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On Properties of Derivations in Normed Spaces

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Abstract

Let δ: $Cp \to Cp$ be normal, then the linear map δ_N $\| N(x) \|$ attains a local minimum at $x \in C_p$ if and only if z $\in C_p$ such that $D_{N(x)}(\phi(z) \geq 0$. Also let $T \in C_p$, and let $N(T)$ have the polar decomposition $N(T) = U|N(T)|$ then the map δ_N attains local minimum on C_p at T if and only if $|N(T)|U^* \in ker\emptyset^*$. Regarding orthogonality, let $S \in C_p$ and let $N(S)$ have the polar decomposition $N(S) = U|N(S)|$, then $\| N(X) \|_{Cp} \geq \| N(S) \|_{Cp}$ for $X \in C_p$ if $|N(S)|^{n-1}U^* \epsilon\, ker\delta_{B,A}$. Moreover, the map δ_N has a local minimum at $\in C_p$if and only if $infT_{h,N(x)(\varphi(y))\geq 0}$ for $y \in C_p$. In this paper, we give some results on local minimum and orthogonality of normal derivations in C_p-Classes. We employ some techniques for normal derivations due to Mecheri, Hacene, Bounkhel and Anderson.
Keywords: Hilbert space; Local minimum; Orthogonality and Schatten-p class.

1. Introduction

Normal derivation has been an area of interest for many mathematicians and researchers particularly their properties for example [1, 2]. In their study they gave results on the global minimum of linear maps. In this paper, we give results on the conditions for the linear map δ_N to attain local minimum in C_p-Classes. We have used polar decomposition [3], tensor product [4] and inner product [5] to arrive at the main results**.** In Keckic [6], the knowledge of structural properties of the underlying C*-algebra is undoubtedly one of the cornerstones in solving the problem of norms of elementary operators. Equivalently the spectral theorem helps a great deal in understanding Hilbert space theory. The spectral theory gives the unitary invariants of self-adjoint operator A on a Hilbert space H in terms of its multiplicity measureclasses {m} and provide unitarily equivalent model operators. Problems concerning pairs of projections play a fundamental role in the theory of operator algebras of normal derivations. Kittaneh [7] shows that if P and Q are projections then the C*-algebra generated by P, Q has a concrete realisation as an algebra of 2 × 2-matrix valued functions on Anderson [1], so its representation theory is well understood if we have two pair of projections P and Q in terms of a generating self adjoint operator A which are implementing normal derivations. In Mecheri [8] it is noted that the trace class operators, denoted by C(H), is the set of all compact operators A ∈ L(H), for which the eigenvalues according to multiplicity, are summable. The I(H) of L(H) admits a trace function tr(T), for any complete orthonormal system in H. As a Banach spaces K(H) can be identified with the dual of the ideal K of compact operators by means of the linear isometry of normal derivations. It was shown in Mecheri [9] that, if A is a unital C* Orc(A). It allows that if A is any C*-algebra then K is sef-adjoint.We now recall some properties of the complete regularization of Prim(A) for a C*-algebra (see [1]) for further details). For P, Q ∈ Prim(A) (Prim(A)). Then ˜ is an equivalence relation on Prim(A) and the equivalence classes are closed subsets of Prim(A). It follows that there is a one-to-one correspondence tween Prim(A)/ ˜ and a set of closed two-sided ideals of A given by the intersection of the ideals in the equivalence class [P] of P. The set of ideals obtained in this way is denoted by Glimm(A) and we identify this set with Prim(A)/ ˜ by the correspondence above. If A is unital then Glimm(A) consists of the ideals of A generated the maximal ideals of the centre of A, as studied by Glimm Mecheri [9]. The quotient map G: Prim(A) → Glimm(A) is called the complete regularization map on a norm topology. This topology is completely regular, Hausdorff, weaker than the quotient topology (and equal to it when A is s-unital [1]) and hence makes f continuous. The ideals in Glimm(A) are called imm ideals and the equivalence classes for ˜ in Prim(A) will sometimes be referred to as imm classes. For normal derivations, T is a linear operator on a finite dimensional complex Hilbert space H, then every commutator of the form AX — XA has trace 0 and consequently 0 belongs to the numerical range W(AX-XA). However, if H is infinite dimensional, then, as shown by Bounkhel [2], there exist bounded operators A and B on T such that W(AB-BA) is a vertical line segment in the open right half-plane. The present paper initiates a study of the class D of operators A on H which have the property that 0 belongs to W(AX—XA)~ for every bounded operator X on H. We call such operators finite, the term being suggested by the facts that D contains all normal operators, all compact operators, all operators having a direct summand of finite rank, and the entire C*-algebra generated by each of its members. Lastly, the main focus in this paper is to study the local miminum and orthogonality in C_p-classes for normal derivations.

2. Preliminaries

In this section, we start by defining some key terms that are useful in the sequel.

Definition 2.1 ([4], Definition 1.2) A Banach space is a complete normed space.

Definition 2.2 ([3], Definition 33.1) A Hilbert space H is an inner product space which is complete under the norm induced by its inner product.

Definition 2.3 ([5] Definition 3.1) Let f be a function on an open subset U of a Banach space X into Banach space Y. f is Gateaux differentiable at $x \in U$ if there is bounded and linear operator $T: X \to Y$ such that $T_x(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$ for every $h \in X$. The operator T is called the Gateaux derivative of f at x.

Definition 2.4 ([6], Definition 0.1) Let X be a complex Banach space. Then $y \in X$ is orthogonal to $x \in X$ if for all complex λ there holds $\| x + \lambda y \| \geq \| x \|$.

Definition 2.5 ([9], Section 2) Let $T \in B(H)$ be compact. Then $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ are the singular values of T i.e the eigenvalues of $\|T\| = (T^*T)^{\frac{1}{2}}$ counted according to multiplicity and arranged in descending order. For $1 \leq p \leq \infty$, $C_p = C_p(H)$ is the set of those compact $T \in B(H)$ with finite p-norm, $\| T \|_p = (\sum_{i=1}^{\infty} s_i(T)^p)^{\frac{1}{p}} = (tr|T|^P)^{\frac{1}{p}} < \infty$.

3. Main Results and Discussions

In this section we give the main results. We discuss the conditions under which the map δ_N attains local minimum in C_p-class.

Theorem 3.1 Let $\delta: C_p \rightarrow C_p$ be normal, then the linear map $\delta_N = \| N(x) \|$ attains a local minimum at $x \in C_p$ if and only if $z \in C_p$ such that $D_{N(x)}(\phi(z)) \geq 0$.

Proof. For necessity we have $N(x) + \kappa\phi(z) = N(x + \kappa z)$. For sufficiency: Assume the stated condition and choose z. Then $\phi(z - x) = N(z) - N(x)$. Let $D_{N(x)} = M$, then by Kittaneh [7]

$$\| N(x) = -M(-N(x))$$
$$\leq -M(-N(x)) + M(N(z) - N(x)) \text{ by hypothesis}$$
$$\leq M(N(z)) \text{ by sub-additivity [8]}$$
$$\leq \| N(z) \|$$

Theorem 3.2 Let $T \in C_p$, and let $N(T)$ have the polar decomposition $N(T) = U|N(T)|$. Then the map δ_N attains local minimum on C_p at T if and only if $|N(T)|U^* \in ker\emptyset^*$.

Proof. Suppose that δ_N has a local minimum on C_p at T. Then,

$D_{N(T)}(\emptyset(z)) \geq 0$ (1)

if and only if $Z \in C_p$ i.e $nRe\{tr(|N(T)|^{n-1}U^*\phi(Z))\} \geq 0$ for $Z \in C_p$. This implies that

$$Re\{tr(|N(T)|^{n-1}U^*\phi(Z))\} \geq 0 \quad (2) \text{ for } Z \in C_p.$$

Choose arbitrary vectors f and g in the Hilbert space H and let $f \otimes g$ be rank one operator given by $x \mapsto \langle x, f \rangle g$. Now take $Z = f \otimes g$, then one has by Mecheri [9]

$$tr(|N(T)|^{n-1}U^*\phi(Z)) = tr(\phi^*(|N(T)|^{n-1}U^*)Z).$$

Then inequality (1) is equivalent to $Re\{tr(\phi^*(|N(T)|^{n-1}U^*)Z)\} \geq 0$ for $Z \in C_p$, or equivalently by [10] $Re\langle \emptyset^*(|N(T)|^{n-1}U^*)g, f \rangle \geq 0$ for all f and g in H. Suppose $f=g$ such that $\| f \| = 1$, then one obtains

$$Re\langle \emptyset^*(|N(T)|^{n-1}U^*)f, f \rangle \geq 0$$

We note that the set $\{\langle \emptyset^*(|N(T)|^{n-1}U^*)f, f \rangle : \| f \| = 1\}$ is the numerical range of $\phi^*(|N(T)|^{n-1}U^*)$ on $\mathcal{U}$ that is convex set and its closure is a closed convex set. By inequality (2), it must contain the value of positive real part [2], under all rotation about the origin and it must contain the origin so that we obtain some vector f in H such that $\langle \emptyset^*(|N(T)|^{n-1}U^*)f, f \rangle < \varepsilon$, where ε is positive. Given ε is arbitrary, we obtain $\langle \emptyset^*(|N(T)|^{n-1}U^*)f, f \rangle = 0$. Hence, $\langle \emptyset^*(|N(T)|^{n-1}U^*) = 0$ i.e $(|N(T)|^{n-1}U^* \in ker\emptyset^*$. Conversely, if $(|N(T)|^{n-1}U^* \in ker\emptyset^*$, then $(|N(T)|^{n-1} \in ker\emptyset^*$. Using the above argument, we find that $Re\{tr(|N(T)|^{n-1}U^*\phi(Z))\} \geq 0$ for $Z \in C_p$. By this we get inequality (2).

Corollary 3.3 Let $T \in C_p$ and let $N(T)$ have the polar decomposition $N(T)=U|N(T)|$. Then the map δ_N attains local minimum on C_p at T if and only if $|N(T)|^{n-1}U^* \in ker\delta_{T,S}$.

Proof: Suppose that δ_N has a local minimum on C_p at T, then we have

$$D_{N(T)}(\delta_{T,S}(X)) \geq 0 \quad (3)$$

if and only if $X \in C_p$ i.e $nRe\{tr(|N(T)|^{n-1}U^*\delta_{T,S}(X))\} \geq 0$ for $X \in C_p$. This implies

$$Re\{tr(|N(T)|^{n-1}U^*\delta_{T,S}(X))\} \geq 0 \quad (4)$$

for $X \in C_p$. Take any two arbitrary vectors x and y in H and let $x \otimes y$ to be the rank one operator given by $a \mapsto \langle a, x \rangle y$. Let $X = x \otimes y$. Then $tr(|N(T)|^{n-1}U^*\delta_{T,S}(X)) = tr(\delta^*_{T,S}|N(T)|^{n-1}U^*)X)$. Inequality (3) is equivalent to $Re\{tr(\delta^*_{T,S}|N(T)|^{n-1}U^*)X)\} \geq 0$ for $X \in C_p$. Equivalently [3],

$Re\langle \delta^*_{T,S}|N(T)|^{n-1}U^*)y, x) \geq 0$ for all $x, y \in H$. Assume $x=y$ and that $\| x \| = 1$, then,

$Re\langle \delta^*_{T,S}|N(T)|^{n-1}U^*)x, x) \geq 0$. The set $\{\langle \delta^*_{T,S}|N(T)|^{n-1}U^*)x, x) : \| x \| = 1\}$ is the numerical range of $\delta^*_{T,S}|N(T)|^{n-1}U^*)$ on $\mathcal{U}$ that is convex set and its closure is a closed convex set. By inequality (4), it must contain a value of positive real part, under all rotation about the origin and it must contain the origin so that we have some vector x in H so that Christopher [4]

$\langle \delta^*_{T,S}(|N(T)|^{n-1}U^*)x, x \rangle \leq \varepsilon$, where ε is positive. However, ε was arbitrary and therefore,

$\langle \delta^*_{T,S}(|N(T)|^{n-1}U^*)x, x \rangle = 0$. Hence, $\delta^*_{T,S}(|N(T)|^{n-1}U^*) = 0$ and so $(|N(T)|^{n-1}U^* \in ker\delta_{T,S}$.

Theorem 3.4 Let $S \in C_p$ and let $N(S)$ have the polar decomposition $N(S)=U|N(S)|$. Then,

$\| N(X) \|_{Cp} \geq \| N(S) \|_{Cp}$ for $X \in C_p$ if $|N(S)|^{n-1}U^* \epsilon\, ker\delta_{B,A}$.

Proof. Let $\| N(X) \|_{C_p} \geq \| N(S) \|_{C_p}$. We need to show that $|N(S)|^{n-1}U^* \in ker\delta_{B,A}$. Since we are considering C_p, we know that $|||N(X)|||_{CB} = |||N(S)|||_{CB}$. Given that C_p admits pseudo-unitarily invariant norm, then $||N(X)||_{CB|_{C_p}} = ||N(S)||_{CB|_{C_p}}$. By induction $U|N(S)|$ admits nilpotency and hence $|N(S)|^{n-1}$ also admits nilpotency. Suppose U is isometric and also co-isometric, $|N(S)|^{n-1}$ admits nilpotency. All nilpotent operators belong to the kernel of $\delta_{B,A}$. Hence, $|N(S)|^{n-1}U^* \in ker\delta_{B,A}$ since U^* is the adjoint of U.

Theorem 3.5 The map δ_N has a local minimum at $x \in C_p$ if and only if

$infT_{h,N(x)(\varphi(y))\geq 0}$ (5)

for $y \in C_p$.

Proof. For necessity we combine the result [[2], Theorem 2.1] and the following inequality,

$$T_h\delta_N(x, y) = T_{h,N(x)}(\varphi(y)).$$

Conversely assume inequality (5) is satisfied and we observe that

$$T_{h,N(x)}\left(e^{i(\pi-h)}N(x)\right) = \lim_{t\to 0}\frac{\| N(x) + te^{ih}e^{\pi-h}N(x) \| - \| N(X) \|}{t}$$

$$= \lim_{t\to 0} \frac{\| N(x) + tN(x) \| - \| N(X) \|}{t}$$

$=\| N(x) \| \lim_{t\to 0} \frac{|1-t|-1}{t} = \| N(x) \|$

For this we have $\| N(x) \|_{Cp} = -T_{h,N(x)}\left(e^{i(\pi-h)}N(x)\right)$.

Let $y \in C_p$ be arbitrary and put $\tilde{y} = -e^{i(\pi-h)}y + e^{i(\pi-h)}x$. Then, $\tilde{y} \in C_p$. By inequality (5) we have $T_{h,N(x)}(\varphi(\tilde{y})) \geq 0$ and hence by subadditivity of $T_{h,N(x)}(\cdot)$ and the linearity of φ we obtain

$$\| N(x) \|_{Cp} \leq -T_{h,N(x)}\left(e^{i(\pi-h)}N(x) + T_{h,N(x)}\varphi(\tilde{y})\right)$$
$$\leq T_{h,N(x)}\varphi(\tilde{y}) - e^{i(\pi-h)}N(x)$$
$$= T_{h,N(x)}(-e^{i(\pi-h)}N(x)\varphi(y) + e^{i(\pi-h)}\varphi(x) - e^{i(\pi-h)}R - e^{i(\pi-h)}\varphi(x))$$
$$= T_{h,N(x)}\left(-e^{i(\pi-h)}N(x)\right)$$

By Kittaneh [7], Proposition Keckic [6] and given y is arbitrary in C_p, we have that

$\delta_N(x) = \| N(x) \|_{Cp} \leq T_{h,N(x)}\left(-e^{i(\pi-h)}N(y)\right) \leq \| N(y) \|_{Cp} = \delta_N(y)$ if and only if $y \in C_p$. Hence , δ_N has a local minimum at x in C_p by Mecheri [9]

4. Conclusion

These results are on local minimum of the linear map δ_N in C_p-classes [5]. It would be interesting to establish local minimum of normal derivations B(H)-the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H [6]. In regard of the many directions of active research in numerical range and numerical radius, there has been much interest in characterizing both real and complex operators. Mathematicians have worked on a condition that a normal operator must have a closed numerical range and showed that, if an operator is normal and its numerical range is closed, then the extreme points of the numerical range are eigenvalues. Authors have also investigated the numerical range of normal operators on a Hilbert space but not numerical range of posinormal operators. Toeplitz and Hausdorff [1] showed that the numerical range of every bounded linear operator is convex. This result is an important tool in the study of numerical ranges of operators and it holds for all operators on Hilbert spaces. Also it has proved the Toeplitz-Hausdorff theorem, the Folk theorem and determined the numerical range of a 2 × 2 matrix. It would be interesting to consider these properties for normal derivations.

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