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Method of Averaging for Some Parabolic Partial Differential Equations

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Abstract

Quantitative and qualitative analysis of the Averaging methods for the parabolic partial differential equation appears as an exciting field of the investigation. In this paper, we generalize some known results due to Krol on the averaging methods and use them to solve the parabolic partial differential equation.

Keywords: Averaging; Averaging method; Partial differential equation; Parabolic partial differential equation.

1. Introduction

The investigation in the field of the qualitative and quantitative analysis of the Averaging methods for the parabolic partial differential equation is more exciting field to be studied. We study the parabolic partial differential equation in this paper using the technique of the averaging method of the linear operator. In section 2, we study the averaging of the linear operator where we generalize some known results due to Krol [1]. We consider the following parabolic partial differential equation in the form:

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon L(x,t,D)u(x,t), \qquad (1)$$

$$u(x,0) = \varphi(x), \qquad (2)$$

where

$$L(x,t,D) = \sum_{|q|\le 2m} a_q(x,t)D^q, \qquad (3)$$

$\varepsilon > 0, x \in \mathcal{R}^n, \mathcal{R}^n$ is the $n-$dimensional Euclidean space, $q = (q_1 \dots q_n)$ is an $n-$dimensional multi index, $|q| = q_1 + \cdots + q_n, D^q = D^{q_1} \dots D^{q_n}, D_j = \frac{\partial}{\partial x_j}$. The coefficients $a_q(x,t)$ and $\varphi(x)$ are bounded continuous with bounded derivatives and $D^q u(x,t)$ are bounded on $x \in \mathcal{R}^n, 0 \le t \le T$. In section 3, we discuss a special case for the problem (1), (2). Compare [2-11].

2. Averaging a Linear Operator

By averaging the coefficients $a_q(x,t)$ over t, we can average the operator L

$$\bar{a}_q(x) = \frac{1}{T}\int_0^T a_q(x,t)D^q dt, \qquad (4)$$

for all $(x,t), x \in \mathcal{R}^n$ producing the averaged operator $\bar{L}(x,D)$ and all the coefficients $\bar{a}_q(x), |q| \le 2m$ are bounded continuous with bounded derivatives on $\mathcal{R}^n$.

$$\bar{L}(x,D) = \frac{1}{T}\int_0^T \sum_{|q|\le 2m} a_q(x,t)D^q \, dt,$$

like as an approximating problem for (1), (2), we take

$$\frac{\partial u^*(x,t)}{\partial t} = \varepsilon \bar{L}(x,D)u^*(x,t), \qquad (5)$$

$$u(x,0) = \varphi(x), \qquad (6)$$

another straightforward analysis display the existence and uniqueness of the solutions of problems (1), (2) and (5), (6) on the time-scale $\frac{1}{\varepsilon}$.

We consider the domain $A = \mathcal{R}^n \times [0,T]$. The norm $\|.\|_\infty$ is defined by the supremum norm on A and denoted by

$$\|u(x,t)\|_\infty = \sup_A |u(x,t)|.$$

2.1. Theorem

Let $u(x,t)$ be the solution of the initial value problem (1), (2) and $u^*(x,t)$ be the solution of the initial value problem (5), (6), then we have the estimate $\|u(x,t)-u^*(x,t)\|_\infty = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$.

Proof. We consider the near-identity transformation:

$$\hat{u}(x,t) = u^*(x,t) + \varepsilon \int_0^t \left(L(x,s,D) - \bar{L}(x,D)\right) ds\, u^*(x,t), \tag{7}$$

we have

$$\|\hat{u}(x,t) - u^*(x,t)\|_\infty = O(\varepsilon) \text{ on the time-scale } \frac{1}{\varepsilon}.$$

Differentiation of the near-identity transformation (7) and using the equations (5), (7), we get

$$\frac{\partial \hat{u}(x,t)}{\partial t} = \frac{\partial u^*(x,t)}{\partial t} + \varepsilon\left(L(x,t,D) - \bar{L}(x,D)\right)u^*(x,t) + \varepsilon \int_0^t \left(L(x,s,D) - \bar{L}(x,D)\right) ds \frac{\partial u^*(x,t)}{\partial t}$$

$$= \varepsilon L(x,t,D)\hat{u}(x,t) - \varepsilon^2 L(x,t,D) \int_0^t \left(L(x,s,D) - \bar{L}(x,D)\right) ds\, u^*(x,t)$$

$$+\varepsilon^2 \bar{L}(x,D) \int_0^t \left(L(x,s,D) - \bar{L}(x,D)\right) ds\, u^*(x,t)$$

$$= \varepsilon L(x,t,D)\hat{u}(x,t)$$

$$+\varepsilon^2 \int_0^t \left[\left(L(x,s,D) - \bar{L}(x,D)\right)\bar{L}(x,D) - L(x,t,D)(L(x,s,D) - \bar{L}(x,D))\right] ds\, u^*(x,t)$$

$$= \varepsilon L(x,t,D)\hat{u}(x,t) + \varepsilon^2 \mathcal{M}(x,t,D)\, u^*(x,t),$$

with initial value $\hat{u}(x,0) = \varphi(x)$, where

$$\mathcal{M}(x,t,D) = \int_0^t \left[\left(L(x,s,D) - \bar{L}(x,D)\right)\bar{L}(x,D) - L(x,t,D)(L(x,s,D) - \bar{L}(x,D))\right] ds.$$

Let

$$\frac{\partial}{\partial t} - \varepsilon L(x,t,D) = \mathcal{L},$$

we obtain

$$\mathcal{L}\, \hat{u}(x,t) = \varepsilon^2 \mathcal{M}(x,t,D)\, u^*(x,t),$$

$\mathcal{L}\, u^*(x,t) = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$,

then

$\mathcal{L}\,(\hat{u}(x,t) - u^*(x,t)) = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$.

Moreover $\hat{u}(x,0) - u^*(x,0) = 0$. To end the proof we use barrier functions see [12].

Let $c = \|\mathcal{M}(x,t,D)\, u^*(x,t)\|_\infty$, we introduce the barrier function

$$B(x,t) = \varepsilon^2 c\, t,$$

and the functions (we omit the arguments)

$$Z_1(x,t) = \hat{u}(x,t) - u(x,t) - B(x,t), Z_2(x,t) = \hat{u}(x,t) - u(x,t) + B(x,t).$$

We get

$$\mathcal{L}\, Z_1(x,t) = \left(\frac{\partial}{\partial t} - \varepsilon L(x,t,D)\right)[\hat{u}(x,t) - u(x,t) - B(x,t)]$$

$$= \varepsilon^2 \mathcal{M}(x,t,D)\, u^*(x,t) - \varepsilon^2 \|\mathcal{M}(x,t,D)\, u^*(x,t)\|_\infty + \varepsilon^3 c\, t\, L(x,t,D)$$

$$\leq 0,$$

$Z_1(x,0) = 0$ and similarly $\mathcal{L}\, Z_2(x,t) \geq 0$, $Z_2(x,0) = 0$.

$Z_1(x,t)$ and $Z_2(x,t)$ are bounded, resulting in $Z_1(x,t) \leq 0$ and $Z_2(x,t) \geq 0$. It follows that

$$-B(x,t) \leq \hat{u}(x,t) - u(x,t) \leq B(x,t),$$

$$-\varepsilon^2 c\, t \leq \hat{u}(x,t) - u(x,t) \leq \varepsilon^2 c\, t,$$

so we can estimate

$$\|\hat{u}(x,t) - u(x,t)\|_\infty \leq \|B(x,t)\|_\infty = O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. We can use the triangle inequality to have

$$\|u(x,t) - u^*(x,t)\|_\infty \leq \|\hat{u}(x,t) - u^*(x,t)\|_\infty + \|\hat{u}(x,t) - u(x,t)\|_\infty$$

$= O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$. ■

3. A Special Case

Consider the partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon L_1(D)u(x,t), \tag{8}$$

$$u(x,0) = \varphi(x), \tag{9}$$

where

$$L_1(D) = (D_1^2 + \cdots + D_n^2)^{2N+1}, \tag{10}$$

N is a sufficiently large positive integer.

Let $C_b(\mathcal{R}^n)$ is the set of all bounded continuous functions on $\mathcal{R}^n$.

Consider the following Cauchy problem [6]:

$$\frac{\partial u(x,t)}{\partial t} = (D_1^2 + \cdots + D_n^2)^{2N+1} u(x,t), \tag{11}$$

$$u(x,0) = \varphi(x) \in C_b(\mathcal{R}^n). \tag{12}$$

The solution of the Cauchy problem (11), (12) is given by:

$$u(x,t) = \int_{\mathcal{R}^n} G(x-y,t)\varphi(y)dy,$$

where the function G is the fundamental solution of the Cauchy problem (11), (12) and $dy = dy_1 \dots dy_n$.

For sufficiently large N, we find $\gamma \in (0,1)$ and a constant $M > 0$ such that:

$$\max_x |D^q u(x,t)| \le \frac{M}{t^\gamma} \max_x |\varphi(x)|,$$

for all $t > 0, |q| \le m$. Like as an approximating problems for (8), (9), we take

$$\frac{\partial u^*(x,t)}{\partial t} = \varepsilon \bar{L}_1(D) u^*(x,t), \tag{13}$$

$$u^*(x,0) = \varphi(x), \tag{14}$$

where

$$\bar{L}_1(D) = L_1(D),$$

another straightforward analysis displays the existence and uniqueness of the solutions of problems (8), (9) and (13), (14) on the time-scale $\frac{1}{\varepsilon}$.

3.1. Theorem

Let $u(x,t)$ be the solution of the initial value problem (8), (9) and $u^*(x,t)$ be the solution of the initial value problem (13), (14), then we have the estimate $\|u(x,t) - u^*(x,t)\|_\infty = 0$.

Proof. By using the near-identity transformation (7), we have

$$\|\hat{u}(x,t) - u^*(x,t)\|_\infty = 0.$$

Differentiation of the near-identity transformation (7) and using the equations (7), (13), we have

$$\frac{\partial \hat{u}(x,t)}{\partial t} = \frac{\partial u^*(x,t)}{\partial t} = \varepsilon L(x,t,D)\hat{u}(x,t),$$

with initial value $\hat{u}(x,0) = \varphi(x)$,

Let

$$\frac{\partial}{\partial t} - \varepsilon L_1(D) = \mathcal{L}_1,$$

we get

$$\mathcal{L}_1\, \hat{u}(x,t) = 0,$$

$\mathcal{L}_1\, (\hat{u}(x,t) - u^*(x,t)) = 0.$

Moreover $\hat{u}(x,0) - u^*(x,0) = 0$.We introduce the barrier function

$$B_1(x,t) = 0,$$

and the functions (we omit the arguments)

$Z_3(x,t) = \hat{u}(x,t) - u(x,t) - B_1(x,t), Z_4(x,t) = \hat{u}(x,t) - u(x,t) + B_1(x,t).$

We get

$$\mathcal{L}_1\, Z_3(x,t) = \left(\frac{\partial}{\partial t} - \varepsilon L_1(D)\right)[\hat{u}(x,t) - u(x,t) - B_1(x,t)] = 0,$$

$Z_3(x,0) = 0$ and similarly $\mathcal{L}_1\, Z_4(x,t) = 0$, $Z_4(x,0) = 0$.

$Z_3(x,t)$ and $Z_4(x,t)$ are bounded, resulting in $Z_3(x,t) = 0$ and $Z_4(x,t) = 0$. It follows that

$$\hat{u}(x,t) - u(x,t) = 0,$$

so

$\|\hat{u}(x,t) - \mathrm{u}(x,t)\|_\infty = 0.$

We can use the triangle inequality to obtain

$$\|u(x,t) - u^*(x,t)\|_\infty = 0. \blacksquare$$

4. Conclusion

This paper is focused on generalizing some known results due to Krol on the averaging methods to solve the parabolic partial differential equation. As a special case Cauchy problem is solved for the parabolic partial differential equation.

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