Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 6, Issue. 6, pp: 66-76, 2020
URL: https://arpgweb.com/journal/journal/17
DOI: https://doi.org/10.32861/ajams.66.66.76

Academic Research Publishing Group

Original Research Open Access

Computational Algorithm for the Numerical Solution of Systems of Volterra Integro-Differential Equations

Falade Kazeem Iyanda (Corresponding Author)
Department of Mathematics, Kano University of Science and Technology, P.M.B 3244 Wudil Kano State, Nigeria
Email: faladekazeem2016@kustwudil.edu.ng

Tiamiyu Abd`gafar Tunde
Department of Mathematics, Federal University of Technology, Minna Niger State, Nigeria

Article History
Received: May 9, 2020
Revised: June 11, 2020
Accepted: June 18, 2020
Published: June 22, 2020

Abstract

In this paper, we employ variational iterative method (VIM) to develop a suitable Algorithm for the numerical solution of systems of Volterra integro-differential equations. The formulated algorithm is used to solve first and second order linear and nonlinear system of Volterra integrodifferential equations which demonstrated a good numerical approach to overcome lengthen computational and integral simplification involves. Moreover, the comparison of the exact solution with the approximated solutions are made and approximate solutions $p(x)$ $q(t)$ proved to converge to the exact solutions $p(x)$ $q(t)$ respectively. The results reveal that the formulated algorithm are simple, effective and faster than analytical approach of solving Volterra integro-differential equations.

Keywords: Variational iterative method; System of volterra integro-differential equations; Computational algorithm; Exact solutions.

1. Introduction

In mathematical modeling of real-life problems, we need to deal with functional equations e.g. partial differential equations, integral and integro-differential equation, stochastic equations and others. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. Numerical modeling of integral and integro-differential equations have been paid attention by many scholars. Several numerical methods have been developed for the solution of the integro-differential equations. The iterated Galerkin methods have been proposed in Volk [1]. Compact finite difference method has been used for integro-differential equations by Zhao and Corless [2]. Moreover, in Brunner, *et al.* [3], there are found mixed interpolation collocation methods to solve first- and second-order Volterra linear integro-differential equations. For methods using a quadrature rule, degenerate kernels, interpolation or extrapolation, Homotopy perturbation, Taylor expansion, chebyshev collocation and wavelet-galerkin [4]. Volterra studied the hereditary influences when he was examining a population growth model and the general form of the second kind system of Volterra integrodifferential equation can be written as

$$\begin{cases} p^{(i)}(x) = f_1(x) + \int_0^t \left(K_1(x,t)p(t) + \widehat{K_1}(x,t)q(t) + \cdots\right)dt \\ q^{(i)}(x) = f_2(x) + \int_0^t \left(K_2(x,t)p(t) + \widehat{K_2}(x,t)q(t) + \cdots\right)dt \end{cases} \tag{1}$$

subject to initial conditions

$$\begin{cases} p(x_0) = \alpha \\ q(x_0) = \beta \end{cases} \tag{2}$$

The unknown functions $p(x)$,$q(x)$,..., that will be determined, occur inside the integral sign whereas the derivatives of $p(x)$,$q(x)$,...appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $\widehat{K_i}(x,t)$ and the function $f_i(x)$ are given real-valued functions and α, β are constants.

In recent years, there are various numerical and analytical methods proposed for the solutions of systems of integral and integrodifferential equations. For example, the linear and nonlinear systems of integrodifferential equations have been solved by Haar functions Maleknejad, *et al.* [5], Maleknejad and Tavassoli [6] used the hybrid Legendre functions, the Chebyshev polynomial method [7], the Bessel collocation method [8, 9], the Taylor collocation method [10], the homotopy perturbation method [11, 12], the variational iteration method [13], the differential transformation method [14], and the Taylor series method [15]. Biazar, *et al.* [16], have obtained the solutions of systems of Volterra integral equations of the first kind by the Adomian method. In addition, the homotopy perturbation method has been used for systems of Abel's integral equations [17], in Fuayip and Nurbol [18] solving systems of volterra integral and integrodifferential equations with proportional delays by differential transformation method and Falade [19] proposed exponentially fitted collocation approximate technique for the numerical solutions of higher order integrodifferential equations.

The main objective of this paper is to utilize MAPLE 18 codes to formulate a suitable algorithm based on variational iterative method (VIM) discussed in Abdul-Majid [20] which promise to overcome the lengthen integral simplification and make it possible to obtain accurate numerical results.

The study divided into five sections: In the first section, brief introduction on integral and integro-differential equation was discussed. Section two, description of variational iterative method and formulation of algorithm was explained. In the next section, application of algorithm was tested on five examples of linear and nonlinear problems discussed in the literatures while section four, results graphs are provided and last section, conclusion was discussed.

2. Description of Numerical Technique

2.1. Variational Iteration Method (VIM)

The variational iteration method (VIM) provides rapidly convergent successive approximations of the exact solution in a closed form solution [21]. It handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution.

The correction functional for the Volterra system of integro-differential equations (1) are given by

$$\begin{cases} p_{n+1}(x) = p_n(x) + \int_0^x \lambda(t)\left(p^{(i)}(t) - f_1(t) - \int_0^t K(t,r)\hat{p}_n(r)\,dr\right)dt \\ q_{n+1}(x) = q_n(x) + \int_0^x \lambda(t)\left(q^{(i)}(t) - f_2(t) - \int_0^t K(t,r)\hat{q}_n(r)\,dr\right)dt \end{cases} \tag{3}$$

It has two essential approaches, one needs first to determine the Lagrange multiplier

λ that can be identified optimally via integration by parts and by using a restricted variation. Having λ determined, an iteration formula, without restricted variation, should be used for the determination of the successive approximations $p_{n+1}(x),\ n \geq 0$ and $q_{n+1}(x), n \geq 0$ of the solutions $p(x)$ and $q(x)$.The zeroth approximations $p_0(x)$ and $q_0(x)$ can be any selective functions. The initial conditions are preferably used to select these approximations $p_0(x)$ and $q_0(x)$ shall be consider. Thurs, the solutions are given by

$$\begin{cases} p(x) = \lim_{n\to\infty} p_n(x) \\ q(x) = \lim_{n\to\infty} q_n(x) \end{cases} \tag{4}$$

where for i = 1,2, , n and the Lagrange multiplier, therefore, can be identified as

$$\lambda_i(t) = \frac{(-1)^m}{(m-1)!}(t-x)^{m-1} \tag{5}$$

where m is the highest order of the differential equation.

2.2. Formulation of Algorithm

Base on correction functional for the Volterra system of integro-differential in (3), we develop four steps algorithm on MAPLE 18 software package as follow:

Restart:
Step 1
$F[x_1] := f_1(x);$
$F[1] := eval(F[x_1], [x = t]);$
$F[x_2] := f_2(x);$
$F[2] := eval(F[x_2], [x = t]);$
$p[x_0] := \alpha;\ P[x_0] := \alpha; p1[x_0] := \alpha;$
$q[x_0] := \beta;\ Q[x_0] := \beta; q1[x_0] := \beta;$
$\lambda_i(t) = \frac{(-1)^m}{(m-1)!}(t-x)^{m-1};$
$N := \mathbb{R}^+;$
Step 2
for n from 0 to N do
$G[1] := int(K_1(t,r) * (p1[x_n]^2) + K_2(t,r) * (q1[x_n]^2), [r = 0 \ldots t]);$
$G[2] := int(K_1(t,r) * (p1[x_n]^2) + K_2(t,r) * (q1[x_n]^2), [r = 0 \ldots t]);$
$A :=\ p[n] + int(\ \lambda_i * (Diff(P[n],t) - F[1] - G[1]), [t = 0 \ldots x]);$
$B :=\ q[n] + int(\ \lambda_i * (Diff(Q[n],t) - F[2] - G[2]), [t = 0 \ldots x]);$
$p[n+1] := value(A);$
$q[n+1] := value(B);$
$P[n+1] := eval(p[n+1], [x = t]);$
$Q[n+1] := eval(q[n+1], [x = t]);$

$p1[n+1] := eval(p[n+1],[x=r]);$
$q1[n+1] := eval(q[n+1],[x=r]);$
end do
Step 3
for $\boldsymbol{n}$ ***from*** **0** ***to N do***
$p[n+1] := collect(value(p[n+1]),x);$
$p_x[n+1] := convert(series(p[n+1],x,s)`polynom`);$
$q[n+1] := collect(value(q[n+1]),x);$
$q_x[n+1] := convert(series(q[n+1],x,s)`polynom`);$
end do
Step 4
for $\boldsymbol{n}$ ***from*** **0** ***to N do***
$p[n+1] := p_x[n+1];$
$q[n+1] := q_x[n+1];$
end do where $N, s \in \mathbb{R}^+$

3. Numerical Examples

In this section, we present five examples to show applicability and efficiency of the formulated algorithm for solving Volterra-integro-differential equations. Results obtained are found to be in good agreement with the exact solution.

3.1. Example 1

Consider the following the system of nonlinear Volterra-integro-differential equations [20]

$$\begin{cases} p'(x) = 2x + \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x \left((x-2t)(p(t)^2 + q(t)\right)dt, & p(0) = 1 \\ q'(x) = -2x - \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x ((x-2t)(p(t) + q(t)^2)dt, & q(0) = 1 \end{cases} \tag{6}$$

Exact solution is given
$$\begin{cases} p(x) = 1 + x^2 \\ q(x) = 1 - x^2 \end{cases} \tag{7}$$

Compare (6) with proposed algorithm, we have the following:

$$\begin{cases} f_1(x) = 2x + \frac{1}{6}x^4 + \frac{2}{15}x^6 \\ f_2(x) = -2x - \frac{1}{6}x^4 + \frac{2}{15}x^6 \\ K_1(t,r) = (t-2r) \\ K_2(t,r) = (t-2r) \\ \alpha = 1 \\ \beta = 1 \\ \lambda = -1 \\ N = 2 \\ s = 5 \end{cases}$$

Compute the above parameters into algorithm, we obtain the following successive approximations solutions

$$\begin{cases} p_0(x) = 1 \\ q_0(x) = 1 \\ p_1(x) = 1 + x^2 + \frac{1}{30}x^5 + \frac{2}{105}x^7, \\ q_1(x) = 1 - x^2 - \frac{1}{30}x^5 + \frac{2}{105}x^7, \\ p_2(x) = 1 + x^2 - \frac{1}{2016}x^8, \\ q_2(x) = 1 - x^2 - \frac{1}{2016}x^8, \\ p_3(x) = 1 + x^2 \\ q_3(x) = 1 - x^2 \end{cases}$$

Therefore, is converges to exact solution

$$\begin{cases} p(x) = 1 + x^2 \\ q(x) = 1 - x^2 \end{cases} \tag{8}$$

3.2. Example 2

Consider the following first order system of linear Volterra integro-differential equations [22]

$$\begin{cases} p^{/}(x) = 2x^2 + \int_0^t \big((x-t)p(t) + (x-t)q(t)\big)dt\,, & p(0) = 1 \\ q^{/}(x) = -3x^2 - \dfrac{x^2}{10} + \int_0^t \big((x-t)p(t) - (x-t)q(t)\big)dt\,, & q(0) = 1 \end{cases} \tag{9}$$

Exact solution is given $\begin{cases} p(x) = 1 + x^3 \\ q(x) = 1 - x^3 \end{cases}$ (10)

Compare (9) with proposed algorithm, we have the following:

$$\begin{cases} f_1(x) = 2x^2 \\ f_2(x) = -3x^2 - \dfrac{x^2}{10} \\ K_1(t,r) = (t-r) \\ K_2(t,r) = (t-r) \\ \alpha = 1 \\ \beta = 1 \\ \lambda = -1 \\ N = 1 \\ s = 3 \end{cases}$$

Compute the above parameters into algorithm, we obtain the following successive approximations solutions

$$\begin{cases} p_0(x) = 1 \\ q_0(x) = 1 \\ p_1(x) = 1 + x^3 \\ q_1(x) = 1 - x^3 \\ p_2(x) = 1 + x^3 \\ q_2(x) = 1 - x^3 \end{cases}$$

Converges to exact solution

$$\begin{cases} p(x) = 1 + x^3 \\ q(x) = 1 - x^3 \end{cases} \tag{11}$$

3.3. Example 3

Consider the following the system of nonlinear Volterra integro-differential equations [23]

$$\begin{cases} \begin{cases} p^{//}(x) = \dfrac{7}{3}e^x - e^{2x} - \dfrac{1}{3}e^{4x} \\ \int_0^x e^{x-t}(p(t)^2 + q(t)^2)dt\,, \quad p(0) = 1 \quad p^{/}(0) = 1 \end{cases} \\ \begin{cases} q^{//}(x) = \dfrac{2}{3}e^x + 3e^{2x} + \dfrac{1}{3}e^{4x} \\ \int_0^x e^{x-t}(p(t)^2 - q(t)^2)dt\,, \quad q(0) = 1 \quad q^{/}(0) = 2 \end{cases} \end{cases} \tag{12}$$

Exact solution is given $\begin{cases} p(x) = e^x \\ q(x) = e^{2x} \end{cases}$ (13)

Compare (12) with proposed algorithm, we have the following:

$$\begin{cases} f_1(x) = \dfrac{7}{3}e^x - e^{2x} - \dfrac{1}{3}e^{4x} \\ f_2(x) = \dfrac{2}{3}e^x + 3e^{2x} + \dfrac{1}{3}e^{4x} \\ K_1(t,r) = e^{t-r} \\ K_2(t,r) = e^{t-r} \\ \alpha = 1 + x \\ \beta = 1 + 2x \\ \lambda = t - x \\ N = 2 \\ s = 8 \end{cases}$$

Compute the above parameters into algorithm, we obtain the following successive approximations solutions

$$\begin{cases} p_0(x) = 1 + x \\ q_0(x) = 1 + 2x \\ p_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{3}{40}x^5 - \frac{9}{80}x^6 - \frac{353}{5040}x^7 + \cdots \\ q_1(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{19}{60}x^5 + \frac{7}{40}x^6 + \frac{43}{504}x^7 + \cdots \\ p_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots \\ q_2(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \cdots \\ p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots \\ q_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \cdots \end{cases}$$

Converges to exact solution

$$\begin{cases} p(x) = e^x \\ q(x) = e^{2x} \end{cases} \tag{14}$$

3.4. Example 4

Consider the following the system of nonlinear Volterra integro-differential equations [20]

$$\begin{cases} \begin{cases} p^{//}(x) = \cosh(x) - \frac{1}{2}\sinh^2(x) - \frac{x^4}{6} - \frac{x^2}{2} + \\ \int_0^x \left((x-t)p(t)^2 + (x-t)q(t)^2\right)dt, \quad p(0) = 1 \quad p^{/}(0) = 1 \end{cases} \\ \begin{cases} q^{//}(x) = -(1+4x)\cosh(x) + 8\sinh(x) - 4x + \\ \int_0^x \left((x-t)p(t)^2 - (x-t)q(t)^2\right)dt, \quad q(0) = -1 \quad q^{/}(0) = 1 \end{cases} \end{cases} \tag{15}$$

Exact solution is given

$$\begin{cases} p(x) = x + \cosh(x) \\ q(x) = x + \sinh(x) \end{cases} \tag{16}$$

Compare (15) with proposed algorithm, we have the following:

$$\begin{cases} f_1(x) = \cosh(x) - \frac{1}{2}\sinh^2(x) - \frac{x^4}{6} - \frac{x^2}{2} \\ f_2(x) = -(1+4x)\cosh(x) + 8\sinh(x) - 4x \\ K_1(t,r) = (t-r) \\ K_2(t,r) = (t-r) \\ \alpha = 1 + x \\ \beta = -1 + x \\ \lambda = t - x \\ N = 2 \\ s = 8 \end{cases}$$

Compute the above parameters into algorithm, we obtain the following successive approximations solutions

$$\begin{cases} p_0(x) = 1 + x \\ q_0(x) = 1 + 2x \\ p_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{120}x^6 - \frac{1}{2688}x^8 + \cdots \\ q_1(x) = -1 + x - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{720}x^6 - \frac{1}{420}x^7 - \frac{1}{40320}x^8 + \cdots \\ p_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \cdots \\ q_2(x) = -1 + x - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{720}x^6 - \frac{1}{40320}x^8 + \cdots \\ p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \cdots \\ q_3(x) = -1 + x - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{720}x^6 - \frac{1}{40320}x^8 + \cdots \end{cases}$$

Converges to exact solution

$$\begin{cases} p(x) = x + \cosh(x) \\ q(x) = x + \sinh(x) \end{cases} \tag{17}$$

3.5. Example 5

Consider the following second order system of nonlinear Volterra integro-differential equations [20]

$$\begin{cases} \begin{cases} p^{//}(x) = e^x + (x-1)\dfrac{e^{2x}}{2} + (3x-1)\dfrac{e^{4x}}{4} + (x+1)\dfrac{3}{4} + \\ \displaystyle\int_0^x \left((x-2t)p(t)^2 + (x-4t)q(t)^2\right)dt, \quad p(0)=1 \quad p^/(0)=1 \end{cases} \\ \begin{cases} q^{//}(x) = 4e^{2x} + (3x-1)\dfrac{e^{4x}}{4} + (5x-1)\dfrac{e^{6x}}{6} + (x+1)\dfrac{5}{12} + \\ \displaystyle\int_0^x \left((x-4t)p(t)^2 + (x-6t)w(t)^2\right)dt, \quad q(0)=1 \quad q^/(0)=2 \end{cases} \\ \begin{cases} w^{//}(x) = 9e^{3x} + (x-1)\dfrac{e^{2x}}{2} + (5x-1)\dfrac{e^{6x}}{6} + (x+1)\dfrac{2}{3} + \\ \displaystyle\int_0^x \left((x-6t)w(t)^2 + (x-2t)p(t)^2\right)dt, \quad w(0)=1 \quad w^/(0)=3 \end{cases} \end{cases} \tag{18}$$

Exact solution is given
$$\begin{cases} p(x) = e^x \\ q(x) = e^{2x} \\ w(x) = e^{3x} \end{cases} \tag{19}$$

Compare (18) with proposed algorithm, we have the following:

$$\begin{cases} f_1(x) = e^x + (x-1)\dfrac{e^{2x}}{2} + (3x-1)\dfrac{e^{4x}}{4} + (x+1)\dfrac{3}{4} \\ f_2(x) = 4e^{2x} + (3x-1)\dfrac{e^{4x}}{4} + (5x-1)\dfrac{e^{6x}}{6} + (x+1)\dfrac{5}{12} \\ f_3(x) = 9e^{3x} + (x-1)\dfrac{e^{2x}}{2} + (5x-1)\dfrac{e^{6x}}{6} + (x+1)\dfrac{2}{3} \\ K_1(t,r) = [(t-2r),(t-4r),(t-6r)] \\ K_2(t,r) = [(t-4r),(t-6r),(t-2r)] \\ \alpha = 1+x \\ \beta = 1+2x \\ \gamma = 1+3x \\ \lambda = t-x \\ N = 2 \\ s = 8 \end{cases}$$

Compute the above parameters into algorithm, we obtain the following successive approximations solutions

$$\begin{cases} p_0(x) = 1+x \\ q_0(x) = 1+2x \\ w_0(x) = 1+3x \\ p_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{23}{240}x^6 + \frac{81}{560}x^7 + \frac{3649}{40320}x^8 + \cdots \\ q_1(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{19}{36}x^6 + \frac{617}{630}x^7 + \frac{13}{15}x^8 + \cdots \\ w_1(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5 + \frac{197}{144}x^6 + \frac{421}{336}x^7 + \frac{37729}{40320}x^8 \ldots \\ p_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{270}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \cdots \\ q_2(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \frac{2}{315}x^8 + \cdots \\ w_2(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5 + \frac{81}{80}x^6 + \frac{243}{560}x^7 + \frac{729}{4480}x^8 \ldots \\ p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{270}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \cdots \\ q_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \frac{2}{315}x^8 + \cdots \\ w_3(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5 + \frac{81}{80}x^6 + \frac{243}{560}x^7 + \frac{729}{4480}x^8 \ldots \end{cases}$$

Converges to exact solution
$$\begin{cases} p(x) = e^x \\ q(x) = e^{2x} \\ w(x) = e^{3x} \end{cases} \tag{20}$$

4. Numerical Results

Table-1. Numerical solution of system of nonlinear Volterra integro-differential equations Example 1

x	**Exact Solution** $p(x)$	**VIM Solution** $p(x)$	**Exact solution** $q(x)$	**VIM Solution** $q(x)$
0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.01000000	1.01000000	0.99000000	0.99000000
0.2	1.04000000	1.04000000	0.96000000	0.96000000
0.3	1.09000000	1.09000000	0.91000000	0.91000000
0.4	1.16000000	1.16000000	0.84000000	0.84000000
0.5	1.25000000	1.25000000	0.74000000	0.74000000
0.6	1.36000000	1.36000000	0.64000000	0.64000000
0,7	1.49000000	1.49000000	0.51000000	0.51000000
0.8	1.64000000	1.64000000	0.36000000	0.36000000
0.9	1.81000000	1.81000000	0.19000000	0.19000000
1.0	2.00000000	2.00000000	0.00000000	0.00000000

Table-2. Numerical solution of system of nonlinear Volterra integro-differential equations Example 2

x	**Exact Solution** $p(x)$	**VIM Solution** $p(x)$	**Exact Solution** $q(x)$	**VIM Solution** $q(x)$
0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.00100000	1.00100000	0.99900000	0.99900000
0.2	1.00800000	1.00800000	0.99200000	0.99200000
0.3	1.02700000	1.02700000	0.97300000	0.97300000
0.4	1.06400000	1.06400000	0.93600000	0.93600000
0.5	1.12500000	1.12500000	0.87500000	0.87500000
0.6	1.21600000	1.21600000	0.78400000	0.78400000
0,7	1.34300000	1.34300000	0.65700000	0.65700000
0.8	1.51200000	1.51200000	0.48800000	0.48800000
0.9	1.72900000	1.72900000	0.27100000	0.27100000
1.0	2.00000000	2.00000000	0.00000000	0.00000000

Table-3. Numerical solution of system of nonlinear Volterra integro-differential equations Example 3

x	**Exact Solution** $p(x)$	**VIM solution** $p(x)$	**Exact solution** $q(x)$	**VIM solution** $q(x)$
0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105170918	1.221402758	1.221402758
0.2	1.221402758	1.221402758	1.491824698	1.491824698
0.3	1.349858808	1.349858808	1.822118800	1.822118800
0.4	1.491824698	1.491824698	2.225540928	2.225540928
0.5	1.648721271	1.648721271	2.718281828	2.718281828
0.6	1.822118800	1.822118800	3.320116923	3.320116923
0,7	2.013752707	2.013752707	4.055199967	4.055199967
0.8	2.225540928	2.225540928	4.953032424	4.953032424
0.9	2.459603111	2.459603111	6.049647464	6.049647464
1.0	2.718281828	2.718281828	7.389056099	7.389056099

Table-4. Numerical solution of system of nonlinear Volterra integro-differential equations Example 4

x	**Exact Solution** $p(x)$	**VIM Solution** $p(x)$	**Exact solution** $q(x)$	**VIM solution** $q(x)$
0	1.000000000	1.000000000	-1.00000000	-1.00000000
0.1	1.105004168	1.105004168	-0.90500416	-0.90500416
0.2	1.220066756	1.220066756	-0.82006675	-0.82006675
0.3	1.345338514	1.345338514	-0.74533851	-0.74533851
0.4	1.481072372	1.481072372	-0.68107237	-0.68107237
0.5	1.627625965	1.627625965	-0.62762596	-0.62762596
0.6	1.785465218	1.785465218	-0.58546521	-0.58546521
0,7	1.955169006	1.955169006	-0.55516900	-0.55516900
0.8	2.137434946	2.137434946	-0.53743494	-0.53743494
0.9	2.333086385	2.333086385	-0.53308638	-0.53308638
1.0	2.543080635	2.543080635	-0.54308063	-0.54308063

Table-5. Numerical solution of system of nonlinear Volterra integro-differential equations Example 5

x	Exact Solution $p(x)$	VIM solution $p(x)$	Exact solution $q(x)$	VIM solution $q(x)$	Exact solution $w(x)$	VIM solution $w(x)$
0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105170918	1.221402758	1.221402758	1.349858800	1.349858800
0.2	1.221402758	1.221402758	1.491824698	1.491824698	2.459603111	2.459603111
0.3	1.349858808	1.349858808	1.822118800	1.822118800	2.459601938	2.459601938
0.4	1.491824698	1.491824698	2.225540928	2.225540928	3.320116923	3.320116923
0.5	1.648721271	1.648721271	2.718281828	2.718281828	4.481689070	4.481689070
0.6	1.822118800	1.822118800	3.320116923	3.320116923	6.049647464	6.049647464
0,7	2.013752707	2.013752707	4.055199967	4.055199967	8.166169913	8.166169913
0.8	2.225540928	2.225540928	4.953032424	4.953032424	11.02317638	11.02317638
0.9	2.459603111	2.459603111	6.049647464	6.049647464	14.87973172	14.87973172
1.0	2.718281828	2.718281828	7.389056099	7.389056099	20.08553692	20.08553692

4.1. Graph Representation

Figure-1. Numerical solution of system of Volterra-integro-differential equations Example 1

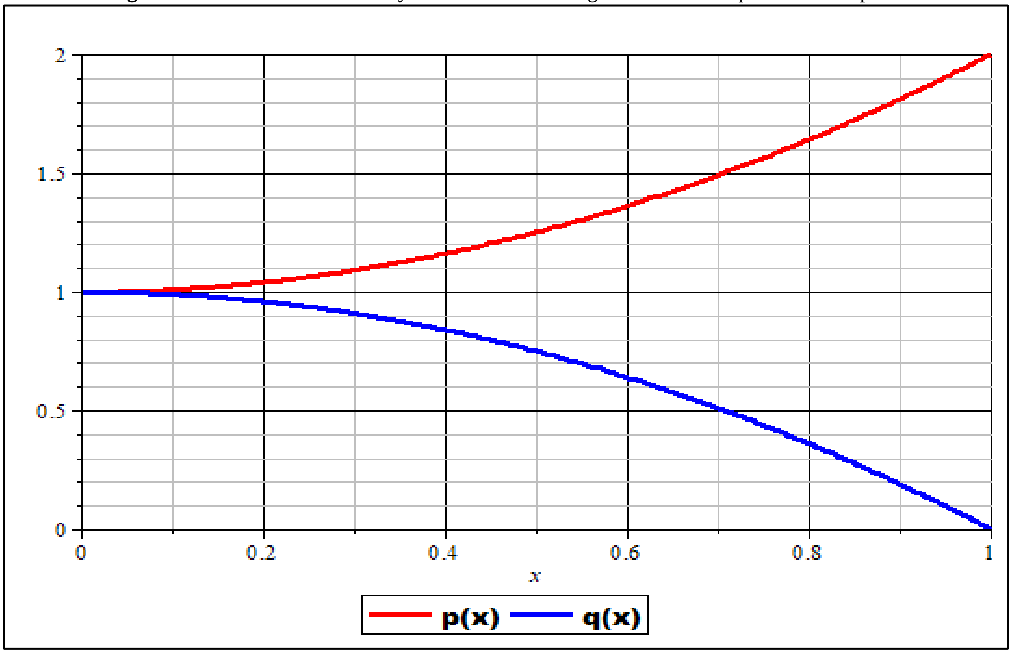


Figure-2. Numerical solution of system of Volterra-integro-differential equations Example 2

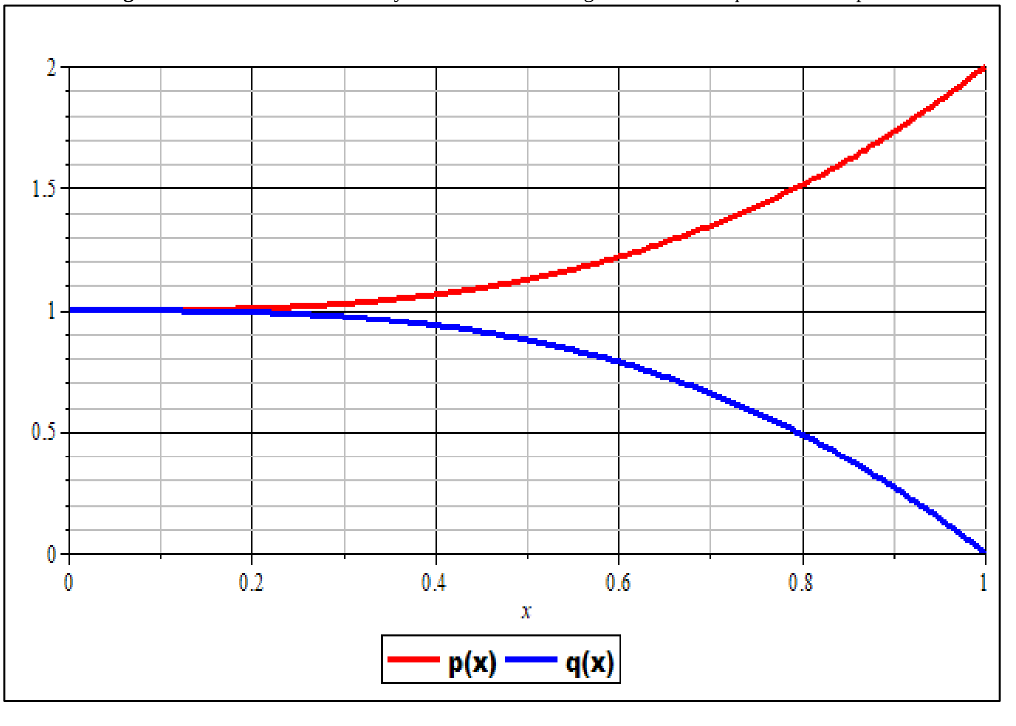


Figure-3. Numerical solution of system of Volterra-integro-differential equations Example 3

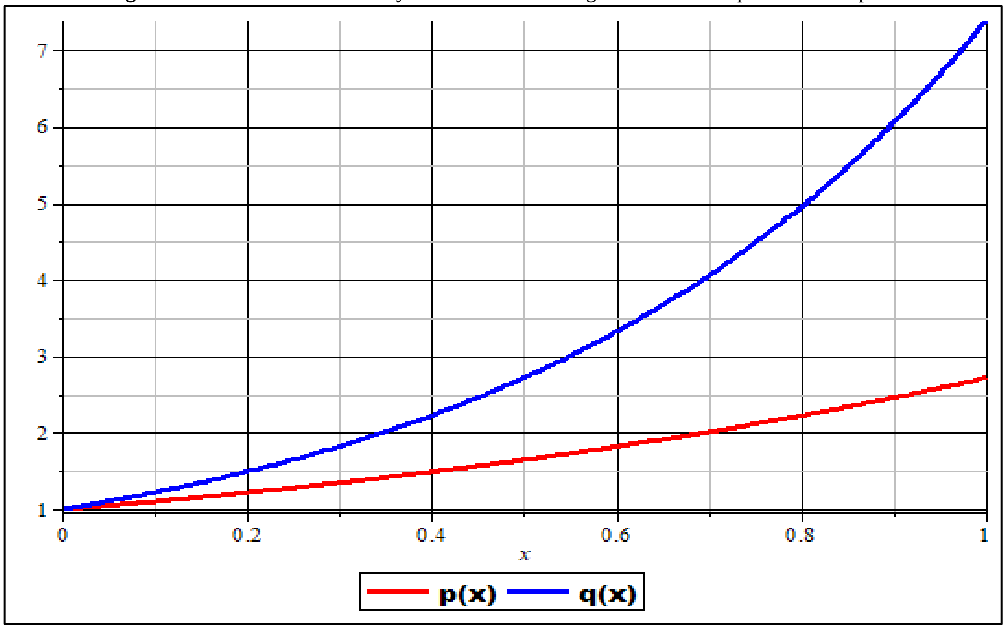


Figure-4. Numerical solution of system of Volterra-integro-differential equations Example 4

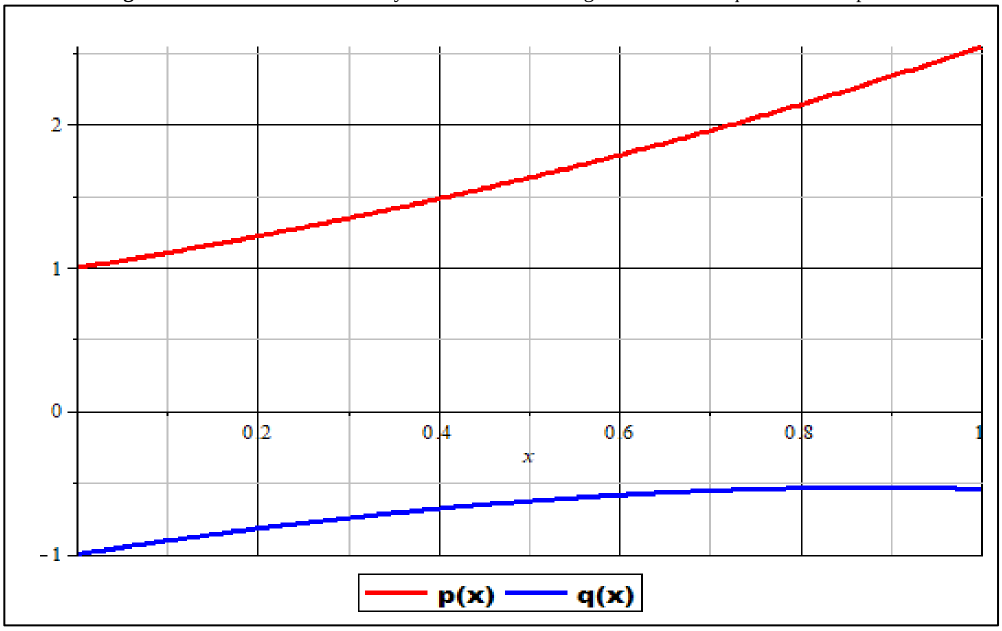
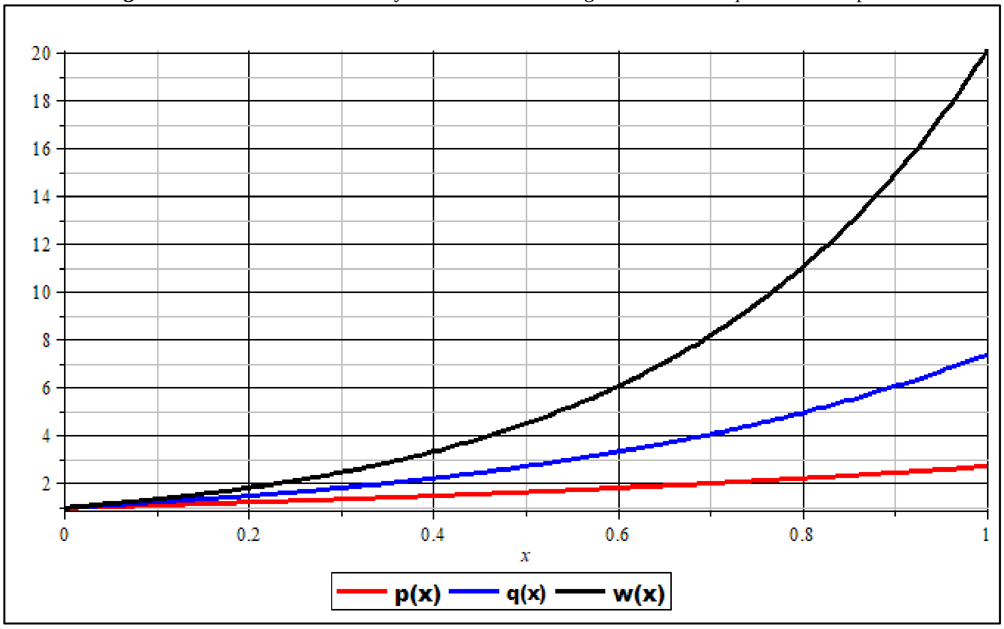


Figure-5. Numerical solution of system of Volterra-integro-differential equations Example 5



5. Conclusion

In this paper, an algorithm based on variational iteration method was developed and applied to solve system of linear and nonlinear system of volterra integro-differential equations. The comparison of the solutions obtained and the exact solutions shows that the formulated algorithm is more efficient and practical for solving linear and non-linear systems of integro-differential equations. Therefore, this computation approach is very effective for calculating the exact solutions of systems of integro-differential equations and recommend for usage in related problems in engineering and applied sciences.

References

[1] Volk, W., 1988. "The iterated Galerkin methods for linear integro-differential equations." *Journal of Computational and Applied Mathematics,* pp. 2163-74. Available: https://doi.org/10.1016/0377-0427(88)90388-3

[2] Zhao, J. and Corless, R. M., 2006. "Compact finite difference method for integro-differential equations." *Applied Mathematics and Computation,* vol. 177, Available: https://doi.org/10.1016/j.amc.2005.11.007

[3] Brunner, H., Makroglou, A., and Miller, R. K., 1997. "Mixed interpolation collocation methods for first and second order Volterra integro-differential equations with periodic solution." *Applied Numerical Mathematics,* vol. 23, pp. 381-402. Available: https://doi.org/10.1016/S0168-9274(96)00075-X

[4] Christopher and Baker, T. H., 1980. "Structure of recurrence relations in the study of stability in the numerical treatment of Volterra integral and integro-differential equation." *Journal of Integral Equations,* vol. 2, pp. 11-29.

[5] Maleknejad, K., Mirzaee, F., and Abbasbandy, S., 2004. "Solving linear integro-differential equations system by using rationalized Haar functions method." *Applied Mathematics and Computation,* vol. 155, pp. 317–328.

[6] Maleknejad, K. and Tavassoli, K. M., 2004. "Solving linear integrodifferential equation system by Galerkin methods with hydrid functions." *Applied Mathematics and Computation,* vol. 159, pp. 603–612.

[7] Dascioglu, A. and Sezer, M., 2005. "Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integrodifferential equations." *Journal of the Franklin Institute,* vol. 342, pp. 688–701.

[8] Ahin, N. S., Uzbas¸ı, S. Y., and ulsu, M. G., 2011. "A collocation approach for solving systems of linear Volterra integral equations with variable coefficients." *Computers and Mathematics with Applications,* vol. 62, pp. 755–769.

[9] Uzbas, S. Y., Ahin, N. S., and Sezer, M., 2011. "Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases." *Computers and Mathematics with Applications,* vol. 61, pp. 3079–3096.

[10] Ulsu, M. G. and Sezer, M., 2006. "Taylor collocation method for solution of system of high-order linear Fredholm-Volterra integrodifferential equations." *International Journal of Computer Mathematics,* vol. 83, pp. 429-448.

[11] Yusufo, E. and Glu, 2007. "An efficient algorithm for solving integrodifferential equations system." *Applied Mathematics and Computation,* vol. 192, pp. 51–55.

[12] Yusufo, E. and glu, 2008. "A homotopy perturbation algorithm to solve a system of Fredholm-Volterra type integral equations." *Mathematical and Computer Modelling,* vol. 47, pp. 1099–1107.

[13] Saberi-Nadjafi, J. and Tamamgar, M., 2008. "The variational iteration method: a highly promising method for solving the system of integro-differential equations." *Computers and Mathematics with Applications,* vol. 56, pp. 346–354.

[14] Arikoglu, A. and Ozkol, I., 2008. "Solutions of integral and integrodifferential equation systems by using differential transform method." *Computers and Mathematics with Applications,* vol. 56, pp. 2411–2417.

[15] Sorkunand, H. H. and Yalc¸inbas, S., 2010. "Approximate solutions of linear Volterra integral equation systems with variable coefficients." *Applied Mathematical Modelling,* vol. 34, pp. 3451–3464.

[16] Biazar, J., Babolian, E., and Islam, R. M., 2003. "Solution of a system of Volterra integral equations of the first kind by Adomian method." *Applied Mathematics and Computation,* vol. 139, pp. 249–258.

[17] Kumar, S., Singh, O. P., and Dixit, S., 2011. "Homotopy perturbation method for solving system of generalized Abel's integral equations." *Applications and Applied Mathematics,* vol. 6, pp. 2009–2024.

[18] Fuayip, Y. and Nurbol, I., 2014. "Solving systems of volterra integral and integrodifferential equations with proportional delays by differential transformation method." *Hindawi Publishing Corporation Journal of Mathematics,* p. 5. Available: http://dx.doi.org/10.1155/2014/725648

[19] Falade, K. I., 2019. "Solving integro-differential equations using exponentially fitted collocation approximate technique EFCAT." *Middle East Journal of Science (2019),* vol. 5, pp. 73-85. Available: http://dergipark.org.tr/mejs

[20] Abdul-Majid, W., 2011. *Linear and nonlinear integral equations methods and applications*. Higher education press Beijing, pp. 452- 455.

[21] He, J. H., 2000. "Variational iteration method for autonomous ordinary differential systems." *Appl. Math. Computation,* vol. 114, pp. 115–123.

[22] Esmail, H. and Elham, A., 2013. "Solving systems of linear volterra integrodifferential equations by using sinc-collocation method." *International Journal of Mathematical Engineering and Science,* vol. 7, pp. 5-6.

[23] Kourosh, P. and Mehdi, D., 2018. "System of nonlinear volterra integro-differential equations of arbitrary order." *Bol. Soc. Paran. Mat.*, vol. 36, pp. 33–54.