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A Generalized Transmuted Moment Exponential Distribution: Properties and Application

Amal S. Hassan
Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt

Ibrahim B. Abdul-Moniem (Corresponding Author)
Department of Statistics, Higher Institute of Management Sciences in Sohag, Sohag, Egypt
Email: ibtaib@hotmail.com

Khater A. E. Gad
Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt

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Abstract

This paper introduces a new generalization of moment exponential (or length biased) distribution. The new model is referred to as generalized transmuted moment exponential distribution. This model contains some new existing distributions. Structural properties of the suggested distribution including closed forms for ordinary and incomplete moments, quantile and generating functions and Rényi entropy are derived. Maximum likelihood estimation is employed to obtain the parameter estimators of the new distribution. We illustrate the importance of the new model by means of three applications to real data sets.

Keywords: Transmuted distribution; Generalized transmuted-G; The moment exponential; Moments; Order statistics; Maximum likelihood.

1. Introduction

In recent times, many generators have been proposed by extending some useful classical distributions. Such generated families of distributions have been extensively used for modeling and analyzing lifetime data in many applied sciences such as reliability, engineering, actuarial sciences, demography, economics, hydrology, biological studies, insurance, medicine and finance, among others. However, there still remain many real world phenomena involving data, which do not follow any of the classical statistical distributions.

A class of distributions called transmuted distributions has been provided by Shaw and Buckley [1]. A random variable X is said to have a transmuted distribution, if its cumulative distribution function (cdf) is given by

$$F_T(x)=G(x)\left[1+\lambda-\lambda G(x)\right],\ |\lambda|\le 1, \qquad (1)$$

The probability density function (pdf) corresponding to (1) is given by

$$f_T(x)=\left[1+\lambda-2\lambda G(x)\right]g(x), \qquad (2)$$

where $g(x)$ and $G(x)$ are the pdf and cdf of base distribution. A more general form for (1) with two extra shape parameters, called generalized transmuted (GT) distribution, has been introduced by Nofal, *et al.* [2]. The cdf of GT distribution is defined by

$$F_{GT}(x)=\left[G(x)\right]^a\left\{1+\lambda-\lambda\left[G(x)\right]^b\right\},\qquad a,b>0. \qquad (3)$$

The pdf corresponding to (3) is

$$f_{GT}(x)=g(x)\left[G(x)\right]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[G(x)\right]^b\right\}. \qquad (4)$$

The exponential distribution is an important statistical model and widely applied in several fields [3]. Due to its benefit, various generalizations and extensions of the exponential distribution are available in the literature such as; exponentiated exponential [4], beta exponential [5], beta generalized exponential distribution [6], moment exponential [7], exponentiated moment exponential [8], generalized exponentiated moment exponential [9], Marshall-Olkin generalized exponential [10], Marshall-Olkin length-biased exponential [11], exponential Slashed moment exponential [12], alpha power transformed extended exponential [13] exponentiated length biased exponential [14] and Kumerswmay moment exponential [15], Weibull moment exponential [16] among others.

The moment exponential (ME) (or length biased) distribution was proposed by Dara and Ahmad [7] and discussed hazard and reversed hazard rate functions with the next pdf:

$$g(x) = \beta^2 x e^{-\beta x}; \qquad x \geq 0 \text{ and } \beta > 0 \tag{5}$$

The cdf corresponding to (5) is

$$G(x) = 1 - (1+\beta x) e^{-\beta x}; \qquad x \geq 0 \text{ and } \beta > 0 \tag{6}$$

Properties, extensions and applications of the ME distribution mentioned in the context of reliability analysis have been discussed by Dara and Ahmad [7].

In this article we offer a new generalization of the ME distribution called the generalized transmuted moment exponential (GTME) distribution. The fundamental motivation of this generalization is

i. Providing highly flexible life distribution which contains as sub models; some new existing distributions,

ii. To permit different degrees of kurtosis and asymmetry and,

iii. To provide significant improvement in data modeling.

This article is orgnizied as follows. We define the new distribution in Section 2. Section 3 contains some structrual properties of the new distribution. In Section 4, maximum likelihood estimators are derived and numerical study is given. Application to real data is provided in Section 5 and the article ends with a conclusion.

2. Generalized Transmuted Moment Exponential Distribution

Here, we present the GTME distribution and its sub-models. So, a random variable X is said to have GTME distribution with vector parameters ϖ where, $\varpi = (a, b, \beta, \lambda)$ if its cdf is defined by substituting (5), (6) in (3) as follows

$$F(x;\varpi) = \left[1-(1+\beta x)e^{-\beta x}\right]^a \left\{1+\lambda-\lambda\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}, \quad x > 0, \tag{7}$$

and its pdf is as follows

$$f(x;\varpi) = \beta^2 x e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}, \ x > 0. \tag{8}$$

Hence, we denote a random variable having the pdf (8) by $X \sim$ GTME (a, b, β, λ). Special sub-models of the GTME distribution are recorded in Table 1.

Table-1. Sub-models of the GTME distribution

No.	Distribution	β	a	b	λ	Author
1	TME	β	1	1	λ	New
2	EME(a+b)	β	a	b	1-	[8]
3	EME(a)	β	a	b	0	[8]
4	ME	β	1	1	0	[9]

The survival function (SF), and hazard rate function (HR) are, respectively, given by

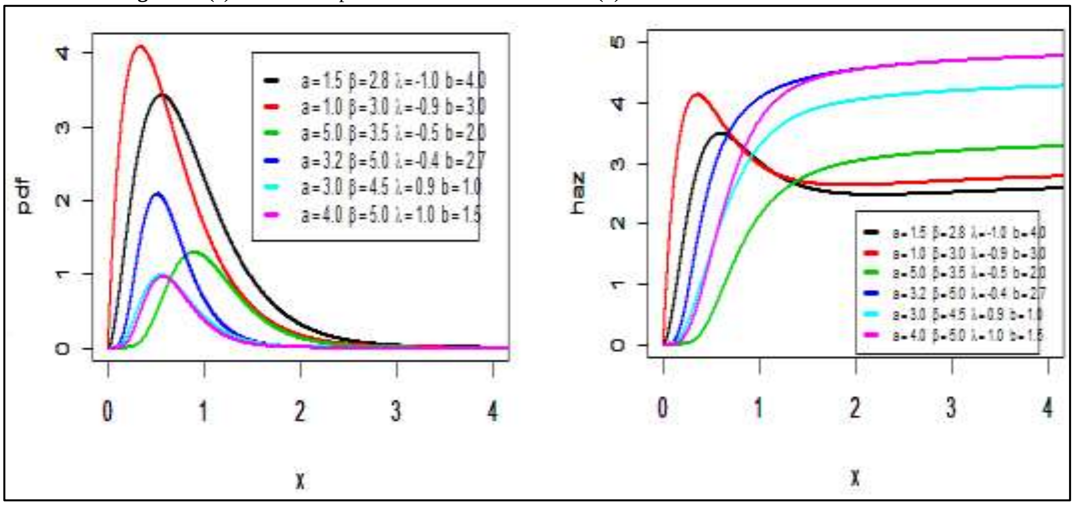
$$\bar{F}(x;\varpi) = 1 - \left[1-(1+\beta x)e^{-\beta x}\right]^a \left\{1+\lambda-\lambda\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\},$$

and,

$$h(x;\varpi) = \frac{\beta^2 x e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}}{1-\left[1-(1+\beta x)e^{-\beta x}\right]^a\left\{1+\lambda-\lambda\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}}.$$

Some descriptive pdf and hrf plots of GTME model are illustrated below for specific parameter choices of ϖ (see Figure 1).

Figure-1. (a) Plots of the pdf of the GTME distribution (b) Plots of the hrf the GTME distribution



(a) (b)

From Figure 1 (a), we conclude that pdf of GTME distribution can be uni-model and right skewed. Also, the hrf of GTME distribution can be increasing, decreasing and up-side down as seen from Figure 1 (b).

Lemma 1. The limit of the GTME density function is given by

$$\lim_{x\to 0} f\,(x;\varpi)=0$$

$$\lim_{x\to \infty} f\,(x;\varpi)=0$$

$$\lim_{x\to \frac{1}{\beta}} f\,(x;\varpi)=\beta e^{-1}\left(1-2e^{-1}\right)^{a-1}\left[a(1+\lambda)-\lambda(a+b)\left(1-2e^{-1}\right)^{b}\right]$$

Proof. It is easy to demonstrate the result from the density function (8).

Additionally, the limit of the GTME hazard function as $x\rightarrow$zero is zero and $x\rightarrow\infty$ is ∞ as seen below

$$\lim_{x\to 0} h(x;\varpi)=0$$

$$\lim_{x\to \infty} h(x;\varpi)=\infty$$

It is straightforward to prove this result.

3. Statistical Properties

The statistical properties of the GTME distribution including moments, quantile function, incomplete moments, mode and Rényi entropy are discussed in the following sub-sections.

3.1. Moments

In this subsection, the r^{th} moment about zero of X is derived. From (8), we can write

$$\mu_r'=a\beta^2(1+\lambda)\int_0^\infty x^{r+1}e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a-1}dx-\beta^2\lambda(a+b)\int_0^\infty x^{r+1}e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a+b-1}dx$$

$$=I_1-I_2.$$

First, to obtain I_1, we employ the binomial expansion, hence,

$$I_1=a(1+\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a-1}{i}\binom{i}{j}(-1)^i\beta^{j+2}\int_0^\infty x^{r+j+1}e^{-\beta(i+1)x}dx.$$

So, I_1 is given by

$$I_1=a(1+\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a-1}{i}\binom{i}{j}\frac{(-1)^i\beta^{-r}\Gamma(r+j+2)}{(i+1)^{r+j+2}}.$$

By similar way I_2 is as follows

$$I_2 == \lambda(a+b)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a+b-1}{i}\binom{i}{j}\frac{(-1)^i \beta^{-r}\Gamma(r+j+2)}{(i+1)^{r+j+2}}.$$

Then, μ_r' can be written as

$$\mu_r' = \frac{A(i,j)}{(i+1)^{r+j+2}}\beta^{-r}\Gamma(r+j+2),$$

$$A(i,j)=\left[a(1+\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{a-1}{i}\binom{i}{j}-\lambda(a+b)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{a+b-1}{i}\binom{i}{j}\right] \quad (9)$$

In particular, the mean and variance are as follows

$$\mu_1' = \frac{A(i,j)}{(i+1)^{j+3}\beta}\Gamma(j+3),$$

Then the variance is

$$\sigma^2 = \frac{A(i,j)}{(i+1)^{j+4}\beta^2}\Gamma(j+4)-\left[\frac{A(i,j)}{(i+1)^{j+3}\beta}\Gamma(j+3)\right]^2$$

The coefficient of skewness (Sk) and coefficient of kurtosis (Ku) are obtained by using the well-known relationships. Table 2 contains values of mean (μ_1'),variance (σ^2), SK, and Ku of GTME distribution for certain values of parameters.

Table-2. μ_1', σ^2, Sk and Ku of GTME distribution

λ		$a=2$, $b=1$, $\beta=6$	$a=3$, $b=2$, $\beta=4$	$a=4$, $b=3$, $\beta=2$
-1	μ_1'	0.535	0.952	2.102
	σ^2	0.062	0.141	0.562
	Sk	1.139	1.088	1.069
	Ku	5.168	5.056	5.024
-0.5	μ_1'	0.497	0.878	1.938
	σ^2	0.063	0.146	0.589
	Sk	1.136	1.052	1.014
	Ku	5.140	4.938	4.855
0.5	μ_1'	0.420	0.729	1.609
	σ^2	0.056	0.123	0.482
	Sk	1.326	1.286	1.262
	Ku	5.811	5.780	5.769
1	μ_1'	0.381	0.654	1.445
	σ^2	0.048	0.094	0.348
	Sk	1.465	1.406	1.329
	Ku	6.557	6.669	6.564

From Table 2, we conclude that, as the values of a , b increase and β decrease then the values of μ_1' and σ^2 are increasing, whereas, the values of SK and Ku are decreasing. As the values of λ increase then the values μ_1' and σ^2 are decreasing, whereas, the values of SK and Ku are increasing. Also, we conclude that the distribution is skewed to right and leptokurtic.

3.2. Incomplete Moments

The s^{th} incomplete moment of *X*, denoted by $\varphi_s(t)$, is given by

$$\varphi_s = a\beta^2(1+\lambda)\int_0^t x^{s+1}e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a-1}dx - \beta^2\lambda(a+b)\int_0^\infty x^{s+1}e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a+b-1}dx$$
$$= J_1 - J_2,$$

where, J_1 and J_2 are obtained as follows

$$J_1 = a\beta^2(1+\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a-1}{i}\binom{i}{j}(-1)^i\beta^j\int_0^t x^{s+j+1}e^{-\beta(i+1)x}dx,$$

which leads to

$$J_1 = a(1+\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a-1}{i}\binom{i}{j}\frac{(-1)^i\beta^{-s}\gamma\left(s+j+2,\beta(i+1)t\right)}{(i+1)^{s+j+2}}.$$

where, $\gamma\left(s+j+2,\beta(i+1)t\right)$ is the lower incomplete gamma function. By similar way, J_2 is obtained as follows

$$J_2 = \lambda(a+b)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a+b-1}{i}\binom{i}{j}\frac{(-1)^i\beta^{-s}\gamma\left(s+j+2,\beta(i+1)t\right)}{(i+1)^{s+j+2}}.$$

Then, $\varphi_s(t)$ of GTME distribution is given by

$$\varphi_s(t) = A(i,j)\frac{(-1)^i\beta^{-s}\gamma\left(s+j+2,\beta(i+1)t\right)}{(i+1)^{s+j+2}}, \tag{10}$$

where $\gamma\left(s+j+2,\beta(i+1)t\right)$ and $A(i,j)$ as given above. The first incomplete moment of the GTME model, $\varphi_1(t)$, can be obtained by setting $s=1$ in (10).

Another application of the first incomplete moment is related to mean residual life and mean waiting time given by $m_1(t) = \left[1-\varphi_1(t)\right]/\mathrm{R(t)}$ and $m_1(t) = 1-\left[\varphi_1(t)/F(\mathrm{t})\right]$, respectively.

Note that: the s^{th} complete moment of GTME distribution can be obtained as $t \to \infty$.

3.3. Moments of the Residual and Reversed Residual Life

The n^{th} moment of the residual life (MRL), $m_n(x) = E\{(X-x)^n | X > x\},\ n = 1,2,...$ uniquely determines $F(x)$, see Navarro, *et al.* [17]. It is given by

$$m_n(t) = \frac{1}{1-F(t)}\int_t^\infty (x-t)^n\,dF(x)$$
$$= \frac{\beta^2}{1-F(t;\varpi)}\sum_{l=0}^{n}\binom{n}{l}(-1)^l t^l\int_t^\infty x^{n-l+1}e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}dx$$
$$= \frac{\beta^2}{1-F(t;\varpi)}\sum_{l=0}^{n}\binom{n}{l}(-1)^l t^l\left[M_1-M_2\right],$$

where M_1, and M_2 are obtained as

$$M_1 = a(1+\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a-1}{i}\binom{i}{j}\frac{(-1)^i\beta^{-n+l-2}\Gamma\left(n-l+j+2,\beta(i+1)t\right)}{(i+1)^{n-l+j+2}},$$

and

$$M_2 = \lambda(a+b)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{a+b-1}{i}\binom{i}{j}\frac{(-1)^i\beta^{-n+l-2}\Gamma\left(n-l+j+2,\beta(i+1)t\right)}{(i+1)^{n-l+j+2}}.$$

Hence, the n^{th} MRL can be written as

$$m_n(t) = D(l,i,j)\frac{\beta^{-n+l-2}\Gamma\left(n-l+j+2,\beta(i+1)t\right)}{(i+1)^{n-l+j+2}},$$

where, $\Gamma(z,t)$ is the lower incomplete gamma function and

$$D(l,i,j)=\frac{\beta^2}{1-F(t;\varpi)}\sum_{l=0}^{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\binom{n}{l}(-1)^l t^l\left\{a(1+\lambda)\binom{a-1}{i}-\lambda(a+b)\binom{a+b-1}{i}\right\}.$$

The mean inactivity time (MIT) or mean waiting time also called the mean reversed residual life function is given by $m_1(x)=E\{(X-x)^n|X>x\},$ and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0 , X). The MIT of the GTME distributions can be obtained easily by setting in the above equation.

3.4. Quantile and Median

The GTME distribution can be easily simulated by inverting cdf (7) as follows: if U follows uniform distribution on (0, 1), then

$$u-\left[1-(1+\beta Q(u))e^{-\beta Q(u)}\right]^a\left\{1+\lambda-\lambda\left[1-(1+\beta Q(u))e^{-\beta Q(u)}\right]^b\right\}=0 \tag{11}$$

By solving the nonlinear Equation (11), numerically, the GTME random variable X can be generated. The percentage points at 25%, 50% and 75% of some specific choices of the parameters are given in Table 3.

Table-3. Percentage points for a,b,β and λ

λ	$a=2$, $b=1$, $\beta=6$			$a=3$, $b=2$, $\beta=4$			$a=4$, $b=3$, $\beta=2$		
-1	25%	50%	75%	25%	50%	75%	25%	50%	75%
	0.086	0.514	0.825	0.134	0.265	0.329	0.481	0.591	0.891
0.5	0.141	0.578	0.855	0.027	0.277	0.606	0.156	0.273	0.678
1	0.305	0.769	0.935	0. 351	0.467	0.698	0.231	0.357	0.674

We detect from Table 3 that as the values of a, b increase and β decrease, for fixed values of λ, the values of percentage points increase.

3.5. Rényi Entropy

Rényi entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. For any real parameter $\gamma>0$ and $\gamma\neq 1$, the Rényi entropy is defined as

$$I_R(\gamma)=\frac{1}{\gamma-1}\log\int_R f^{\gamma}(x)dx;\quad \gamma>0 \text{ and } \gamma\neq 1.$$

Now using the density function (8), we obtain the integrated part as follows

$$\int_R f^{\gamma}(x)dx=\int_0^{\infty}\beta^{2\gamma}x^{\gamma}e^{-\gamma\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma(a-1)}\left\{a(1+\lambda)-\lambda(a+b)\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}^{\gamma}dx.$$

By using the binomial expansion, then

$$\int_R f^{\gamma}(x)dx=\sum_{i,j=0}^{\infty}\sum_{l=0}^{j}\binom{\gamma}{i}\binom{\gamma(a-1)+bi}{j}\binom{j}{l}(-1)^{i+j}\beta^{l+2\gamma}\left[a(1+\lambda)\right]^{\gamma-i}\left[\lambda(a+b)\right]^i\int_0^{\infty}x^{\gamma+l}e^{-\beta x(\gamma+j)}dx,$$

which yields,

$$\int_R f^{\gamma}(x)dx=E(i,j,l)\frac{\left[\lambda(a+b)\right]^i\Gamma(\gamma+l+1)}{\left[\beta(\gamma+j)\right]^{\gamma+l+1}},$$

where

$$E(i,j,l)=\sum_{i,j=0}^{\infty}\sum_{l=0}^{j}\binom{\gamma}{i}\binom{\gamma(a-1)+bi}{j}\binom{j}{l}(-1)^{i+j}\beta^{l+2\gamma}\left[a(1+\lambda)\right]^{\gamma-i}\left[\lambda(a+b)\right]^i.$$

Therefore, the Rényi entropy of GTME distribution is given by:

$$I_R(\gamma)=\frac{1}{\gamma-1}\log\left\{E(i,j,l)\frac{[\lambda(a+b)]^i\Gamma(\gamma+l+1)}{[\beta(\gamma+j)]^{\gamma+l+1}}\right\}.$$

3.6. Order Statistics

In this sub-section, we drive the single order statistics for GTME distribution. Let $x_1, x_2,\ldots,x_n$ be n independent and identically distributed GTME random variables. Further, let $x_{(1)}, x_{(2)},\ldots,x_{(n)}$ denote the order statistics from these n variables. Then, the pdf of the r^{th} order statistic, say $f_{r:n}(x)$, is given by

$$f_{r:n}(x)=c_{r:n}\left[F(x)\right]^{r-1}f(x)\left[1-F(x)\right]^{n-r},$$

where, $c_{r:n}=\frac{n!}{(r-1)!(n-r)!}$. By using the binomial expansion, then r^{th} order statistic of GTME distribution is given by

$$f_{r:n}(x)=c_{r:n}\beta^2\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i xe^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a(r+i)-1}\left\{1+\lambda-\lambda\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}^{r+i-1}$$
$$\left\{a(1+\lambda)-\lambda(a+b)\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}.$$

the k^{th} moments of r^{th} order statistics for GTME distribution is

$$\mu_{r:n}^{(k)}=c_{r:n}\beta^2\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\int_0^\infty x^{k+1}e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{a(r+i)-1}$$
$$\left\{1+\lambda-\lambda\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}^{r+i-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-(1+\beta x)e^{-\beta x}\right]^b\right\}dx$$

By using the binomial expansion

$$\mu_{r:n}^{(k)}=c_{r:n}\beta^2\sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}\binom{n-r}{i}\binom{r+i-1}{j}(-1)^{i+j}(1+\lambda)^{r+i-j-1}\lambda^j\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^l\beta^m$$
$$\left\{a(1+\lambda)\binom{a(r+i)+j-1}{l}-\lambda(a+b)\binom{a(r+i)+b+j-1}{l}\right\}\int_0^\infty x^{k+m+1}e^{-\beta x(l+1)}dx$$

The k^{th} moments of r^{th} order statistics for GTME distribution is

$$\mu_{r:n}^{(k)}=c_{r:n}\beta^2\sum_{i=0}^{n-r}\sum_{j=0}^{r+i-1}\binom{n-r}{i}\binom{r+i-1}{j}(-1)^{i+j}(1+\lambda)^{r+i-j-1}\lambda^j\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^l\beta^m$$
$$\left\{a(1+\lambda)\binom{a(r+i)+j-1}{l}-\lambda(a+b)\binom{a(r+i)+b+j-1}{l}\right\}\frac{\Gamma(k+m+2)}{[\beta(l+1)]^{k+m+2}}$$

4. Parameter Estimation

In this section, the maximum likelihood (ML) estimators of the unknown parameters of the GTME model are derived. Also, numerical study is provided

4.1. Maximum Likelihood Estimators

In the statistical literature, various methodologies for parameter estimation were proposed while the ML method is the most commonly utilized. We investigate the estimation of the parameters of the GTME distribution by ML for complete data. Let, $x_1,..,x_n$ be a random sample of size n of this distribution with set of parameter vector $\varpi=(a, b, \beta, \lambda)$, then the log-likelihood function, say $\ell(\varpi)$ can be written as

$$\ell(\varpi)=2n\ln\beta+\sum_{i=1}^n\ln x_i-\beta\sum_{i=1}^n x_i+(a-1)\sum_{i=1}^n\ln(1-z_i)+\sum_{i=1}^n\ln\left\{a(1+\lambda)-\lambda(a+b)\left[1-z_i\right]^b\right\},$$

where, $z_i=(1+\beta x_i)e^{-\beta x_i}$ The partial derivatives of $\ell(\varpi)$, denoted by $\ln\ell$ are given by

$$\frac{\partial \ln \ell}{\partial \lambda}=\sum_{i=1}^{n}\frac{a-(a+b)\left[1-z_i\right]^b}{a(1+\lambda)-\lambda(a+b)\left[1-z_i\right]^b},$$

$$\frac{\partial \ln \ell}{\partial b}=\sum_{i=1}^{n}\frac{-\lambda\left[1-z_i\right]^b\left\{(a+b)\ln\left[1-z_i\right]+1\right\}}{a(1+\lambda)-\lambda(a+b)\left[1-z_i\right]^b},$$

$$\frac{\partial \ln \ell}{\partial a}=\sum_{i=1}^{n}\ln(1-z_i)+\sum_{i=1}^{n}\frac{(1+\lambda)-\lambda\left[1-z_i\right]^b}{a(1+\lambda)-\lambda(a+b)\left[1-z_i\right]^b},$$

$$\frac{\partial \ln \ell}{\partial \beta}=\frac{2n}{\beta}-\sum_{i=1}^{n}x_i+(a-1)\sum_{i=1}^{n}\frac{\beta x_i^2 e^{-\beta x_i}}{1-z_i}-\sum_{i=1}^{n}\frac{b\lambda\beta(a+b)x_i^2 e^{-\beta x_i}(1-z_i)^{b-1}}{\left\{a(1+\lambda)-\lambda(a+b)^b(1-z_i)^b\right\}}.$$

The ML estimators of the model parameters are determined by solving the non-linear equations $\partial \ln \ell/\partial\lambda=0,$ $\partial \ln \ell/\partial b=0$, $\partial \ln \ell/\partial a=0$, and $\partial \ln \ell/\partial \beta=0$, numerically by employing an iterative technique.

4.2. Simulation Study

Here, an empirical investigation is formed to evaluate the performance of ML estimate for GTME model by using R software. Behavior of estimates is assessed via biases and mean square errors (MSEs) for different sample sizes. The numerical example is characterized as follows:

- 1000 random samples of sizes *n* = 20, 40 and 100 are generated from GTME distribution.
- Certain values of parameters (a,b,β,λ) are selected as Set1= (1.5,4,3,-0.9), Set 2= (2,3.5,2.5,-0.5), Set 3= (2.5,3,2, ,-1) and Set4=(3,2,1,1).
- For each *n* and for each set of parameters, ML estimates of $a,b,\beta,$ and λ are obtained by iterative technique.
- The biases and MSEs for each *n* are calculated (see Table 4).

In general, we conclude that the MSEs for the estimates of the parameters decrease as the sample size increases.

Table-4. MSE and Bias of GTME distribution

	$\mathbf{a=1.5}$		$\boldsymbol{b=4}$		$\boldsymbol{\beta=3}$		$\boldsymbol{\lambda=-0.9}$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	0.526	0.277	1.423	2.025	0.673	0.453	0.613	0.376
40	0.175	0.021	0.712	0.507	0.235	0.055	0.543	0.295
100	0.103	0.013	0.236	0.056	0.114	0.013	0.243	0.059
	$a=2$		$b=3.5$		$\beta=2.5$		$\lambda=-0.5$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	0.247	0.062	0.929	0.863	.8201	3.314	1.635	2.674
40	0.204	0.042	0.456	0.208	0.986	0.973	0.968	0.937
100	0.123	0.016	0.114	0.013	0.567	0.323	0.634	0.402
	$a=2.5$		$b=3$		$\beta=2$		$\lambda=-1$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	0.277	0.077	0.715	0.511	0.517	0.267	0.439	0.193
40	0.205	0.043	0.502	0.250	0.331	0.109	0.345	0.119
100	0.150	0.023	0.219	0.078	0.189	0.036	0.237	0.056
	$a=3$		$b=2$		$\beta=1$		$\lambda=1$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	0.135	0.018	0.358	0.128	0.475	0.225	0.295	0.087
40	0.112	0.012	0.233	0.055	0.367	0.135	0.187	0.034
100	0.107	0.011	0.196	0.038	0.221	0.048	0.110	0.012

5. Real Data Analysis

In this section, three applications to real data sets are employed to illustrate the importance and potentiality of the GTME distribution.

Data Set I: Waiting Times in a Bank

The first data set consists of 100 observations on waiting time (inminutes) before the customer received service in a bank see [18].

Data Set II: March precipitation (in inches)

The second data set and represents thirty successive values of March precipitation (in inches) in Minneapolis/St Paul see; [19]

Data Set III: Exceedances of Wheaton River Flood

The third data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark [20]

First, we investigate the quality of adjustment of the GTME distribution when compared to some other models. For comparative study, we consider five models, namely ME, expinentiated ME (EME), Weibull ME (WME), Kumaraswamy exponentiated Burr XII distribution (KEBXII) pioneered by Mead and Afify [21] and transmuted exponentiated generalized Weibull (TExGW) as presented by Yousof, *et al.* [22]. We consider minus 2logL, Akaike information criterion (AIC), Corrected AIC Criterion (AICc), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). The best distribution corresponds to smallest values of the regarded measures.

Table 5, 6 and 7 contains ML estimates, the values of -2logL, AIC, BIC and HQIC statistics for the data set. From these results, it is evident that the GTME distribution is the best distribution for fitting these data set compared to other distributions considered here. It is a strong competitor to other distributions commonly used in literature for fitting lifetime data.

Table-5. Analytical results of the GTME model and other competing models for data I

Parameters	**Distributions**					
	ME	**EME**	**WME**	**TExGW**	**KEBXII**	**GTME**
$\hat{a}$	0.202	5.13	1.542	0.313	0.338	1.093
$\hat{b}$	-	3.80	0.812	1.036	15.245	22.082
$\hat{\beta}$	-	-	2.515	1.803	54.744	0.228
$\hat{\lambda}$	-	-	-	0.069	1.574	-0.047
θ	-	-	-	-	0.4	-
-2logL	700.7	666.52	635.2	634.43	1437.88	634.34
KS	0.8948	0.793	0.1988	0.2914	0.2765	0.15784
AIC	702.70	670.55	670.55	642.43	1448	640.347
BIC	705.31	675.76	675.76	652.851	1461	648.162
AICc	702.74	670.67	670.67	643.54	1449	640.942
HQIC	703.75	672.66	672.66	646.453	1453	643.855
P-value	0.1032	0.037	0.1012	0. 044	0.1304	0.160

Table-6. Analytical results of the GTME model and other competing models for data II

Parameters	**Distributions**					
	ME	**EME**	**WME**	**TExGW**	**KEBXII**	**GTME**
$\hat{a}$	1.194	0.885	1.753	0.313	0.123	1.631
$\hat{b}$	-	-	0.952	1.036	68.429	6.811
$\hat{\beta}$	-	1.983	0.623	1.803	1551	1.613
$\hat{\lambda}$	-	-	-	0.069	0.152	-0.094
θ	-	-	-	-	5.175	-
-2logL	78.48	613.22	80.28	77.21	491.3	76.13
KS	0.8645	0.892	0.1831	0.2386	0.1852	0.16419
AIC	84.477	617.22	86.287	85.209	501.312	84.167
BIC	89.681	620.02	90.491	90.814	508.318	88.772
AICc	85.401	617.66	86.09	86.809	503.812	85.067
HQIC	85.822	618.11	87.632	87.002	503.553	85.16
P-value	0.1055	0.108	0.0169	0.0261	0.1048	0.1108

Table-7. Analytical results of the GTME model and other competing models for data III

Parameters	**Distributions**					
	ME	**EME**	**WME**	**TExGW**	**KEBXII**	**GTME**
$\hat{a}$	-	0.876	0.232	0.119	1.427	16.875
$\hat{b}$	-	-	4.374	1.173	0.313	0.024
$\hat{\beta}$	1.047	2.38	70.155	2.793	81.996	2.784
$\hat{\lambda}$	-	-	-	0.203	3.63	-0.98
θ	-	-	-	-	3.292	-
-2logL	52.42	42.52	47.03	40.082	203.86	31.45
KS	0.6829	0.5819	0.4795	0.5241	0.5961	0.1745
AIC	54.412	53.023	214.84	48.082	213.868	37.447
BIC	55.408	56.01	48.506	52.065	218.847	40.435
AICc	54.634	58.23	47.22	50.749	218.154	38.947
HQIC	54.607	53.606	47.22	48.86	214.84	38.03
P-value	0.0211	0.0101	0.0201	0.0105	0.0439	0.0955

Based on Table 5, 6 and 7, it is clear that GTME distribution provides the overall best fit and therefore could be chosen as the more adequate model than other models for explaining the considered data set. More information can be provided in Figures 3, 5 and 7. Also QQ- plots and PP-plots are shown in Figures 4, 6 and 8 for the three real data.

Figure-3. Estimated pdf, cdf, and sf of GTME model for data Set I

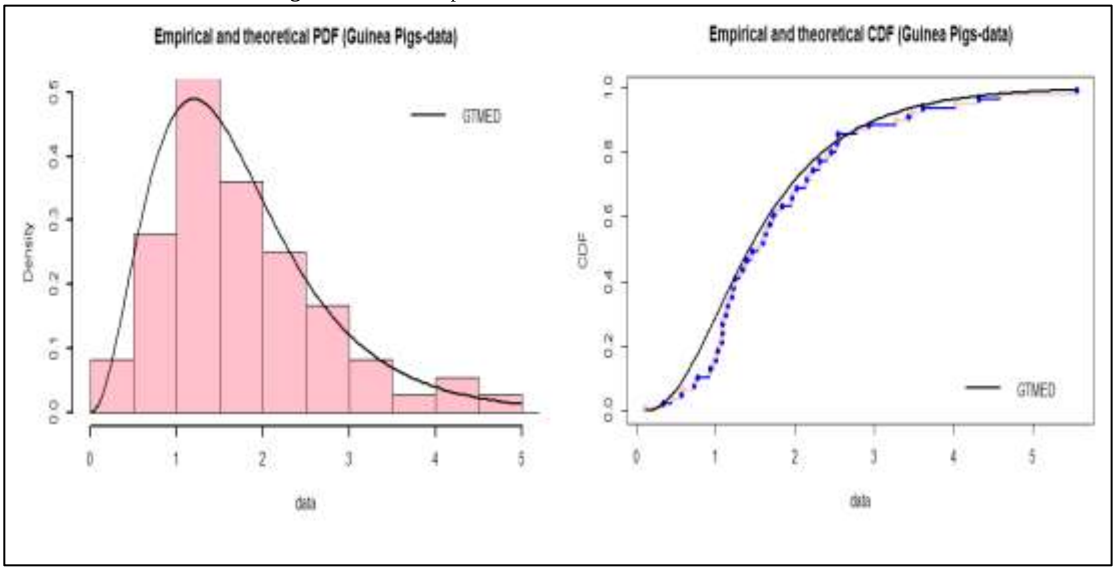


Figure-4. PP plot and QQ plot of GTME model for data Set I

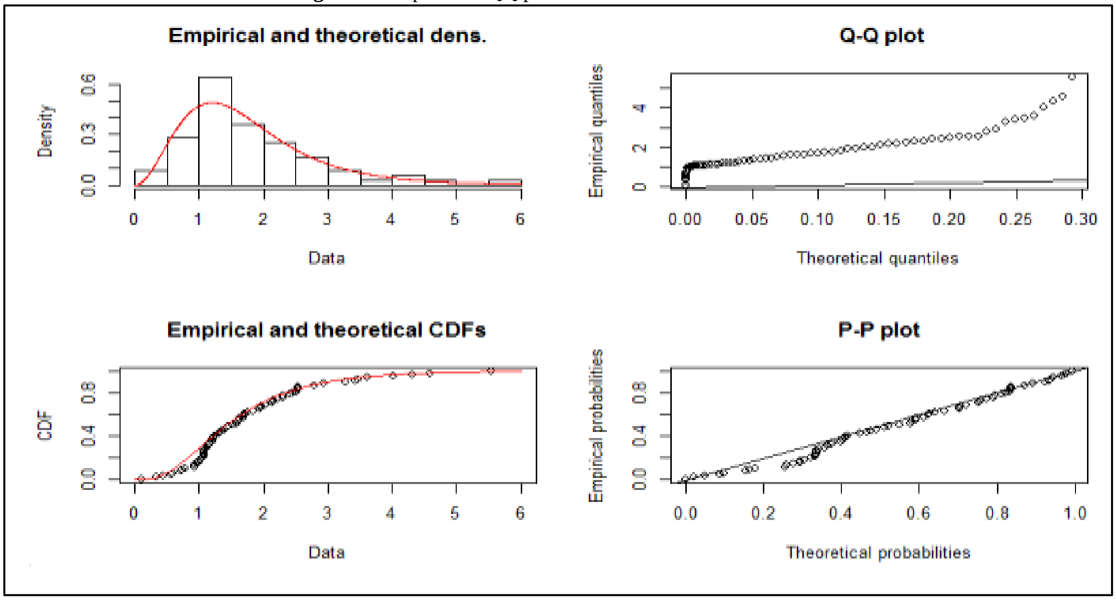


Figure-5. Estimated pdf, cdf, and sf of GTME model for data Set II

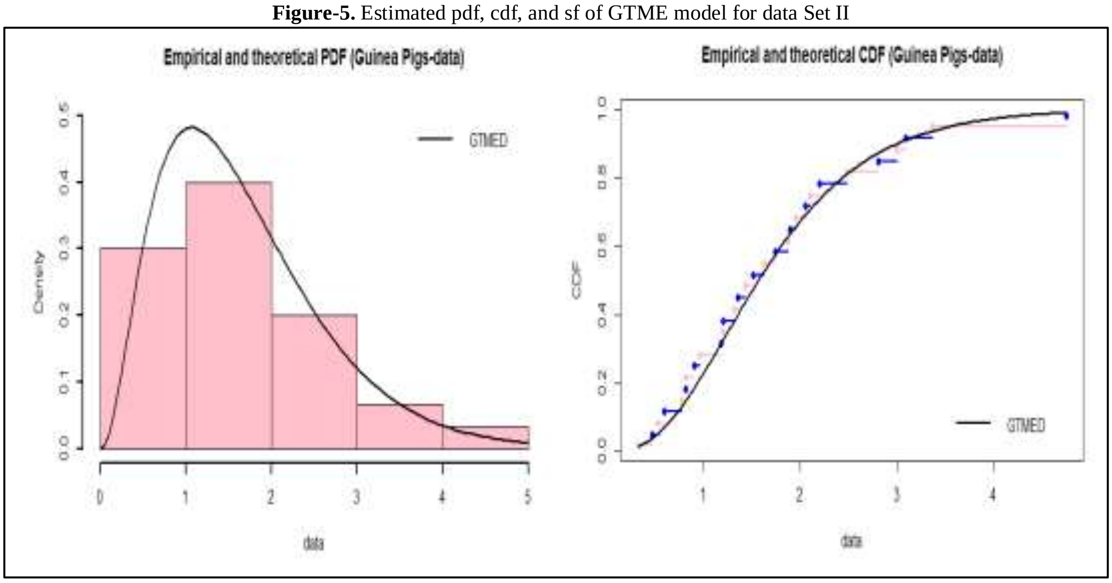


Figure-6. PP plot and QQ plot of GTME model for data Set II

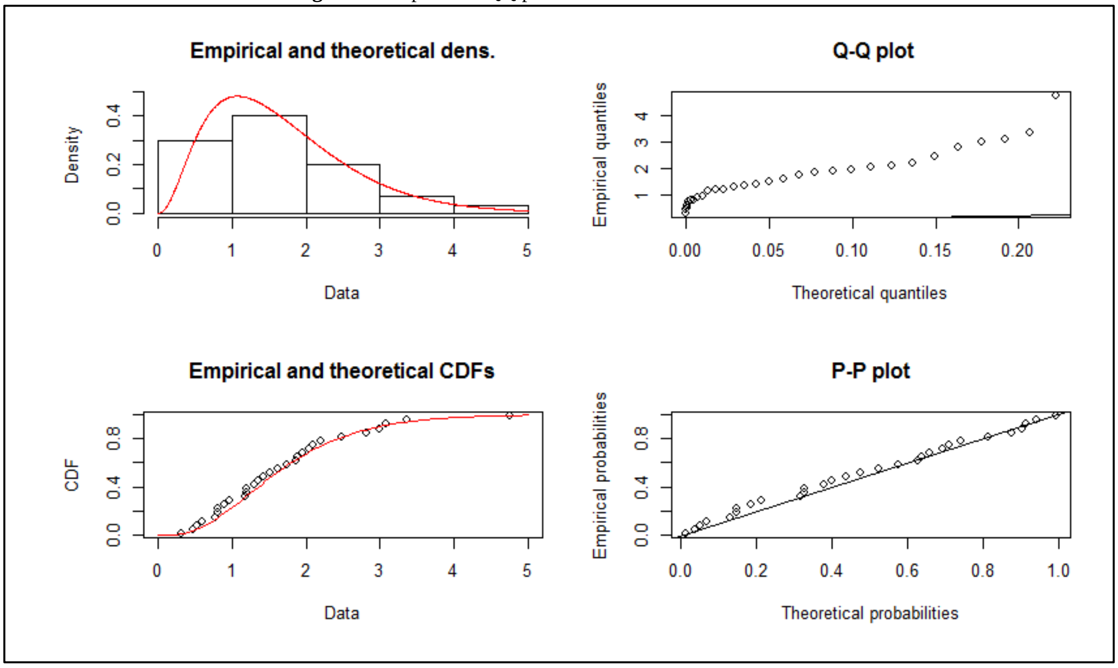


Figure-7. Estimated pdf, cdf, and sf of GTME model for data Set III

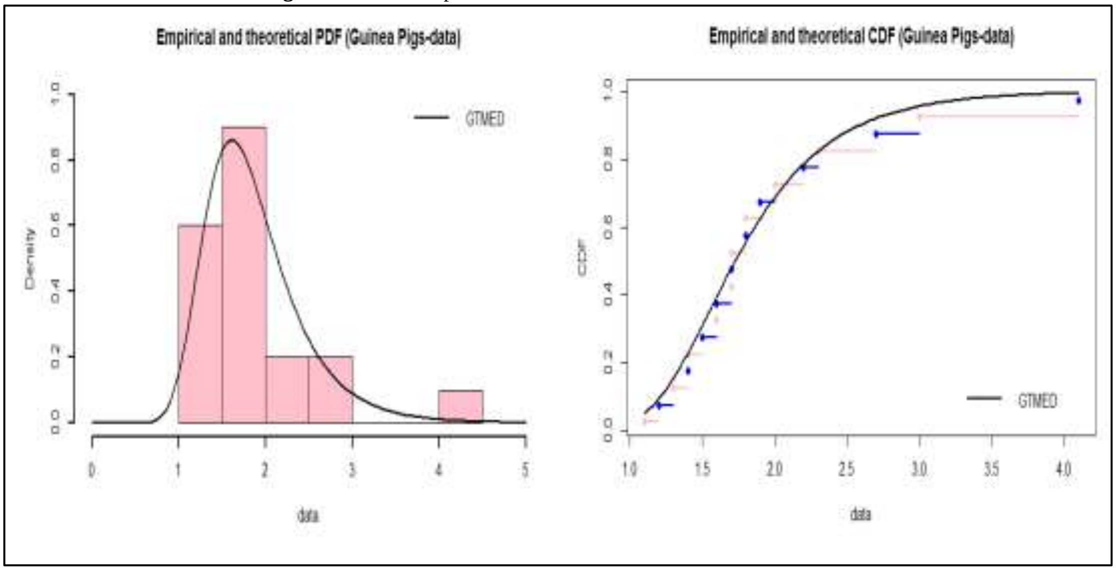
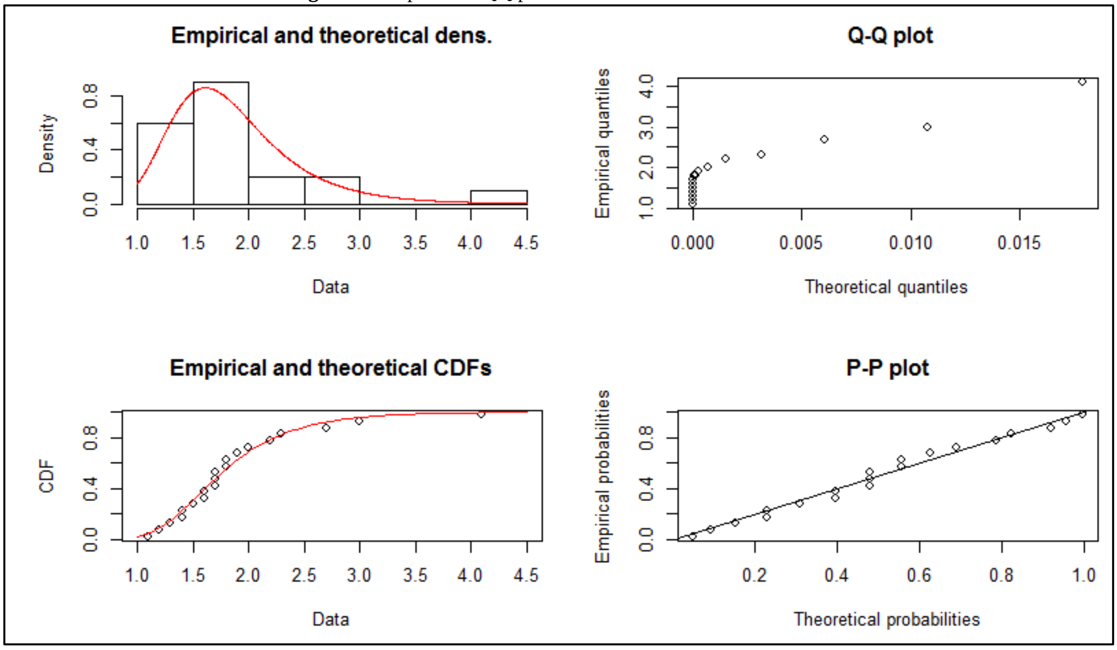


Figure-8. PP plot and QQ plot of GTME model for data Set III



From Figures 3 4 5 6 7 8, we conclude that the GTME distribution provides better fits the other competitive models. We wish that the proposed model may be an alternative model for a wider range of statistical research.

6. Concluding Remarks

In this paper, we study the so-called a generalized transmuted moment exponential distribution. The GTME model includes moment exponential, and exponentiated moment exponential distributions and at the same time transmuted moment exponential as new model. Some structural properties of the GTME distribution are derived. Estimation of the population parameters is achieved via maximum likelihood procedure. Simulation study and application to real data sets are provided.

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