n}$. Then the following properties hold:

(1). $(AB)^{-1} = B^{-1}A^{-1}$ if A and B are invertible;1

(2). $(A^T)^R = (A^R)^T$;

(3). $(AB)^T = B^T A^T$;

(4). U is a orthogonal matrix if and only if U^R is a orthogonal matrix.

Proof. (1)and (2) can be easily verified.

(3)From $((AB)^T)^R = ((AB)^R)^T = (A^R B^R)^T = (B^R)^T (A^R)^T = (B^T)^R (A^T)^R = (B^T A^T)^R$,we have $(AB)^T = B^T A^T$.

(4) If U is a orthogonal matrix, i.e., $UU^T = I$. By (1) of Theorem 2.1 and (2) of this theorem, we have $U^R (U^T)^R = U^R (U^R)^T = I^R = I_{4n}$, that is, U^R is a orthogonal matrix, and vice versa.

3. Some Applications of the Real Representation

Various quaternion matrix equations have been studied in a large number of papers. In He, *et al.* [18], Structure [19], He, *et al.* [20], the authors discussed some quaternion matrix equation and equations by means of matrix decomposition, rank equality, real representation and so on. In Zhang, *et al.* [2], Jiang, *et al.* [21], the authors studied some split quaternion matrix equations by real or complex representation. In Kösal and Tosun [14], the authors proposed the real matrix representation of canonical elliptic quaternion matrices and considered their equations. In this section, we study some applications of the real representation, including linear equation, inverse and MP inverse.

Let $A \in \mathbf{Q}_{\mathbf{hc}}{}^{m\times n}, b \in \mathbf{Q}_{\mathbf{hc}}{}^{m}$. If linear equation

$$Ax = b \qquad (3.1)$$

has the solution $x \in \mathbf{Q}_{\mathbf{hc}}{}^{n}$, then we have

$$A^R x^R = b^R$$

and therefore real linear equation

$$A^R y = b^R(1:4m, 1) \qquad (3.2)$$

has the solution $y \in \mathbf{R}^n$, where the symbol $M(i:j, k:l)$ represents the submatrix of M containing the intersection of rows i to j and columns k to l.

On the other hand, if (3.2) has a solution $y \in \mathbf{R}^n$, then we have

$$A^R Q_n y = Q_m A^R Q_n Q_n y = Q_m b^R(1:4m, 1),$$
$$A^R R_n y = R_m A^R R_n R_n y = R_m b^R(1:4m, 1),$$
$$A^R S_n y = S_m A^R S_n S_n y = S_m b^R(1:4m, 1),$$

and then

$$A^R(y, Q_n y, R_n y, S_n y)$$
$$= (b^R(1:4m,1), Q_m b^R(1:4m,1), R_m b^R(1:4m,1), S_m b^R(1:4m,1))$$
$$= b^R.$$

From Theorem 2.3, we know that $(y, Q_n y, R_n y, S_n y)$ is the real representation matrix of a canonical hyperbolic quaternion matrix(marked as x), and obtain $Ax = b$ ..

In conclusion, we have the following result.

3.1. Theorem

Let $A \in \mathbf{Q_{hc}}^{m\times n}, b \in \mathbf{Q_{hc}}^{m}$. Then equation (3.1) has a solution in $\mathbf{Q_{hc}}^{n}$ if and only if the real linear equation (3.2) has a solution in $\mathbf{R}^{4n}$. And if real linear equation (3.2) has the solution $y \in \mathbf{R}^{4n}$, then

$$(I_n, I_n i, I_n j, I_n k)y$$

is the solution of (3.1).

For the inverse, we have the following result.

3.2. Theorem

Let $A \in \mathbf{Q_{hc}}^{n\times n}$. Then A is invertible if and only if A^R is invertible. And when A is invertible, $(A^{-1})^R = (A^R)^{-1}$.

Proof. If A is invertible, there exists $B \in \mathbf{Q_{hc}}^{n\times n}$ such that $AB = I$. By (a) of Theorem 2.1, we have $A^R B^R = I^R = I_{4n}$, that is, A^R is invertible.

If A^R is invertible, there exists $B \in \mathbf{R}^{4n\times 4n}$ such that $A^R B = I_{4n}$. By (4) of Theorem 2.1, we have

$$A^R Q_n B Q_n = Q_n A^R Q_n Q_n B Q_n = Q_n I_{4n} Q = I_{4n}.$$

Similarly, $A^R S_n B S_n = I_{4n}, A^R R_n B R_n = I_{4n}$. So we can get

$$A^R \frac{B + Q_n B Q_n + S_n B S_n + R_n B R_n}{4} = I_{4n}.$$

By Theorem 2.2, we know that

$$\frac{B + Q_n B Q_n + S_n B S_n + R_n B R_n}{4}$$

is the real representation matrix of a canonical hyperbolic quaternion matrix. We denote this matrix as $\hat{B}$ and obtain $A^R \hat{B}^R = I_{4n}$, which can derive $A\hat{B} = I$, that is, A is invertible. W

Through the above proof, if B is the inverse of A^R, we can verify that

$$\frac{1}{16}(I_n, I_n i, I_n j, I_n k)(B + Q_n B Q_n + S_n B S_n + R_n B R_n)\begin{pmatrix} I_n \\ I_n i \\ I_n j \\ I_n k \end{pmatrix}$$

is the inverse of A. Through the above proof and the uniqueness of the inverse matrix, we can know that the inverse B of A^R belongs to $\mathbf{Rr}^{4m\times 4n}$. And so we have

$$\frac{1}{4}(I_n, I_n i, I_n j, I_n k)B\begin{pmatrix} I_n \\ I_n i \\ I_n j \\ I_n k \end{pmatrix}$$

is the inverse of A.

By the way, we can get the following interesting conclusion.

3.3. Corollary

Let $D \in \mathbf{Rr}^{4n\times 4n}$. Then its inverse (if exists) also belongs to $\mathbf{Rr}^{4n\times 4n}$, that is, D and its inverse have the same structure.

By summing up the above conclusions, we can naturally get the following result.

3.4. Corollary

Let $A \in \mathbf{Q_{hc}}^{n\times n}, b \in \mathbf{Q_{hc}}^{n}$. Then the following are equivalent.

(1). $Ax = b$ has a unique solution;

(2). A^R is invertible;
(3). A is invertible;
(4) . $rank(A^R)=4n$.

By similar derivation, we can obtain the following further conclusions.

3.5. Theorem

Let $A\in \mathbf{Q_{hc}}^{m\times n}, B\in \mathbf{Q_{hc}}^{n_1\times m_1}, C\in \mathbf{Q_{hc}}^{m\times m_1}$.

(1) $AXB=C$ has a solution in $\mathbf{Q_{hc}}^{n\times n_1}$ if and only if real matrix equation $A^R Y B^R = C^R$ has a solution in $\mathbf{R}^{4n\times 4n_1}$.

(2) If $A^R Y B^R = C^R$ has the solution $Y\in \mathbf{R}^{4n\times 4n_1}$, then

$$\frac{1}{16}(I_n, I_n i, I_n j, I_n k)(Y+Q_n Y Q_{n_1}+S_n Y S_{n_1}+R_n Y R_{n_1})\begin{pmatrix} I_{n_1} \\ I_{n_1} i \\ I_{n_1} j \\ I_{n_1} k \end{pmatrix}$$

is the solution of $AXB=C$.

(3) If $A^R Y B^R = C^R$ has the unique solution Y, then $Y\in \mathbf{Rr}^{4n\times 4n_1}$ and

$$\frac{1}{4}(I_n, I_n i, I_n j, I_n k)Y\begin{pmatrix} I_{n_1} \\ I_{n_1} i \\ I_{n_1} j \\ I_{n_1} k \end{pmatrix}$$

is the unique solution of $AXB=C$.

Next, we define a generalized inverse. Let $A\in \mathbf{Q_{hc}}^{m\times n}$. If $X\in \mathbf{Q_{hc}}^{n\times m}$ satisfies the conditions

$$(i).AXA=A, (ii).XAX=X, (iii).(AX)^T=AX, (iv).(XA)^T=XA, \quad (3.3)$$

we call X as T Moore-Penrose(T-MP) inverse of A .

Let Y is the MP inverse of A^R, that is, Y satisfies the conditions

$$(i).A^R Y A^R = A^R, (ii).YA^R Y = Y, (iii).(A^R Y)^T = A^R Y, (iv).(YA^R)^T = YA^R.$$

From $A^R Y A^R = A^R$, we have

$$Q_m A^R Q_n Q_n Y Q_m Q_m A^R Q_n = Q_m A^R Q_n$$

and then $A^R Q_n Y Q_m A^R = A^R$.

From $(A^R Y)^T = A^R Y$, we have

$$(Q_m A^R Q_n Q_n Y Q_m)^T = Q_m (A^R Q_n Q_n Y)^T Q_m = Q_m A^R Q_n Q_n Y Q_m$$

and then $(A^R Q_n Y Q_m)^T = A^R Q_n Y Q_m$.

For similar derivation, we have

$$Q_n Y Q_m A^R Q_n Y Q_m = Q_n Y Q_m \quad and \quad (Q_n Y Q_m A^R)^T = Q_n Y Q_m A^R,$$

that is, $Q_n Y Q_m$ is also a MP inverse of A^R . For the same reason, $S_n Y S_m$ and $R_n Y R_m$ are both MP inverses of A^R .

It follows from the uniqueness of the MP inverse that

$$Y = Q_n Y Q_m = S_n Y S_m = R_n Y R_m.$$

From Theorem 2.4, we have $Y\in \mathbf{Rr}^{4n\times 4m}$. Let $X\in \mathbf{Q_{hc}}^{n\times m}$ satisfy $X^R = Y$, we can obtain that X satisfies (3.3) and is the T-MP inverse of A .

From the above discussion, we can get the following conclusion.

3.6. Theorem

Let $A\in \mathbf{Q_{hc}}^{m\times n}$.

(1). A has a unique T-MP inverse.

(2). $(A^R)^\dagger \in \mathbf{Rr}^{4n\times 4m}$ and

$$\frac{1}{4}(I_n, I_n i, I_n j, I_n k)(A^R)^\dagger \begin{pmatrix} I_m \\ I_m i \\ I_m j \\ I_m k \end{pmatrix}$$

is the T -MP inverse of A .

By the way, we can also get the following interesting conclusion.

Corollary 3.3. Let $D \in \mathbf{Rr}^{4m\times 4n}$. Then D and its MP inverse have the same structure.

Fianlly, we give an example. Let $A = A_1 + A_2 i + A_3 j + A_4 k$ with

$$A_1 = \begin{pmatrix} 1 & -12 & 0 \\ 0 & 8 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -6 & 0 & 11 \\ 17 & 5 & -7 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & -9 & 8 \\ 10 & 5 & -9 \end{pmatrix}, A_4 = \begin{pmatrix} -6 & 13 & 5 \\ 11 & 2 & -14 \end{pmatrix}.$$

We can get the MP inverse A^R is $(A^R)^\dagger =$

$$\begin{pmatrix}
0.0077 & -0.0211 & 0.0143 & 0.0320 & 0.0124 & 0.0174 & 0.0131 & 0.0079 \\
-0.0137 & -0.0036 & -0.0142 & 0.0053 & 0.0019 & -0.0022 & 0.0359 & 0.0125 \\
0.0123 & 0.0281 & 0.0330 & 0.0133 & 0.0240 & -0.0028 & -0.0025 & -0.0179 \\
0.0143 & 0.0320 & 0.0077 & -0.0211 & 0.0131 & 0.0079 & 0.0124 & 0.0174 \\
-0.0142 & 0.0053 & -0.0137 & -0.0036 & 0.0359 & 0.0125 & 0.0019 & -0.0022 \\
0.0330 & 0.0133 & 0.0123 & 0.0281 & -0.0025 & -0.0179 & 0.0240 & -0.0028 \\
0.0124 & 0.0174 & 0.0131 & 0.0079 & 0.0077 & -0.0211 & 0.0143 & 0.0320 \\
0.0019 & -0.0022 & 0.0359 & 0.0125 & -0.0137 & -0.0036 & -0.0142 & 0.0053 \\
0.0240 & -0.0028 & -0.0025 & -0.0179 & 0.0123 & 0.0281 & 0.0330 & 0.0133 \\
0.0131 & 0.0079 & 0.0124 & 0.0174 & 0.0143 & 0.0320 & 0.0077 & -0.0211 \\
0.0359 & 0.0125 & 0.0019 & -0.0022 & -0.0142 & 0.0053 & -0.0137 & -0.0036 \\
-0.0025 & -0.0179 & 0.0240 & -0.0028 & 0.0330 & 0.0133 & 0.0123 & 0.0281
\end{pmatrix},$$

which belongs to $\mathbf{Rr}^{12\times 8}$. The T -MP inverse of A is $X_1 + X_2 i + X_3 j + X_4 k$ with

$$X_1 = (A^R)^\dagger(1:3,1:2), X_2 = (A^R)^\dagger(4:6,1:2),$$
$$X_3 = (A^R)^\dagger(7:9,1:2), X_4 = (A^R)^\dagger(10:12,1:2).$$

4. Conclusions

In this paper, we construct the real representation of canonical hyperbolic quaternion matrices and systematically study its properties. Then, we discuss the canonical hyperbolic quaternion linear equations and study the judgment and construction of solutions. Next, we give the necessary and sufficient condition for canonical hyperbolic quaternion matrix invertibility. Finally, we define a generalized inverse and initially discuss its existence and uniqueness. Some results are interesting.

We have only initially studied canonical hyperbolic quaternion matrices, and there are still a lot of problems worthy of further discussion. For example, rank, norm, determinant, etc. In the future work, we will pay more attention to the least squares problem.

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