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On The Upper Bound Estimates for the Coefficients of Certain Class Bi-Univalent Functions of Complex Order

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Abstract

In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions of complex order in the unit disk in complex plane. We obtain upper bound estimates for the initial three coefficients of the functions belonging to this class. In this study, the Fekete-Szegö problem for this function class is also investigated.
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1. Introduction

Let A be the class of the functions in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

We denote by S the subclass of A consisting of functions which are also univalent in U. Some of the important subclass of S is the class $\Re(\alpha, \beta)$ that is defined as follows

$$\Re(\alpha,\beta) = \left\{ f \in S : \operatorname{Re}\left[f'(z) + \beta z f''(z) \right] > \alpha, z \in U \right\}, \alpha \in [0,1), \beta \geq 0.$$

Gao and Zhou [1], have researched the class $\Re(\alpha, \beta)$ and showed some mapping properties of this subclass.

In the special case, we have subclass $\Re(\beta)$

$$\Re(\beta) = \left\{ f \in S : \operatorname{Re}\left[f'(z) + \beta z f''(z) \right] > 0, z \in U \right\}, \beta \geq 0$$

for $\alpha = 0$.

Early, by Altıntaş and Özkan [2] were studied a subclass $\Re(\alpha,\beta,\gamma)$ of analytic and bi-univalent functions consisting of the functions $f(z)$ which satisfy the conditions

$$f \in T, \left| \frac{1}{\gamma} \left[f'(z) + \beta z f''(z) - 1 \right] \right| \leq \alpha, z \in U, \beta \in [0,1], \alpha \in (0,1], \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}.$$

Here T is the class of the functions $f(z)$ in the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \cdots = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0,$$

which are analytic in the open unit disk U [2]. Found necessary and sufficient conditions for the functions belonging to this class.

It is well-known that (see, for example [3], every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, z \in U, f(f^{-1}(w)) = w, w \in D = \{ w : |w| < r_0(f) \}, r_0(f) \geq \frac{1}{4},$$

where $f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots, w \in D$.

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent. Let Σ denote the class of bi-univalent functions in U given (1.1).

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In Lewin [4], showed that for every function $f \in \Sigma$ of the form (1.1) the second coefficient satisfies the estimate $|a_2| < 1.51$. In 1967, [5] conjectured that $|a_2| < \sqrt{2}$ for $f \in \Sigma$. In Tan [6], obtained the bound for $|a_2|$, namely, that $|a_2| < 1.485$, which is the best known estimate for functions in the class Σ. In Kedzierawski [7], proved the Brannan-Clunie conjecture for bi-starlike functions. Brannan and Taha [8], obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in the classes of bi-starlike functions of order α and bi-convex functions of order α.

The study of bi-univalent functions was revived, in recently years, by Srivastava, *et al.* [9], and a considerably large number of sequels to the work of Srivastava, *et al.* [9], have appeared in the literature. In particular, several results on coefficient estimates for the initial coefficients $|a_2|, |a_3|$ and $|a_4|$ were proved for various subclasses of Σ [10-17].

Recently Deniz [18], and Kumar, *et al.* [19], both extended and improved the results of Brannan and Taha [8], by generalizing their classes by means of the principle of subordination between analytic functions.

Despite the numerous studies mentioned above, the problem of estimating the coefficients $|a_n| (n = 2, 3, \ldots)$ for the general class functions Σ is still open [12].

One of the important tools in the theory of analytic functions is the functional $H_2(1) = a_3 - a_2^2$ which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$, where μ is some real number [20]. Estimating for the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem. In Keogh and Merkes [21], solved the Fekete-Szegö problem for the classes starlike and convex functions. Someone can see the Fekete-Szegö problem for the classes of starlike functions of order α and convex functions of order α at special cases in the paper of Orhan, *et al.* [22]. On the other hand, recently [23], have obtained Fekete-Szegö inequality for a subclass of bi-univalent functions. Also Zaprawa [24], Zaprawa [25], have studied on Fekete-Szegö problem for some subclasses of bi-univalent functions. In special cases, he gave the Fekete-Szegö problem for the subclasses bi-starlike functions of order α and bi-convex functions of order α.

Motivated by the aforementioned works, we define a new subclass of bi-univalent functions Σ as follows.

1.1. Definition

A function $f \in \Sigma$ *given by (1.1) is said to be in the class* $\Im_\Sigma(\alpha, \beta, \tau)$ *if the following conditions are satisfied*

$$\operatorname{Re}\left\{1 + \frac{1}{\tau}\left[f'(z) + \beta z f''(z) - 1\right]\right\} > \alpha, z \in U, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, \alpha \in [0,1), \beta \geq 0$$

and

$$\operatorname{Re}\left\{1 + \frac{1}{\tau}\left[g'(w) + \beta w g''(w) - 1\right]\right\} > \alpha, w \in D, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}, \alpha \in [0,1), \beta \geq 0$$

where the function $g = f^{-1}$.

1.1.1. Remark

Taking $\tau = 1$ *in Definition 1.1, we have function class* $\Im_\Sigma(\alpha, \beta, 1) = H_\Sigma(\alpha, \beta),\ \alpha \in [0,1), \beta \geq 0$*; that is,*

$$f \in H_\Sigma(\alpha, \beta) \Leftrightarrow \operatorname{Re}\left(f'(z) + \beta z f''(z)\right) > \alpha, z \in U \quad \text{and} \quad \operatorname{Re}\left(g'(w) + \beta w g''(w)\right) > \alpha, w \in D,$$

where $g = f^{-1}$.

1.1.2. Remark

Taking $\beta = 0$ *in Definition 1.1, we have function class* $\Im_\Sigma(\alpha, 0, \tau),\ \alpha \in [0,1),\ \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$*; that is,*

$$f \in \Im_\Sigma(\alpha, 0, \tau) \Leftrightarrow \operatorname{Re}\left\{1 + \frac{1}{\tau}\left[f'(z) - 1\right]\right\} > \alpha, z \in U \quad \text{and} \quad \operatorname{Re}\left\{1 + \frac{1}{\tau}\left[g'(w) - 1\right]\right\} > \alpha, w \in D,$$

where $g = f^{-1}$

1.1.3. Remark

Taking $\beta = 0$, $\tau = 1$ *Definition 1.1, we have function class* $\Im_\Sigma(\alpha, 0, 1) = \Re_\Sigma(\alpha, 0),\ \alpha \in [0,1)$ *; that is,*

$$f \in \Re_\Sigma(\alpha, 0) \Leftrightarrow \operatorname{Re}\left(f'(z)\right) > \alpha, z \in U \text{ and } \operatorname{Re}\left(g'(w)\right) > \alpha, w \in D,$$

where $g = f^{-1}$.

1.1.4. Remark

Taking $\beta = 1$ *in Definition 1.1, we have function class* $\Im_\Sigma(\alpha, 1, \tau),\ \alpha \in [0,1),\ \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$ *; that is,*

$$f \in \Im_\Sigma(\alpha, 1, \tau) \Leftrightarrow \operatorname{Re}\left\{1 + \frac{1}{\tau}\left[f'(z) + zf''(z) - 1\right]\right\} > \alpha, z \in U$$

and

$$\operatorname{Re}\left\{1 + \frac{1}{\tau}\left[g'(w) + wg''(w) - 1\right]\right\} > \alpha, w \in D,$$

where $g = f^{-1}$.

1.1.5. Remark

Taking $\beta = 1$, $\tau = 1$ *in Definition 1.1, we have function class* $\Im_\Sigma(\alpha, 1, 1) = \Re_\Sigma(\alpha, 1),\ \alpha \in [0,1)$ *; that is,*

$$f \in \Re_\Sigma(\alpha, 1) \Leftrightarrow \operatorname{Re}\left(f'(z) + zf''(z)\right) > \alpha, z \in U \text{ and } \operatorname{Re}\left(g'(w) + wg''(w)\right) > \alpha, w \in D,$$

where $g = f^{-1}$.

The class $\Im_\Sigma(\alpha, 0, 1) = \Re_\Sigma(\alpha, 0) = N_\Sigma(\alpha)$ were investigated by Grenander and Szegö [26], and by Çağlar, *et al.* [27].

Recently Frasin [28], investigated subclass $\Im_\Sigma(\alpha, \beta, 1) = H_\Sigma(\alpha, \beta), \alpha \in [0,1), \beta > 0$ with condition $2(1-\alpha)\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{\beta n + 1} \le 1$. He found estimates on two first coefficients for the functions in this class.

The object of the present paper is to find the upper bound estimates for the initial coefficients $|a_2|, |a_3|$ and $|a_4|$ of the functions belonging to the class $\Im_\Sigma(\alpha, \beta, \tau)$ and its special cases. The Fekete-Szegö problem for this function class is also investigated.

To prove our main results, we need require the following lemmas.

1.2. Lemma

(See, for example, [29] *If* $p \in \mathrm{P}$*, then the estimates* $|p_n| \le 2, n = 1, 2, 3, \ldots$ *are sharp, where* P *is the family of all functions* p*, analytic in* U *for which* $p(0) = 1$ *and* $\operatorname{Re}\left(p(z)\right) > 0\ (z \in U)$*, and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in U. \tag{1.2}$$

1.2.1. Lemma

(See, for example, [26] *If the function* $p \in \mathrm{P}$ *is given by the series (1.2), then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some x *and* z *with* $|x| \le 1$ *and* $|z| \le 1$.

2. Coefficient Bound Estimates for the Function Class $\Im_\Sigma(\alpha, \beta, \tau)$

In this section, we prove the following theorem on upper bound estimates for the initial three coefficients of the function class $\Im_\Sigma(\alpha, \beta, \tau)$.

2.1. Theorem

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,\beta,\tau)$, $\alpha\in[0,1),\ \beta\in[0,1],\ \tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. *Then,*

$$|a_2|\le\frac{(1-\alpha)|\tau|}{1+\beta},\qquad |a_3|\le\begin{cases}\dfrac{2(1-\alpha)|\tau|}{3(1+2\beta)}, & \text{if } |\tau|\in(0,\tau_0),\\[2mm] \dfrac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}, & \text{if } |\tau|\in[\tau_0,+\infty),\end{cases}$$

where $\tau_0=\dfrac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$ *and*

$$|a_4|\le\frac{(1-\alpha)|\tau|}{2(1+3\beta)}.$$

Proof. Let $f\in\Im_\Sigma(\alpha,\beta,\tau),\alpha\in[0,1),\beta\in[0,1],\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ and $g=f^{-1}$. Then,

$$1+\frac{1}{\tau}\left[f'(z)+\beta zf''(z)-1\right]=\alpha+(1-\alpha)p(z) \tag{2.1}$$

and

$$1+\frac{1}{\tau}\left[g'(w)+\beta wg''(w)-1\right]=\alpha+(1-\alpha)q(w), \tag{2.2}$$

where functions $p(z)=1+p_1z+p_2z^2+\cdots$ and $q(w)=1+q_1w+q_2w^2+\cdots$ are in the class P.
Comparing the coefficients in (2.1) and (2.2), we have

$$a_2=\frac{\tau(1-\alpha)}{2(1+\beta)}p_1,\quad a_3=\frac{\tau(1-\alpha)}{3(1+2\beta)}p_2,\quad a_4=\frac{\tau(1-\alpha)}{4(1+3\beta)}p_3 \tag{2.3}$$

and

$$-a_2=\frac{\tau(1-\alpha)}{2(1+\beta)}q_1,\quad 2a_2^2-a_3=\frac{\tau(1-\alpha)}{3(1+2\beta)}q_2,\quad -5a_2^3+5a_2a_3-a_4=\frac{\tau(1-\alpha)}{4(1+3\beta)}q_3. \tag{2.4}$$

From the first equality of (2.3) and (2.4), we find

$$\frac{\tau(1-\alpha)}{2(1+\beta)}p_1=a_2=-\frac{\tau(1-\alpha)}{2(1+\beta)}q_1. \tag{2.5}$$

Also, from the second equality of (2.3) and (2.4), considering (2.5), we get

$$a_3=\frac{\tau^2(1-\alpha)^2}{4(1+\beta)^2}p_1^2+\frac{\tau(1-\alpha)}{6(1+2\beta)}\left(p_2-q_2\right). \tag{2.6}$$

Subtracting the third equality of (2.4) from the third equality of (2.3) and considering (2.5) and (2.6), we can easily obtain

$$a_4=\frac{5\tau^2(1-\alpha)^2}{24(1+\beta)(1+2\beta)}p_1\left(p_2-q_2\right)+\frac{\tau(1-\alpha)}{8(1+3\beta)}\left(p_3-q_3\right). \tag{2.7}$$

In view of Lemma 1.2, since (see (2.5)) $p_1=-q_1$, we can write

$$\left.\begin{aligned}2p_2&=p_1^2+\left(4-p_1^2\right)x,\\ 2q_2&=q_1^2+\left(4-q_1^2\right)y\end{aligned}\right\}\Rightarrow p_2-q_2=\frac{4-p_1^2}{2}(x-y) \tag{2.8}$$

and

$$\left.\begin{aligned}4p_3&=p_1^3+2\left(4-p_1^2\right)p_1x-\left(4-p_1^2\right)p_1x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z,\\ 4q_3&=q_1^3+2\left(4-q_1^2\right)q_1y-\left(4-q_1^2\right)q_1y^2+2\left(4-q_1^2\right)\left(1-|y|^2\right)w\end{aligned}\right\}\Rightarrow$$

$$p_3-q_3=\frac{p_1^3}{2}+\frac{p_1\left(4-p_1^2\right)}{2}(x+y)-\frac{p_1\left(4-p_1^2\right)}{4}\left(x^2+y^2\right)+\frac{4-p_1^2}{2}\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right] \quad (2.9)$$

for some x,y and z,w with $|x|\le 1,\ |y|\le 1,\ |z|\le 1$ and $|w|\le 1$.

Since $|p_1|\le 2$, we may assume without any restriction that $t\in[0,2]$, where $t=|p_1|$.
From (2.5), we easily see that

$$|a_2|\le\frac{(1-\alpha)|\tau|}{2(1+\beta)}t,\ t\in[0,2],$$

so

$$|a_2|\le\frac{(1-\alpha)|\tau|}{1+\beta}. \quad (2.10)$$

Substituting the expression (2.8) in (2.6) and using triangle inequality, taking $|x|=\xi,\ |y|=\eta$, we can easily obtain

$$|a_3|\le C_1(t)(\xi+\eta)+C_2(t)=F(\xi,\eta), \quad (2.11)$$

where

$$C_1(t)=\frac{(1-\alpha)|\tau|\left(4-t^2\right)}{12(1+2\beta)}\ge 0,\quad C_2(t)=\frac{(1-\alpha)^2|\tau|^2}{4(1+\beta)^2}t^2\ge 0,\quad t\in[0,2].$$

Now, we need to maximize the function $F(\xi,\eta)$ on the closed square $\Omega=\{(\xi,\eta):\ \xi,\eta\in[0,1]\}$.

Since the coefficients $C_1(t)$ and $C_2(t)$ of the function $F(\xi,\eta)$ is dependent to variable t, we must investigate the maximum of $F(\xi,\eta)$ respect to t taking into account these cases $t=0, t=2$ and $t\in(0,2)$.

Let $t=0$. Then, we write

$$F(\xi,\eta)=C_1(0)(\xi+\eta)=\frac{|\tau|(1-\alpha)}{3(1+2\beta)}(\xi+\eta).$$

It is clear that the maximum of the function $F(\xi,\eta)$ occurs at $(\xi,\eta)=(1,1)$, and in this case

$$\max\{F(\xi,\eta):\xi,\eta\in[0,1]\}=F(1,1)=\frac{2|\tau|(1-\alpha)}{3(1+2\beta)}. \quad (2.12)$$

For $t=2$, the function $F(\xi,\eta)$ is a constant function as follows

$$F(\xi,\eta)=C_2(2)=\frac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}. \quad (2.13)$$

Now, let $t\in(0,2)$. In this case, we can easily see that

$$\max\{F(\xi,\eta):\xi,\eta\in[0,1]\}=F(1,1)=2C_1(t)+C_2(t) \quad (2.14)$$

for all $t\in(0,2)$.

The function $G:(0,2)\to\mathbb{R}$, we will define as follows

$$G(t)=2C_1(t)+C_2(t) \quad (2.15)$$

for fixed value of $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$.

Substituting the value $C_1(t)$ and $C_2(t)$ in (2.15), we obtain

$$G(t)=A(\alpha,\beta;\tau)t^2+B(\alpha,\beta;\tau),$$

Where

$$A(\alpha,\beta;\tau)=\frac{3(1-\alpha)^2|\tau|^2}{12(1+2\beta)(1+\beta)^2}\left[|\tau|-\frac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}\right],\quad B(\alpha,\beta;\tau)=\frac{2(1-\alpha)|\tau|}{3(1+2\beta)}.$$

Now, we must investigate the maximum of the function $G(t)$ in the interval $(0,2)$.
By simple computation, we can easily show

$$G'(t)=2A(\alpha,\beta;\tau)t.$$

It is clear that $G'(t)<0$ if $A(\alpha,\beta;\tau)<0$; that is if $|\tau|\in\left(0,\frac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}\right)$.

Thus, the function $G(t)$ is a decreasing function if $|\tau|\in(0,\tau_0)$, where $\tau_0=\frac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$. Therefore,

$$\max\{G(t):t\in(0,2)\}=G(0+)=2C_1(0)=\frac{2(1-\alpha)|\tau|}{3(1+2\beta)}. \tag{2.16}$$

Also, $G'(t)\geq 0$ if $|\tau|\geq\tau_0$; that is, the function $G(t)$ is an increasing function for $|\tau|\geq\tau_0$. Therefore,

$$\max\{G(t):t\in(0,2)\}=G(2-)=C_2(2)=\frac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}. \tag{2.17}$$

Substituting the expressions (2.8) and (2.9) in (2.7) and using triangle inequality, taking $|x|=\zeta,\ |y|=\varsigma$, we can easily obtain that

$$|a_4|\leq c_1(t)(\zeta^2+\varsigma^2)+c_2(t)(\zeta+\varsigma)+c_3(t)=\phi(\zeta,\varsigma), \tag{2.18}$$

where

$$c_1(t)=\frac{(1-\alpha)(4-t^2)(t-2)|\tau|}{32(1+3\beta)}\leq 0,$$

$$c_2(t)=\frac{(1-\alpha)(4-t^2)|\tau|t\left[5|\tau|(1-\alpha)(1+3\beta)+3(1+\beta)(1+2\beta)\right]}{48(1+\beta)(1+2\beta)(1+3\beta)}\geq 0,$$

$$c_3(t)=\frac{(1-\alpha)|\tau|}{16(1+3\beta)}t^3+\frac{(1-\alpha)|\tau|(4-t^2)}{8(1+3\beta)}\geq 0,\quad t\in[0,2].$$

Now, we need to maximize the function $\phi(\zeta,\varsigma)$ on the closed square $\Omega=\{(\zeta,\varsigma):\ \zeta,\varsigma\in[0,1]\}$.

Since the coefficients $c_1(t)$, $c_2(t)$ and $c_3(t)$ of the function $\phi(\zeta,\varsigma)$ is dependent to variable t, we must investigate the maximum of $\phi(\zeta,\varsigma)$ respect to t taking into account these cases $t=0, t=2$ and $t\in(0,2)$.

Let us $t=0$. Then, we write

$$\phi(\zeta,\varsigma)=-\frac{(1-\alpha)|\tau|}{4(1+3\beta)}(\zeta^2+\varsigma^2)+\frac{(1-\alpha)|\tau|}{2(1+3\beta)}.$$

In this case, we will examine the maximum of the function $\phi(\zeta,\varsigma)$ taking into account the sing of

$$\Delta(\zeta,\varsigma)=\phi_{\zeta\zeta}(\zeta,\varsigma)\phi_{\varsigma\varsigma}(\zeta,\varsigma)-\left[\phi_{\zeta\varsigma}(\zeta,\varsigma)\right]^2.$$

By simple computation, we can easily see that

$$\phi_\zeta'(\zeta,\varsigma)=-\frac{|\tau|(1-\alpha)}{2(1+3\beta)}\zeta,\ \phi_\varsigma'(\zeta,\varsigma)=-\frac{|\tau|(1-\alpha)}{2(1+3\beta)}\varsigma$$

and

$\phi_{\zeta\varsigma}(\zeta,\varsigma)=\phi_{\varsigma\zeta}(\zeta,\varsigma)=0$, $\phi_{\zeta\zeta}''(\zeta,\varsigma)=\phi_{\varsigma\varsigma}''(\zeta,\varsigma)=-\dfrac{|\tau|(1-\alpha)}{2(1+3\beta)},(\zeta,\varsigma)\in\Omega$.

Thus,

$$\Delta(\zeta_0,\varsigma_0)=\left(\frac{|\tau|(1-\alpha)}{4(1+3\beta)}\right)^2>0 \text{ and } \phi_{\zeta\zeta}''(\zeta_0,\varsigma_0)<0;$$

that is, (ζ_0,ς_0) is a maximum point for the function $\phi(\zeta,\varsigma)$, where $(\zeta_0,\varsigma_0)=(0,0)$. Therefore, in the case $t=0$

$$\max\left\{\phi(\zeta,\varsigma):\zeta,\varsigma\in[0,1]\right\}=\phi(0,0)=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}. \tag{2.19}$$

For $t=2$, the function $\phi(\zeta,\varsigma)$ is a constant function as follows

$$\phi(\zeta,\varsigma)=c_3(2)=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}. \tag{2.20}$$

In the case $t\in(0,2)$, we will examine the maximum of the function $\phi(\zeta,\varsigma)$ taking into account the sign of

$$\Lambda(\zeta,\varsigma)=\phi_{\zeta\zeta}(\zeta,\varsigma)\phi_{\varsigma\varsigma}(\zeta,\varsigma)-\left[\phi_{\zeta\varsigma}(\zeta,\varsigma)\right]^2.$$

By simple computation, we can easily see that

$$\phi_{\zeta}'(\zeta,\varsigma)=2c_1(t)\zeta+c_2(t),\ \phi_{\varsigma}'(\zeta,\varsigma)=2c_1(t)\varsigma+c_2(t)$$

and

$$\phi_{\zeta\varsigma}''(\zeta,\varsigma)=\phi_{\varsigma\zeta}''(\zeta,\varsigma)=0,$$

$$\phi_{\zeta\zeta}''(\zeta,\varsigma)=\phi_{\varsigma\varsigma}''(\zeta,\varsigma)=2c_1(t),(\zeta,\varsigma)\in\Omega.$$

Thus, $(\zeta_0,\varsigma_0)=\left(\dfrac{-c_2(t)}{2c_1(t)},\dfrac{-c_2(t)}{2c_1(t)}\right)$ is a critical point of the function $\phi(\zeta,\varsigma)$ if $(\zeta_0,\varsigma_0)\in\Omega$. We assume that $(\zeta_0,\varsigma_0)\in\Omega$. Since

$$\Delta(\zeta_0,\varsigma_0)=4c_1^2(t)>0 \text{ and } \phi_{\zeta\zeta}''(\zeta_0,\varsigma_0)=2c_1(t)<0,$$

(ζ_0,ς_0) is a maximum point for the function $\phi(\zeta,\varsigma)$. Therefore,

$$\max\left\{\phi(\zeta,\varsigma):(\zeta,\varsigma)\in\Omega\right\}=\phi(\zeta_0,\varsigma_0)=c_3(t)-\frac{c_2^2(t)}{2c_1(t)}$$

for all $t\in(0,2)$.

Hence, we can write

$$|a_4|\le\inf\left\{c_3(t)-\frac{c_2^2(t)}{2c_1(t)}:t\in(0,2)\right\}. \tag{2.21}$$

Now, we must investigate the infimum of the function $c_3(t)-\dfrac{c_2^2(t)}{2c_1(t)}$ in the interval $(0,2)$.

Since

$$\inf\left\{c_3(t):t\in(0,2)\right\}=\frac{25(1-\alpha)|\tau|}{54(1+3\beta)},\quad \inf\left\{c_2(t):t\in(0,2)\right\}=0 \text{ and}$$

$$\sup\left\{-c_1(t):t\in(0,2)\right\}=-\inf\left\{c_1(t):t\in(0,2)\right\}=\frac{|\tau|(1-\alpha)}{4(1+3\beta)},$$

$$\inf\left\{c_3(t)-\frac{c_2^2(t)}{2c_1(t)}:t\in(0,2)\right\}\leq\frac{25(1-\alpha)|\tau|}{54(1+3\beta)}. \quad (2.22)$$

Therefore, from (2.19), (2.20) and (2.21), (2.22), we have

$$|a_4|\leq\max\left\{\frac{(1-\alpha)|\tau|}{2(1+3\beta)},\frac{25(1-\alpha)|\tau|}{54(1+3\beta)}\right\}=\frac{(1-\alpha)|\tau|}{2(1+3\beta)}. \quad (2.23)$$

Thus, from (2.10) - (2.14), (2.16), (2.17) and (2.23) the proof of Theorem 2.1 is completed.
In the special cases from Theorem 2.1, we arrive at the following results.

2.2. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,\beta,1)=H_\Sigma(\alpha,\beta)$, $\alpha\in[0,1),\ \beta\in[0,1]$. *Then,*

$$|a_2|\leq\frac{1-\alpha}{1+\beta},$$

$$|a_3|\leq\begin{cases}\dfrac{(1-\alpha)^2}{(1+\beta)^2}, & \text{if } \alpha\in[0,\alpha_0],\\[2mm] \dfrac{2(1-\alpha)}{3(1+2\beta)}, & \text{if } \alpha\in(\alpha_0,1),\end{cases}$$

where $\alpha_0=1-\dfrac{2(1+\beta)^2}{3(1+2\beta)}$ *and*

$$|a_4|\leq\frac{1-\alpha}{2(1+3\beta)}.$$

2.3. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,0,\tau)$, $\alpha\in[0,1),\ \tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. *Then,*

$$|a_2|\leq|\tau|(1-\alpha),$$

$$|a_3|\leq\begin{cases}\dfrac{2|\tau|(1-\alpha)}{3}, & \text{if } |\tau|\in(0,\tau_0),\\[2mm] |\tau|^2(1-\alpha)^2, & \text{if } |\tau|\in[\tau_0,+\infty),\end{cases}$$

where $\tau_0=\dfrac{2}{3(1-\alpha)}$ *and*

$$|a_4|\leq\frac{(1-\alpha)|\tau|}{2}.$$

2.4. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,0,1)=\Re_\Sigma(\alpha,0)=N_\Sigma(\alpha),\ \alpha\in[0,1)$. *Then,*

$$|a_2|\leq1-\alpha,$$

$$|a_3|\leq\begin{cases}(1-\alpha)^2, & \text{if } \alpha\in\left[0,\dfrac{1}{3}\right],\\[2mm] \dfrac{2(1-\alpha)}{3}, & \text{if } \alpha\in\left(\dfrac{1}{3},1\right),\end{cases}$$

and

$$|a_4|\leq\frac{1-\alpha}{2}.$$

2.5. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_{\Sigma}(\alpha,1,\tau)$, $\alpha\in[0,1)$, $\tau\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$. *Then,*

$$|a_2|\le\frac{|\tau|(1-\alpha)}{2},$$

$$|a_3|\le\begin{cases}\dfrac{2|\tau|(1-\alpha)}{9}, & \text{if } |\tau|\in(0,\tau_0),\\[2mm] \dfrac{|\tau|^2(1-\alpha)^2}{4}, & \text{if } |\tau|\in[\tau_0,+\infty),\end{cases}$$

where $\tau_0=\dfrac{8}{9(1-\alpha)}$ *and*

$$|a_4|\le\frac{(1-\alpha)|\tau|}{8}.$$

2.6. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_{\Sigma}(\alpha,1,1)=\Re_{\Sigma}(\alpha,1)$, $\alpha\in[0,1)$. *Then,*

$$|a_2|\le\frac{1-\alpha}{2},$$

$$|a_3|\le\begin{cases}\dfrac{(1-\alpha)^2}{4}, & \text{if } \alpha\in\left[0,\dfrac{1}{9}\right],\\[2mm] \dfrac{2(1-\alpha)}{9}, & \text{if } \alpha\in\left(\dfrac{1}{9},1\right)\end{cases}$$

and

$$|a_4|\le\frac{1-\alpha}{8}.$$

3. Fekete-Szegö Problem for the Function Class $\Im_{\Sigma}(\alpha,\beta,\tau)$

In this section, we will prove the following theorem on the Fekete-Szegö inequality of the function class $\Im_{\Sigma}(\alpha,\beta,\tau)$.

3.1. Theorem

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_{\Sigma}(\alpha,\beta,\tau)$, $\alpha\in[0,1)$, $\beta\in[0,1]$, $\tau\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$ *and* $\mu\in\mathbb{C}$. *Then,*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{2|\tau|(1-\alpha)}{3(1+2\beta)}, & \text{if } |1-\mu|\in[0,\mu_0),\\[2mm] |1-\mu|\dfrac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2}, & \text{if } |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\mu_0=\dfrac{2(1+\beta)^2}{3|\tau|(1-\alpha)(1+2\beta)}$

Proof. Let $f\in\Im_{\Sigma}(\alpha,\beta,\tau)$, $\alpha\in[0,1)$, $\beta\in[0,1]$, $\tau\in\mathbb{C}^{*}=\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$.
From (2.5) and (2.6), we find that

$$a_3-\mu a_2^2=(1-\mu)a_2^2+\frac{\tau(1-\alpha)}{6(1+2\beta)}(p_2-q_2). \tag{3.1}$$

Substituting the expression (2.8) in (3.1) and using triangle inequality, taking $|x|=\theta,\ |y|=\vartheta$, we can easily obtain that

$$\left|a_3-\mu a_2^2\right|\le d_1(t)+d_2(t)\left(\theta+\vartheta\right)=\psi\left(\theta,\vartheta\right), \tag{3.2}$$

where

$$d_1(t)=|1-\mu|\frac{|\tau|^2(1-\alpha)^2}{4\left(1+\beta\right)^2}t^2\ge 0 \quad \text{and} \quad d_2(t)=\frac{|\tau|(1-\alpha)\left(4-t^2\right)}{12\left(1+2\beta\right)}\ge 0.$$

Now, we need to maximize the function $\psi\left(\theta,\vartheta\right)$ on the closed square $\Omega=\left\{\left(\theta,\vartheta\right):\ \theta,\vartheta\in\left[0,1\right]\right\}$. Since the coefficients $d_1(t)$ and $d_2(t)$ of the function $\psi\left(\theta,\vartheta\right)$ is dependent to variable t, we must investigate the maximum of $\psi\left(\theta,\vartheta\right)$ respect to t taking into account these cases $t=0,\ t=2$ and $t\in\left(0,2\right)$.

Let $t=0$. Then,

$$\psi(\theta,\vartheta)=d_2(0)\left(\theta+\vartheta\right)=\frac{|\tau|(1-\alpha)}{3\left(1+2\beta\right)}\left(\theta+\vartheta\right).$$

It is clear that the maximum of the function $\psi(\theta,\vartheta)$ occurs at $\left(\theta,\vartheta\right)=(1,1)$.

Therefore, in the case $t=0$

$$\max\left\{\psi(\theta,\vartheta):\theta,\vartheta\in\left[0,1\right]\right\}=\psi(1,1)=\frac{2|\tau|(1-\alpha)}{3\left(1+2\beta\right)}. \tag{3.3}$$

Now, let $t=2$. In this case, $\psi\left(\theta,\vartheta\right)$ is a constant function as follows

$$\psi\left(\theta,\vartheta\right)=c_1(t,2)=|1-\mu|\frac{|\tau|^2(1-\alpha)^2}{\left(1+\beta\right)^2}. \tag{3.4}$$

In the case $t\in\left(0,2\right)$, we can easily see that

$$\max\left\{\psi\left(\theta,\vartheta\right):\theta,\vartheta\in\left[0,1\right]\right\}=\psi\left(1,1\right)=d_1(t)+2d_2(t) \tag{3.5}$$

for all $t\in\left(0,2\right)$.

Let us define the function $H:\left(0,2\right)\to□$ as follows

$$H(t)=d_1(t)+2d_2(t) \tag{3.6}$$

for fixed $\tau\in□^*=□-\left\{0\right\}$.

Substituting the value $d_1(t)$ and $d_2(t)$ in (3.6), we obtain

$$H(t)=C\left(\alpha,\beta,\mu;\tau\right)t^2+D\left(\alpha,\beta,\mu\right),$$

where

$$C\left(\alpha,\beta,\mu;\tau\right)=\frac{(1-\alpha)^2|\tau|^2}{4\left(1+\beta\right)^2}\left[|1-\mu|-\frac{2\left(1+\beta\right)^2}{3\alpha|\tau|\left(1+2\beta\right)}\right],\quad D\left(\alpha,\beta;\tau\right)=\frac{2|\tau|(1-\alpha)}{3\left(1+2\beta\right)}$$

Now, we must investigate the maximum of the function $H(t)$ in the interval $\left(0,2\right)$.

By simple computation, we can easily show that

$$H'(t)=2C\left(\alpha,\beta,\mu;\tau\right)t.$$

We will examine the sign of the function $H'(t)$ depending on the different cases of the sign of $C\left(\alpha,\beta,\mu;\tau\right)$ as follows.

(ι) Let us $C\left(\alpha,\beta,\mu;\tau\right)\ge 0$. Then $H'(t)\ge 0$, so $H(t)$ is an increasing function. Therefore,

$$\max\left\{H(t):t\in(0,2)\right\}=H(2-)=d_1(2)=|1-\mu|\frac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2}. \quad (3.7)$$

(ii) Let us $C(\alpha,\beta,\mu;\tau)<0$. Then $H'(t)<0$; that is, $H(t)$ is a decreasing function. Therefore,

$$\max\left\{H(t):t\in(0,2)\right\}=H(0+)=2d_2(0)=\frac{2|\tau|(1-\alpha)}{3(1+2\beta)}. \quad (3.8)$$

From (3.7) and (3.8), we conclude that

$$\max\left\{H(t):t\in(0,2)\right\}=|1-\mu|\frac{|\tau|^2(1-\alpha)^2}{(1+\beta)^2} \quad (3.9)$$

if $|1-\mu|\geq\mu_0$ and

$$\max\left\{H(t):t\in(0,2)\right\}=\frac{2|\tau|(1-\alpha)}{3(1+2\beta)} \quad (3.10)$$

if $|1-\mu|<\mu_0$, where $\mu_0=\dfrac{2(1+\beta)^2}{3|\tau|(1-\alpha)(1+2\beta)}$

Thus, from (3.3), (3.4) and (3.9), (3.10), the proof of Theorem 3.1 is completed.
In the special cases from Theorem 2.1, we arrive at the following results.

3.2. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,0,\tau)$, $\alpha\in[0,1)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ *and* $\mu\in\mathbb{C}$*. Then,*

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{2|\tau|(1-\alpha)}{3}, & \text{if } |1-\mu|\in[0,\mu_0),\\ |1-\mu||\tau|^2(1-\alpha)^2, & \text{if } |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\dfrac{2}{3|\tau|(1-\alpha)}$

3.3. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,1,\tau)$, $\alpha\in[0,1)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ *and* $\mu\in\mathbb{C}$*. Then,*

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{2(1-\alpha)|\tau|}{9}, & \text{if } |1-\mu|\in[0,\mu_0),\\ |1-\mu|\dfrac{(1-\alpha)^2|\tau|^2}{4}, & \text{if } |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\mu_0=\dfrac{8}{9|\tau|(1-\alpha)}$

Taking $\mu=0$ and $\mu=1$ in Theorem 3.1, we can easily arrive at the following result.

3.4. Corollary

Let the function $f(z)$ *given by (1.1) be in the class* $\Im_\Sigma(\alpha,\beta,\tau)$, $\alpha\in[0,1)$, $\beta\in[0,1]$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. *Then,*

$$|a_3| \le \begin{cases} \dfrac{2(1-\alpha)|\tau|}{3(1+2\beta)}, & \text{if } |\tau| \in (0, \tau_0), \\ \dfrac{(1-\alpha)^2|\tau|^2}{(1+\beta)^2}, & \text{if } |\tau| \in [\tau_0, +\infty), \end{cases}$$

where $\tau_0 = \dfrac{2(1+\beta)^2}{3(1-\alpha)(1+2\beta)}$ *and*

$$|a_3 - a_2^2| \le \frac{2|\tau|(1-\alpha)}{3(1+2\beta)}$$

3.4.1. Note

The first result of Corollary 3.3 confirms the second inequality of Theorem 2.1.

3.4.2. Remark

Numerous consequences of the results obtained in the Corollary 3.1, 3.2 and 3.3 can indeed be deduced by specializing the various parameters involved.

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