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The Fekete-Szegö Problem for the Logarithmic Function of the Starlike and Convex Functions

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Abstract

In this paper, we discuss the Fekete-Szegö functional $\overline{\overline{H_2}}(1,\mu)=\delta_1\delta_3-\mu\delta_2^2$ which is defined by coefficients of the function $g(z)=\log\left(\frac{f(z)}{z}\right)$ for the analytic and univalent function $f(z)=z+a_2z^2+a_3z^3+\cdots$, $z\in U=\{z\in\mathbb{C}:|z|<1\}$, where μ is a real or complex number, and δ_1, δ_2 and δ_3 are the first three coefficients from the series expansion of the function $g(z)$. Our main purpose in this study is to find the upper bound for $|\delta_1\delta_3-\mu\delta_2^2|$, when $f\in S^*(\alpha)$ or $f\in C(\alpha)$, where $S^*(\alpha)$ and $C(\alpha)$ are, respectively, the class of starlike functions of order α and the class of convex functions of order α for $\alpha\in[0,1)$.

Keywords: Starlike function; Convex function; Fekete-Szegö functional; Logarithmic coefficient.
AMS Subject Classification: 30A10; 30C45; 30C50; 30C55.

1. Introduction

Let $U=\{z\in\mathbb{C}:|z|<1\}$ and $H(U)$ be the analytic functions in U and A the subclass of $H(U)$ functions f having the power series expansion

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}, \tag{1.1}$$

normalized by $f(0)=0=f'(0)-1$. Also, let's S be the subclass of A consisting also univalent functions.

The well-investigated subclasses of S are the class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(\alpha\in[0,1))$, which given as follows

$$S^*(\alpha)=\left\{f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in U\right\} \text{ and } C(\alpha)=\left\{f\in S:\ \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right)>\alpha,\ z\in U\right\}.$$

The function classes $S^*(\alpha)$ and $C(\alpha)$ have been investigated rather extensively in Kim and Srivastava [1], Ravichandran, *et al.* [2], Srivastava, *et al.* [3] Xu, *et al.* [4] and the references therein.

For $\alpha=0$, we obtain well-known subclasses of analytic and univalent functions

$$S^*=\left\{f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U\right\} \text{ and } C=\left\{f\in S:\ \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right)>0,\ z\in U\right\},$$

respectively, starlike and convex function classes [5-7].

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It is easy to see that $S^*(\alpha)\subset S^*$ and $C(\alpha)\subset C$ for each $\alpha\in[0,1)$. Also, between of the classes $C(C(\alpha))$ and $S^*(S^*(\alpha))$ is the relationship $f\in C\Leftrightarrow zf'\in S^*$ (or $f\in C(\alpha)\Leftrightarrow zf'\in S^*(\alpha)$) [5].

Among the important tools in the theory of analytic functions are Hankel determinant, which defined by coefficients of the function $f\in S$ as $H_q(n)=\left|a_j\right|_{j=\overline{n,n+q-1}}^{j=\overline{n,n+q-1}}$, $a_1=1$, $n=1,2,3,\ldots,\ q=1,2,3,\ldots$ [8]. Generally, these determinants was investigated by researchers with $q=2$. The functional $H_2(1)=a_3-a_2^2$ is known as the Fekete-Szegö functional and one usually considers the further generalized functional $H_2(1,\mu)=a_3-\mu a_2^2$, where μ is a number [9]. Finding upper bound for $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem in the theory of analytic functions.

In Koegh and Merkes [10], solved the Fekete-Szegö problem for the classes of starlike and convex functions for some real μ. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions [8, 11-18].

It is well known that logarithmic coefficients δ_n, $n=1,2,3,\ldots$ of a function $f\in S$ are defined by differentiation both sides of the following equality

$$g(z)=\log\left(\frac{f(z)}{z}\right)=2\sum_{n=1}^{\infty}\delta_n z^n,$$

and play a central role in the theory of analytic functions [19].

In Thomas and Derek [19]. By Thomas given sharp estimates for the modulus of the initial three coefficients of the function $g(z)$ when the function f belong to some subclass of the analytic and univalent functions.

Let $f\in S$. We define the determinants $\overline{\overline{H}}_q(n)=\left|\delta_j\right|_{j=\overline{n,n+q-1}}^{j=\overline{n,n+q-1}}$, $n=1,2,3,\ldots,\ q=1,2,3,\ldots$, where δ_j, $j=1,2,3,4,\ldots$ are the coefficients of the function g. The determinants $\overline{\overline{H}}_q(n)$ we next recall the logarithmic Hankel determinants of the function f. Also, we define the functional $\overline{\overline{H}}_2(1)=\delta_1\delta_3-\delta_2^2$, more general $\overline{\overline{H}}_2(1)=\delta_1\delta_3-\mu\delta_2^2$ for some number μ, which we will recall the Fekete-Szegö type functional of the function f. Finding upper bound for $\left|\delta_1\delta_3-\mu\delta_2^2\right|$, we will recall as the Fekete-Szegö type problem for the function f.

In this paper, we obtain the estimates for $\left|\delta_1\delta_3-\delta_2^2\right|$, while f is either in $S^*(\alpha)$ or in $C(\alpha)$.

In order to prove our main results, we need the following lemma [20] concerning functions in the class P, i. e. analytic functions p such that $p(0)=1$ and $\mathrm{Re}(p(z))>0$ for all $z\in U$. That is, $p\in\mathrm{P}$ have the power series expansion as follows

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots,\ z\in U.$$

1.1. Lemma

Let $p\in\mathrm{P}$, *then* $|p_n|\le 2$ *for every* $n=1,2,3,\ldots$. *These inequalities are sharp for each* $n=1,2,3,\ldots$. *Moreover,*

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$

$$4p_3=p_1^3+2\left(4-p_1^2\right)p_1x-\left(4-p_1^2\right)p_1x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

for some complex $x,\ z$ *with* $|x|\le 1$, $|z|\le 1$.

2. Bounds of $\left|\delta_1\delta_3-\delta_2^2\right|$ for the Starlike Functions

In this section, we investigate Fekete-Szegö type problem for the function $f\in S^*(\alpha)$.

The logarithmic coefficients δ_n, $n = 1, 2, 3, \ldots$ of a function $f \in S$ are defined by the following equality with differentiation of both sides

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \delta_n z^n, \qquad (2.1)$$

and play a central role in the theory of analytic functions [19].

2.1. Theorem

Let the function $f(z)$ *given by (1.1) be in the class* $S^*(\alpha)$, $\alpha \in [0,1)$. *Then,*

$$\left|\delta_1\delta_3 - \delta_2^2\right| \le \frac{(1-\alpha)^2}{4}.$$

Proof. Let $f \in S^*(\alpha)$, $\alpha \in [0,1)$. Then,

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)\,p(z), \qquad (2.2)$$

where $p \in \mathrm{P}$.
From (2.2), we have

$$1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 + \cdots = 1 + (1-\alpha)\left(p_1 z + p_2 z^2 + p_3 z^3 + \cdots\right)$$

Comparing coefficients of z, z^2 and z^3, we get

$$a_2 = (1-\alpha)\,p_1,\quad 2a_3 - a_2^2 = (1-\alpha)\,p_2,\quad 3a_4 - 3a_2a_3 + a_2^3 = (1-\alpha)\,p_3. \qquad (2.3)$$

Differentiating both sides of (2.1) and upon simplification, we have

$$a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 + \cdots = 2\delta_1 z + 4\delta_2 z^2 + 6\delta_3 z^3 + \cdots.$$

Comparing the coefficients of z^n for δ_n, $n = 1, 2, 3$, we get

$$\delta_1 = \frac{a_2}{2},\quad \delta_2 = \frac{1}{4}\left(2a_3 - a_2^2\right),\quad \delta_3 = \frac{1}{6}\left(3a_4 - 3a_2a_3 + a_2^3\right). \qquad (2.4)$$

Substituting the values δ_1, δ_2 and δ_3 from (2.4) in $\delta_1\delta_3 - \delta_2^2$, we can write

$$\delta_1\delta_3 - \delta_2^2 = \frac{a_2}{12}\left(3a_4 - 3a_2a_3 + a_2^3\right) - \frac{1}{16}\left(2a_3 - a_2^2\right)^2.$$

Also, using (2.3) in the expression $\delta_1\delta_3 - \delta_2^2$, we obtain

$$\delta_1\delta_3 - \delta_2^2 = \frac{(1-\alpha)^2}{48}\left(4p_1p_3 - 3p_2^2\right). \qquad (2.5)$$

We now use Lemma 1.1 to express the coefficients p_2 and p_3 in term of p_1 to obtain, after simplification, normalizing the coefficient p_1 so that $p_1 = t \in [0,2]$, setting $|x| = \xi \in [0,1]$, and finally using the triangle inequality,

$$\left|\delta_1\delta_3 - \delta_2^2\right| \le \frac{(1-\alpha)^2}{48}\left\{\frac{t^4}{4} + \frac{\left(4-t^2\right)t^2\xi}{2} + \frac{\left(4-t^2\right)\left(12+t^2\right)}{4}\xi^2\right.$$

$$\left. + 2\left(4-t^2\right)\left(1-\xi^2\right)t\right\} := \frac{(1-\alpha)^2}{48}\phi(t,\xi) \text{ (say).}$$

Now, we need to maximize the function $\phi(t,\xi)$ in the square $\Omega = \{(t,\xi):\ t \in [0,2] \text{ and } \xi \in [0,1]\}$.

It is easily verified that differentiating the function $\phi(t,\xi)$ with respect to t and then ξ and equating to zero shows that the only admissible extremum points are $(0,0)$ or $(2,-1)$. Since $(2,-1) \notin \Omega$ and $\phi(0,0) = 0$, both of these points are not the maximum points of the function.

Therefore, we must investigate the maximum of the function $\phi(t,\xi)$ on the boundary of the closed square Ω.

For $t=0$, $\xi\in[0,1]$ we have

$$\phi(0,\xi)=12\xi^2\leq 12. \quad (2.6)$$

For $t=2$, $\xi\in[0,1]$, we obtain

$$\phi(2,\xi)=4. \quad (2.7)$$

Now, let $\xi=0$ and $t\in[0,2]$. Then,

$$\phi(t,0)=\frac{t}{4}\left(t^3-8t^2+32\right).$$

By simple computation, we find

$$\phi'(t,0)=t^3-6t^2+8,\ t\in[0,2].$$

From this, we can easily verified that $t_0=1.3054$, where t_0 is a numerical solution of the equation $t^3-6t^2+8=0$, is a critical point of the function $\phi(t,0)$. Since $\phi'(t,0)>0$ when $t\in[0,t_0)$ and $\phi'(t,0)<0$ when $t\in(t_0,2]$, the point t_0 is a maximum point of the function $\phi(t,0)$. So that,

$$\max\left\{\phi(t,0):\ t\in[0,2]\right\}=\phi(t_0,0). \quad (2.8)$$

Finally, for $\xi=1$ and $t\in[0,2]$, we write

$$\phi(t,1)=-\frac{t^4}{2}+12.$$

It is clear that $t=0$ is a critical point for the function $\phi(t,1)$. Since $\phi^{(n)}(0,1)=0$ for $n=1,2,3$ and $\phi^{\prime v}(0,1)\neq 0$, $\phi^{\prime v}(0,1)=-12<0$, then $t=0$ is a maximum point for the function $\phi(t,1)$. Therefore,

$$\max\left\{\phi(t,1):\ t\in[0,2]\right\}=\phi(0,1)=12. \quad (2.9)$$

From (2.6)-(2.10), we obtain

$$\max\left\{\phi(t,\xi):\ (t,\xi)\in\Omega\right\}=\max\left\{4,\ 12,\ \phi(t_0,0)\right\}. \quad (2.10)$$

Since $\phi(t_0,0)<12$, from (2.11), we write

$$\max\left\{\phi(t,\xi):\ (t,\xi)\in\Omega\right\}=12.$$

Thus, the proof of Theorem 2.1 is completed.

Choosing $\alpha=0$ in Theorem 2.1, we arrive at the following result.

2.2. Corollary

Let the function $f(z)$ given by (1.1) be in the class S^*. Then,

$$\left|\delta_1\delta_3-\delta_2^2\right|\leq\frac{1}{4}.$$

3. Bounds of $\left|\delta_1\delta_3-\delta_2^2\right|$ for the Convex Functions

In this section, we investigate Fekete-Szegö type problem for the function $f\in C(\alpha)$.

3.1. Theorem

Let the function $f(z)$ given by (1.1) be in the class $C(\alpha)$, $\alpha\in[0,1)$. Then,

$$\left|\delta_1\delta_3-\delta_2^2\right|\leq\frac{(1-\alpha)^2\left(5\alpha^2-12\alpha+24\right)}{144\left(\alpha^2-2\alpha+5\right)}$$

Proof. Let $f\in C(\alpha)$, $\alpha\in[0,1)$. Then,

$$\frac{\left(zf'(z)\right)'}{f'(z)}=\alpha+(1-\alpha)p(z), \tag{3.1}$$

where the function $p\in \mathrm{P}$.

Replacing $f'(z)$, $\left(zf'(z)\right)'$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$\sum_{n=2}^{\infty}n(n-1)a_nz^{n-1}=(1-\alpha)\left(1+\sum_{n=2}^{\infty}na_nz^{n-1}\right)\times\sum_{n=1}^{\infty}p_nz^n.$$

Upon simplification, we obtain

$$\sum_{n=2}^{\infty}n(n-1)a_nz^{n-1}=(1-\alpha)\{p_1z+(p_2+2p_1a_2)z^2+(p_3+2p_2a_2+3p_1a_3)z^3$$
$$+\cdots+(p_{n-1}+2p_{n-2}a_2+3p_{n-3}a_3+\cdots+(n-1)p_1a_{n-1})z^{n-1}+\cdots\}. \tag{3.2}$$

Equating the coefficients of z^n, $n=1,2,3,\ldots$, we get

$$n(n-1)a_n=(1-\alpha)\left[p_{n-1}+2p_{n-2}a_2+3p_{n-3}a_3+\cdots+(n-1)p_1a_{n-1}\right],\ n=2,3,4,\ldots;$$

that is,

$$a_n=\frac{1-\alpha}{n(n-1)}\sum_{k=1}^{n-1}kp_{n-k}a_k,\ n=2,3,4,\ldots. \tag{3.3}$$

From (3.3), we have

$$a_2=\frac{1-\alpha}{2}p_1,\ a_3=\frac{1-\alpha}{6}(p_2+2p_1a_2),\ a_4=\frac{1-\alpha}{12}(p_3+2p_2a_2+3p_1a_3). \tag{3.4}$$

Substituting the values of a_2 and a_3 in the next equalities in (3.4), after simplifying, we get

$$a_3=\frac{1-\alpha}{6}\left[p_2+(1-\alpha)p_1^2\right],\ a_4=\frac{1-\alpha}{24}\left[2p_3+3(1-\alpha)p_1p_2+(1-\alpha)^2p_1^3\right]. \tag{3.5}$$

Considering the value of a_2 from (3.4) and the values of a_3 and a_4 from (3.5) in (2.4), we obtain the following expression for δ_1, δ_2 and δ_3

$$\delta_1=\frac{1-\alpha}{4}p_1,\ \delta_2=\frac{1-\alpha}{48}\left[4p_2+(1-\alpha)p_1^2\right],\ \delta_3=\frac{1-\alpha}{48}\left[2p_3+(1-\alpha)p_1p_2\right]. \tag{3.6}$$

Substituting the values of δ_1, δ_2 and δ_3 from (3.6) in the expression $\delta_1\delta_3-\delta_2^2$, we can write

$$\delta_1\delta_3-\delta_2^2=\frac{(1-\alpha)^2}{2304}\times\left[24p_1p_3+4(1-\alpha)p_1^2p_2-16p_2^2-(1-\alpha)^2p_1^4\right]. \tag{3.7}$$

We now use Lemma 1.1 to express the coefficients p_2 and p_3 in term of p_1 in the right hand side of (3.7), we have

$$24p_1p_3+4(1-\alpha)p_1^2p_2-16p_2^2-(1-\alpha)^2p_1^4$$
$$=(3-\alpha^2)p_1^4+2(3-\alpha)(4-p_1^2)p_1^2x-2(8+p_1^2)(4-p_1^2)x^2+12(4-p_1^2)\left(1-|x|^2\right)p_1z.$$

Normalizing the coefficient p_1 so that $p_1=t\in[0,2]$, setting $|x|=\xi\in[0,1]$, and finally using the triangle inequality to last equality, we obtain

$$\left|24p_1p_3+4(1-\alpha)p_1^2p_2-16p_2^2-(1-\alpha)^2p_1^4\right|$$
$$\le(3-\alpha^2)t^4+2(3-\alpha)(4-t^2)t^2\xi+2(t-2)(t-4)(4-t^2)\xi^2$$
$$+12(4-t^2)t=F(t,\xi)\ \text{(say)}. \tag{3.8}$$

Where

$$F(t,\xi)=(3-\alpha^2)t^4+2(3-\alpha)(4-t^2)t^2\xi+2(t-2)(t-4)(4-t^2)\xi^2+12(4-t^2)t \quad (3.9)$$

We next maximize the function $F(t,\xi)$ on the closed rectangle $[0,2]\times[0,1]$. Differentiating the function $F(t,\xi)$ partially with respect to ξ, we get

$$F'_{\xi}(t,\xi)=2(4-t^2)\left[(3-\alpha)t^2+2(t-2)(t-4)\xi\right].$$

Since $F'_{\xi}(t,\xi)\geq 0$ on the closed rectangle $[0,2]\times[0,1]$ for all $\alpha\in[0,1)$, the function $F(t,\xi)$ is an increasing function of ξ and hence it cannot have a maximum value at any point in the interior of the interval $[0,1]$. So that,

$$\max\{F(t,\xi):\ \xi\in[0,1]\}=F(t,1)=\varphi(t)\ \text{(say)} \quad (3.10)$$

for fixed $t\in[0,2]$.

In view of (3.10) and (3.9), after simplifying, we get

$$\varphi(t)=-(\alpha^2-2\alpha+5)t^4+8(2-\alpha)t^2+64,\quad t\in[0,2]. \quad (3.11)$$

We now use elementary calculus to find the maximum of the function $\varphi(t)$ on the interval $[0,2]$. By simple computation, we find

$$\varphi'(t)=-4t\left[(\alpha^2-2\alpha+5)t^2-4(2-\alpha)\right]. \quad (3.12)$$

Considering $\varphi'(t)=0$ from (3.12) we can easily see that $t_1=0$ and $t_2=\sqrt{\dfrac{4(2-\alpha)}{\alpha^2-2\alpha+5}}$ (it is easily verified that $t_2\in(0,2)$ for all $\alpha\in[0,1)$) are two admissible critical points for the function $\varphi(t)$.

We use the second derivative test to find extremum point of the function $\varphi(t)$. Differentiating (3.12), we get

$$\varphi''(t)=12(-\alpha^2+2\alpha-5)t^2+16(2-\alpha). \quad (3.13)$$

From the equation (3.13), we can easily see that $\varphi''(0)=16(2-\alpha)>0$; that is, the point $t_1=0$ is a minimum point for the function $\varphi(t)$.

We now discuss the case $t_2=\sqrt{\dfrac{4(2-\alpha)}{\alpha^2-2\alpha+5}}$. Using the value t_2 in (3.13), after simplification, we obtain

$$\varphi''(t_2)=-32(2-\alpha)<0$$

Hence, by the second derivative test, $\varphi(t)$ has a local maximum value at the point t_2. Therefore,

$$\max\{\varphi(t):\ t\in(0,2)\}=\varphi(t_2)=\frac{16(5\alpha^2-12\alpha+24)}{\alpha^2-2\alpha+5}. \quad (3.14)$$

Also, since $\varphi(2)=16(3-\alpha^2)\leq 16\max\limits_{0\leq\alpha<1}(3-\alpha^2)\leq 48<\varphi(t_2)$, the function $\varphi(t)$ has the maximum value on the interval $[0,2]$ in the point t_2.

Considering this fact, (3.14), (3.10) and (3.8), we get

$$\left|24p_1p_3+4(1-\alpha)p_1^2p_2-16p_2^2-(1-\alpha)^2p_1^4\right|\leq\frac{16(5\alpha^2-12\alpha+24)}{\alpha^2-2\alpha+5}. \quad (3.15)$$

From the expression (3.7) and inequality (3.15), by simplification, we obtain

$$\left|\delta_1\delta_3-\delta_2^2\right|\leq\frac{(1-\alpha)^2(5\alpha^2-12\alpha+24)}{144(\alpha^2-2\alpha+5)}.$$

Thus, the proof of Theorem 3.1 is completed.

Choosing $\alpha = 0$ in Theorem 3.1, we have the following result.

3.2. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* C. *Then,*

$$\left|\delta_1\delta_3 - \delta_2^2\right| \le \frac{1}{30}.$$

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