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Bi-Periodic Pell Sequence

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Abstract

In this study, we introduce a new generalization of the Pell numbers which is called bi-periodic Pell sequences. We then proceed to find the Binet formula as well as the generating function for this sequence. The well-known Cassini, Catalan and the D'ocagne's identities as well as some related binomial summation and sum formulas are also given. The convergence properties of the consecutive terms of this sequence is also examined.

Keywords: Pell sequence; Binet formula; Generating functions.

1. Introduction

There have been so many papers in literature about special integer sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Pell, Pell Lucas etc...It is easily seen that their numerous applications in almost every field of science and art. Therefore the researchers investigated different generalizations of these integer sequences. For example they generalized the sequences with respect to parity of terms. By using this method, Edson and Yayenie first introduced bi-periodic Fibonacci sequence into literature in 2009 in Edson and Yayenie [1]. They gave the generating function, Binet formula, Catalan and D'ocagne properties for the bi periodic Fibonacci sequence etc...And then Yayenie found interesting properties of this sequence in Yayenie [2]. And also Jun and Choi in Jun and Choi [3] gave some properties of this sequence by defining a matrix related to bi periodic Fibonacci sequence. Then just like the bi-periodic Fibonacci sequence, in the year 2014, Bilgici defined the bi-periodic Lucas sequence and gave some different properties and relationship between the two said sequences in Bilgici [4].

In Coskun and Taskara [5] the authors defined bi-periodic Fibonacci matrix sequence and found nth general term and Binet formula, generating function and sum formula for this matrix sequence. Similarly in Coskun, *et al.* [6], the authors investigated properties of bi-periodic Lucas matrix sequence. In Uygun and Owusu [7], the authors defined bi-periodic Jacobsthal sequences similar to the bi-periodic Fibonacci and Lucas sequences and then proceed to find the basic properties such as Binet formula, generating function, D' Ocagne...In Uygun and Owusu [8], the authors defined bi-periodic Jacobsthal matrix sequences. Now in this paper, we define a new generalization of Pell numbers which we shall call bi-periodic Pell sequences in a way that is similar to the bi-periodic Fibonacci and Lucas sequences and then proceed to find the Binet formula as well as its generating function. After examining the convergence properties of the consecutive terms of this sequence, Cassini, Catalan and the D'ocagne's identities as well as some related binomial summation formulas and sum formulas are also given.

1.1. Bi-Periodic Pell Sequence

Definition 1 The Pell sequence denoted by $\{p_n\}_{n=0}^{\infty}$ is defined recursively by

$$p_n = 2p_{n-1} + p_{n-2}, \tag{1}$$

with initial conditions $p_0 = 0$, $p_1 = 1$ for $n \geq 2$ in Horadam [9].

Definition 2 For any two non-zero real numbers a and b, the bi-periodic Pell sequence denoted by $\{P_n\}_{n=0}^{\infty}$ is defined recursively by

$$P_0 = 0,\ P_1 = 1,\ P_n = \begin{cases} 2aP_{n-1} + P_{n-2} & if\ n\ is\ even \\ 2bP_{n-1} + P_{n-2} & if\ n\ is\ odd \end{cases} \qquad n \geq 2 \tag{2}$$

When a = b = 1, we have the classic Pell sequence. If we set a = b = k, for any positive number, we get the k-Pell sequence.

The first five elements of the bi-periodic Pell sequence are

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$$P_0 = 0,\ P_1 = 1,\ P_2 = 2a,\ P_3 = 4ab+1,\ P_4 = 8a^2b+4a.$$

he recurrence equation of the bi-periodic Pell sequence is given as

$$x^2 - 2abx - ab = 0$$

And the roots of this equation are

$$\alpha = ab + \sqrt{a^2b^2+ab},\ \beta = ab - \sqrt{a^2b^2+ab} \qquad (3)$$

2. Main Results

Lemma 3 The bi-periodic Pell sequence $\{P_n\}_{n=0}^{\infty}$ satisfies the following properties

$$P_{2n} = (4ab\ +2)P_{2n-2} - P_{2n-4}$$

$$P_{2n+1} = (4ab\ +2)P_{2n-1} - P_{2n-3}$$

Proof: Definition 2 is used for the proof. Therefore

$$\begin{aligned} P_{2n} &= 2aP_{2n-1} + P_{2n-2} \\ &= 2a(2bP_{2n-2} + P_{2n-3}) + P_{2n-2} \\ &= (4ab+1)P_{2n-2} + 2aP_{2n-3} \\ &= (4ab+1)P_{2n-2} + (P_{2n-2} - P_{2n-4}) \\ &= (4ab+2)P_{2n-2} - P_{2n-4} \end{aligned}$$

Lemma 4 α and β are defined by (3) satisfied the following properties

$$(2\alpha+1)(2\beta+1) = 1 \qquad (4)$$

$$\alpha+\beta = 2ab, \qquad \alpha\beta = -ab \qquad (5)$$

$$(2\alpha+1) = \frac{\alpha^2}{ab}, \qquad (2\beta+1) = \frac{\beta^2}{ab}, \qquad (6)$$

$$-(2\alpha+1)\beta = \alpha, \qquad -(2\beta+1)\alpha = \beta \qquad (7)$$

Proof. By using the given definition of α and β, the identities above can easily to be proved using the basic rules and properties of algebra.

Theorem 5 We denote the generating function of the bi-periodic Pell sequence by *P(x)* and is obtained as

$$P(x) = \frac{x + 2ax^2 - x^3}{1-(4ab+2)x^2 + x^4} \qquad (8)$$

Proof. The generating function for P(x) is represented in power series by

$$P(x) = \sum_{m=0}^{\infty} P_m x^m = P_0 + P_1x + \dots + P_kx^k + \dots$$

The generating function is divided into two parts as even and odd terms. First for the even part

$$P(x) = P_0 + P_2x^2 + \sum_{i=2}^{\infty} P_{2i}x^{2i} = 0 + 2ax^2 + \sum_{i=2}^{\infty} P_{2i}x^{2i}$$

and,

$$-(4ab+2)x^2P_0(x) = -(4ab+2)\sum_{i=0}^{\infty} P_{2i}x^{2i+2} = -(4ab+2)\sum_{i=2}^{\infty} P_{2i-2}x^{2i}$$

$$x^4P_0(x) = \sum_{i=1}^{\infty} P_{2i}x^{2i+4} = \sum_{i=2}^{\infty} P_{2i-4}x^{2i}$$

Now by Lemma 3, it is obtained

$$[1-(4ab+2)x^2 + x^4]P_0(x) = 2ax^2 + \sum_{i=2}^{\infty}(P_{2i} - (4ab+2)P_{2i-2} + P_{2i-4})x^{2i}$$

$$P_0(x) = \frac{2ax^2}{1-(4ab+2)x^2 + x^4}$$

Now, generating function with odd terms is multiplied by $-(4ab+2)x^2$ and x^4 respectively, therefore

$$P_1(x) = P_1 x + P_3 x^3 + \sum_{i=2}^{\infty} P_{2i+1} x^{2i+1}$$

$$-(4ab+2)x^2 P_1(x) = -(4ab+2)x^2 P_1 x - (4ab+2)\sum_{i=2}^{\infty} P_{2i-1} x^{2i+1}$$

and

$$x^4 P_1(x) = \sum_{i=1}^{\infty} P_{2i-3} x^{2i+1}$$

$$P_1(x) = \frac{x + (4ab+1)x^3 - (4ab+2)x^3}{1-(4ab+2)x^2 + x^4} = \frac{x - x^3}{1-(4ab+2)x^2 + x^4}$$

Now $P(x)$ is obtained by adding $P_0(x)$, $P_1(x)$

$$P(x) = \frac{x + 2ax^2 - x^3}{1-(4ab+2)x^2 + x^4}$$

Theorem 6 We are able to express the generalized Pell numbers as a function of the roots α and β with the aid of Binet's formulas. The Binet formula for the bi-periodic Pell sequence is given by

$$P_n = \frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{9}$$

Where $\lfloor \alpha \rfloor$ is the floor function of a, $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function

Proof. It must be noted that the parity function can also be expressed as

$$\xi(m) = \begin{cases} 0, & if\ m\, is\, even \\ 1, & if\ m\, is\, odd \end{cases}$$

It can also be seen from the previous theorem that the generating function for the bi-periodic Pell sequence $\{P_n\}$ is given by

$$P(x) = \frac{x + 2ax^2 - x^3}{(x^2 - (2\alpha+1))(x^2 - (2\beta+1))} = \frac{Ax + B}{x^2 - (2\alpha+1)} + \frac{Cx + D}{x^2 - (2\beta+1)}$$

$$= I_1 + I_2$$

The Maclaurin series expansion of the function $\frac{A - Bz}{z^2 - C}$ is expressed in the form

$$\frac{A - Bz}{z^2 - C} = \sum_{n=0}^{\infty} BC^{-n-1} z^{2n+1} - \sum_{n=0}^{\infty} AC^{-n-1} z^{2n}$$

and hence following the same order, the generating function I_1 can be expanded as

$$I_1 = \sum_{n=0}^{\infty} -\alpha(2\alpha+1)^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty} \alpha(2\alpha+1)(2\alpha+1)^{-n-1} x^{2n}$$

$$I_2 = \sum_{n=0}^{\infty} \beta(2\alpha+1)^{-n-1} x^{2n+1} + \sum_{n=0}^{\infty} \alpha(2\beta+1)(2\beta+1)^{-n-1} x^{2n}$$

By using the identitiy (6), it is seen that

$$P(x) = \left[\frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{(2\alpha+1)^{n+1}} + \frac{\beta}{(2\beta+1)^{n+1}} \right) x^{2n+1} - \sum_{n=0}^{\infty} \left(\frac{\alpha}{(2\alpha+1)^n} - \frac{\alpha}{(2\beta+1)^n} \right) x^{2n} \right]$$

For the even part

$$P(x) = \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \left(\left(\frac{-\alpha}{\beta} \right)^n - \left(\frac{-\beta}{\alpha} \right)^n \right) x^{2n}$$

$$= \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \alpha(-1)^n \frac{\alpha^{2n} - \beta^{2n}}{(-ab)^n} x^{2n+1}$$

$$= \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1} - \beta^{2n+1}}{(ab)^n} x^{2n+1}$$

Hence by comparing the above with $P(x)=\sum_{n=0}^{\infty} P_n x^n$, it follows that,

$$P_n(x)=\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor}}\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

which completes the proof. ■

Theorem 7 The limit of every two consecutive terms of bi-periodic Pell sequence is generalized as

$$\lim_{n\to\infty}\frac{P_{2n+1}}{P_{2n}}=\frac{\alpha}{a}, \qquad \lim_{n\to\infty}\frac{P_{2n}}{P_{2n-1}}=\frac{\alpha}{b},$$

Proof: Taking into account that $|\beta|<\alpha$ and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0$, we have

$$\lim_{n\to\infty}\frac{P_{2n+1}}{P_{2n}}=\lim_{n\to\infty}\frac{\frac{1}{(ab)^n}\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right)P_{2n+1}}{\frac{a}{(ab)^n}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{\alpha}\frac{\alpha^{2n+1}\left(1-\left(\frac{\beta}{\alpha}\right)^{2n+1}\right)}{\alpha^{2n}\left(1-\left(\frac{\beta}{\alpha}\right)^{2n}\right)}=\frac{\alpha}{a}$$

$$\lim_{n\to\infty}\frac{P_{2n}}{P_{2n-1}}=\lim_{n\to\infty}\frac{\frac{a}{(ab)^n}\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}}{\frac{1}{(ab)^{n-1}}\frac{\alpha^{2n-1}-\beta^{2n-1}}{\alpha-\beta}}$$

$$=\lim_{n\to\infty}\frac{\alpha}{ab}\frac{\alpha^{2n}}{\alpha^{2n-1}}\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n}}{1-\left(\frac{\beta}{\alpha}\right)^{2n-1}}=\frac{\alpha}{b}$$

which completes the proof. From this theorem we can conclude that the biperiodic Pell sequence does not converge. ■

Theorem 8 The relationship between positive terms and their corresponding negative terms of the bi-periodic Pell sequence is obtained as

$$P_{-n}=(-1)^{n+1}P_n$$

Proof: By using Binet.s formula, the result is easily seen

$$P_{-n}=\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\frac{\frac{1}{\alpha^n}-\frac{1}{\beta^n}}{\alpha-\beta}=\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\frac{\beta^n-\alpha^n}{(-ab)^n(\alpha-\beta)}$$

$$=\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor}}\frac{\beta^n-\alpha^n}{(\alpha-\beta)}=(-1)^{n+1}P_n$$

Corollary 9 Characterictic polynomial is also obtained by using the limit of consecutive terms as

$$\frac{\alpha}{a} = \lim_{n\to\infty} \frac{P_{2n+1}}{P_{2n}} = \lim_{n\to\infty} \frac{2bP_{2n} + P_{2n-1}}{P_{2n}}$$
$$= \lim_{n\to\infty} \left(2b + \frac{P_{2n-1}}{P_{2n}}\right) = \left(2b + \frac{b}{\alpha}\right)$$

And by algebraic operations

$$\alpha^2 - 2ab\alpha - ab = 0.$$

Theorem 10 Related binomial summation formulas for bi-periodic Pell sequences are given as

$$\sum_{n=0}^{\infty} \binom{n}{k} 2^k a^{\xi(k)} (ab)^{\left\lfloor \frac{k}{2} \right\rfloor} P_k = P_{2n},$$
$$\sum_{n=0}^{\infty} \binom{n}{k} 2^k a^{\xi(k+1)} (ab)^{\left\lfloor \frac{k+1}{2} \right\rfloor -1} P_{k+1} = P_{2n+1}.$$

Proof.

$$\sum_{n=0}^{\infty} \binom{n}{k} 2^k a^{\xi(k)} (ab)^{\left\lfloor \frac{k}{2} \right\rfloor} P_k$$
$$= \frac{a}{\alpha-\beta} \sum_{n=0}^{\infty} \binom{n}{k} (2\alpha)^k - \frac{a}{\alpha-\beta} \sum_{n=0}^{\infty} \binom{n}{k} (2\beta)^k$$
$$= \frac{a}{\alpha-\beta} \left((2\alpha+1)^n - (2\beta+1)^n\right)$$
$$= \frac{a}{\alpha-\beta} \left[\left(\frac{\alpha^2}{ab}\right)^n - \left(\frac{\beta^2}{ab}\right)^n\right]$$
$$= \frac{a}{(ab)^n} \frac{\alpha^{2n} - \beta^{2n}}{\alpha-\beta} = P_{2n}$$

Similarly

$$\sum_{n=0}^{\infty} \binom{n}{k} 2^k a^{\xi(k+1)} (ab)^{\left\lfloor \frac{k+1}{2} \right\rfloor} P_{k+1}$$
$$= \frac{1}{b(\alpha-\beta)} \sum_{n=0}^{\infty} \binom{n}{k} 2^k \left(\alpha^{k+1} - \beta^{k+1}\right)$$
$$= \frac{a}{b(\alpha-\beta)} \sum_{n=0}^{\infty} \binom{n}{k} (2\alpha)^k - \frac{\beta}{b(\alpha-\beta)} \sum_{n=0}^{\infty} \binom{n}{k} (2\beta)^k$$
$$= \frac{\alpha\left(\frac{\alpha^2}{ab}\right)^n - \beta\left(\frac{\beta^2}{ab}\right)^n}{(\alpha-\beta)}$$
$$= \alpha \frac{\alpha^{2n+1} - \beta^{2n+1}}{(ab)^n (\alpha-\beta)} = P_{2n+1}$$

■

Theorem 11 (Catalan Identity)

For every numbers *n* and *r*, belonging to the set of positive integers ($\mathbb{Z}^+$), with $n \geq r$ we have

$$\frac{1}{a^2}\left(\frac{a}{b}\right)^{\xi(n+r)} P_{n-r} P_{n+r} - \frac{1}{a^2}\left(\frac{a}{b}\right)^{\xi(n)} P_n^2 = \frac{(-1)^{n-r}}{(ab)^r} P_r^2 \tag{10}$$

Proof. First of all let us take into account the following properties

$$\xi(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor,$$

$$\left\lfloor\frac{n-1}{2}\right\rfloor-\xi(n)=\left\lfloor\frac{n}{2}\right\rfloor$$
$$\xi(n+r)+\left\lfloor\frac{n-r}{2}\right\rfloor+\left\lfloor\frac{n+r}{2}\right\rfloor=n,$$
$$\xi(n+r)-\left\lfloor\frac{n-r+1}{2}\right\rfloor+\left\lfloor\frac{n+r+1}{2}\right\rfloor=-n,$$

By using Binet's formula, for the first part of the left handside of the equality is obtained

$$I_1=\frac{1}{a^2}\left(\frac{a}{b}\right)^{\xi(n+r)}P_{n-r}P_{n+r}$$
$$=\frac{1}{a^2}\left(\frac{a}{b}\right)^{\xi(n+r)}\frac{a^{1-\xi(n-r)}a^{1-\xi(n+r)}}{(ab)^{\left\lfloor\frac{n-r}{2}\right\rfloor}(ab)^{\left\lfloor\frac{n+r}{2}\right\rfloor}}\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}$$
$$=\frac{a^{\xi(n+r)-\left\lfloor\frac{n-r+1}{2}\right\rfloor-\left\lfloor\frac{n+r+1}{2}\right\rfloor}}{b^{\xi(n+r)+\left\lfloor\frac{n-r}{2}\right\rfloor+\left\lfloor\frac{n+r}{2}\right\rfloor}(\alpha-\beta)^2}\left(\alpha^{2n}+\beta^{2n}-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}\right)$$
$$=\frac{(ab)^{-n}\left(\alpha^{2n}+\beta^{2n}-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}\right)}{(\alpha-\beta)^2}$$

for the second part of the left handside of the equality, similarly

$$I_2=\frac{1}{a^2}\left(\frac{a}{b}\right)^{\xi(n)}P_n^2=\frac{1}{a^2}\left(\frac{a}{b}\right)^{\xi(n)}\frac{a^{2-2\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2$$
$$=\frac{a^{-\xi(n)-2\left\lfloor\frac{n}{2}\right\rfloor}}{b^{\xi(n)+2\left\lfloor\frac{n}{2}\right\rfloor}}\frac{\alpha^n-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}=(ab)^{-n}\frac{\alpha^n-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}$$

By these results, we get

$$I_1-I_2=\frac{(ab)^{-n}}{4ab(ab+1)}\left(-\alpha^{n-r}\beta^{n+r}-\alpha^{n+r}\beta^{n-r}+2(\alpha\beta)^n\right)$$
$$=\frac{(ab)^{-n}}{4ab(ab+1)}(\alpha\beta)^n\left(\frac{\beta^r}{\alpha^r}+\frac{\alpha^r}{\beta^r}+2\right)$$
$$=\left(\frac{\alpha\beta}{ab}\right)^n\frac{\left(\alpha^r-\beta^r\right)}{(\alpha-\beta)^2(\alpha\beta)^r}$$
$$=\frac{(-1)^{n-r}}{(ab)^r}P_r^2$$

By combinig the results, the proof is completed.

Theorem 12 (Cassini property or Simpson property)

For any number *n* belonging to the set of positive integers; we have

$$\left(\frac{a}{b}\right)^{\xi(n+1)}P_{n-1}P_{n+1}-\left(\frac{a}{b}\right)^{\xi(n)}P_n^2=\frac{(-1)^{n-1}a}{b} \qquad (11)$$

Proof. The proof is seen easily by chosing $r = 1$ in Catalan's Identity. ■

Theorem 13 (D'ocagne's property)

For any numbers *m* and *n*, belonging to the set of positive integers, with $m \geq n$, we have

$$a^{\xi(mn+m)}b^{\xi(mn+n)}P_mP_{n+1}-a^{\xi(mn+n)}b^{\xi(mn+m)}P_{m+1}P_n=a^{\xi(m-n)}(ab)^nP_{m-n} \qquad (12)$$

Proof. The following equalities have to be noted.

$$\xi(m)+\xi(n+1)-2\xi(mn+m)=\xi(m+1)+\xi(n)-2\xi(mn+n)=1-\xi(m-n) \quad (13)$$

$$\xi(m-n)=\xi(mn+m)+\xi(mn+n) \quad (14)$$

$$\frac{m-n-\xi(m-n)}{2}=\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor-\xi(mn+n) \quad (15)$$

$$\frac{m-n-\xi(m-n)}{2}=\left\lfloor\frac{m+1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor-\xi(mn+m) \quad (16)$$

By using the extended Binet.s formula, (13), (14), (15), (16), we have

$$\begin{aligned}A&=a^{\xi(mn+m)}b^{\xi(mn+n)}P_mP_{n+1}\\&=a^{\xi(mn+m)}b^{\xi(mn+n)}\frac{a^{1-\xi(m)}a^{1-\xi(n+1)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\\&=\frac{ab^{\xi(mn+n)}a^{1-\xi(m)-\xi(n+1)+\xi(mn+m)}a^{1-\xi(n+1)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor}}\frac{\alpha^{m+n+1}+\beta^{m+n+1}-\beta^m\alpha^{n+1}-\alpha^m\beta^{n+1}}{(\alpha-\beta)^2}\\&=\frac{a}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\frac{\alpha^{m+n+1}+\beta^{m+n+1}-(\alpha\beta)^n\left(\beta\alpha^{m-n}+\alpha\beta^{m-n}\right)}{(\alpha-\beta)^2}\end{aligned}$$

and

$$\begin{aligned}B&=a^{\xi(mn+n)}b^{\xi(mn+m)}P_{m+1}P_n\\&=a^{\xi(mn+n)}b^{\xi(mn+m)}\frac{a^{1-\xi(m+1)}a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\frac{\alpha^n-\beta^n}{\alpha-\beta}\\&=\frac{ab^{\xi(mn+m)}a^{1-\xi(m+1)-\xi(n)+\xi(mn+n)}a^{1-\xi(n+1)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor}}\frac{\alpha^{m+n+1}+\beta^{m+n+1}-\beta^n\alpha^{m+1}-\alpha^n\beta^{m+1}}{(\alpha-\beta)^2}\\&=\frac{a}{(ab)^{\frac{m+n-\xi(m-n)}{2}}}\frac{\alpha^{m+n+1}+\beta^{m+n+1}-\beta^n\alpha^{m+1}-\alpha^n\beta^{m+1}}{(\alpha-\beta)^2}\end{aligned}$$

$$\frac{m-n-\xi(m-n)}{2}=\left\lfloor\frac{m-n}{2}\right\rfloor \quad (17)$$

From the above results and (17), we obtain

$$\begin{aligned}A-B&=a^{\xi(mn+m)}b^{\xi(mn+n)}P_mP_{n+1}-a^{\xi(mn+n)}b^{\xi(mn+m)}P_{m+1}P_n\\&=\frac{a}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\frac{(\alpha-\beta)\left(\alpha^n\beta^m-\alpha^m\beta^n\right)}{(\alpha-\beta)^2}\\&=\frac{a}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\frac{(\alpha\beta)^n\left(\beta^{m-n}-\alpha^{m-n}\right)}{(\alpha-\beta)^2}\\&=\frac{(-1)^{n+1}a(ab)^n}{a^{1-\xi(m-n)}}\frac{a^{1-\xi(m-n)}}{(ab)^{\left\lfloor\frac{m-n}{2}\right\rfloor}}\frac{\alpha^{m-n}-\beta^{m-n}}{(\alpha-\beta)}\\&=(-1)^{n+1}a^{\xi(m-n)}P_{m-n}\end{aligned}$$

Theorem 14 The sum of first *n* terms of bi-periodic of Pell sequence is given as

$$\sum_{k=0}^{n-1}P_k=\frac{a^{\xi(n)}b^{-\xi(n)}P_n+a^{1-\xi(n)}b^{\xi(n)}P_{n-1}-(-1)^n a}{2ab}$$

Proof. Let *n* be even. By using Binet formula for bi-periodic of Pell sequence, we get

$$\sum_{k=0}^{n-1} P_k = \sum_{k=0}^{\frac{n-2}{2}} P_{2k} + \sum_{k=0}^{\frac{n-2}{2}} P_{2k+1}$$
$$= \sum_{k=0}^{\frac{n-2}{2}} \left\{ \frac{\alpha^{2k} - \beta^{k}}{(ab)^k (\alpha-\beta)} + \frac{\alpha^{2k+1} - \beta^{2k+1}}{(ab)^k (\alpha-\beta)} \right\}$$

If we use the property of geometric series, we get

$$= a\left(\frac{\alpha^n - (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2 - ab)} - \frac{\beta^n - (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2 - ab)} \right)$$
$$+ \left(\frac{\alpha^{n+1} - \alpha(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2 - ab)} - \frac{\beta^{n+1} - \beta(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2 - ab)} \right)$$

After some algebraic operations we have

$$= \frac{a}{-4(\alpha-\beta)(ab)^{\frac{n}{2}+2}} \left(a^2b^2(\alpha^{n-2} - \beta^{n-2}) - ab(\alpha^n - \beta^n) + (ab)^{\frac{n}{2}}(\alpha^2 - \beta^2) \right)$$
$$+ \frac{1}{(\alpha-\beta)(ab)^{\frac{n}{2}+2}} \left(a^2b^2(\alpha^{n-1} - \beta^{n-1}) - ab(\alpha^n - \beta^n) + (ab)^{\frac{n}{2}}(\alpha^2 - \beta^2) \right)$$
$$= \frac{P_{n-2} - P_n + 2a + P_{n-1} - P_{n+1}}{-4ab}$$
$$= \frac{bP_n + aP_{n-1} - a}{2ab}$$

If n is odd, the result is as the following

$$\sum_{k=0}^{n-1} P_k = \sum_{k=0}^{\frac{n-1}{2}} P_{2k} + \sum_{k=0}^{\frac{n-3}{2}} P_{2k+1} = \frac{aP_n + bP_{n-1} + a}{2ab}$$

If the results combine,

$$\sum_{k=0}^{n-1} P_k = \frac{a^{\xi(n)} b^{1-\xi(n)} P_n + a^{1-\xi(n)} b^{\xi(n)} P_{n-1} - (-1)^n a}{2ab}$$

Theorem 15 For any positive integer n, we have

$$\sum_{k=0}^{n-1} \frac{P_k}{x^k} = \frac{1}{\left(x^4 - (4ab+2)x^2 + 1\right)} \left\{ \begin{array}{l} \left(\frac{P_{n-2}}{x^{n-2}} - \frac{P_n}{x^{n-4}} \right) x^{\xi(n)} + \left(\frac{P_{n-1}}{x^{n-1}} - \frac{P_{n+1}}{x^{n-3}} \right) x^{1-\xi(n)} \\ +2ax^2 - x^2 + x^4 \end{array} \right\}$$

and

$$\sum_{k=0}^{\infty} \frac{P_k}{x^k} = \frac{2ax^2 - x^2 + x^4}{x^4 - (4ab+2)x^2 + 1}$$

3. Corollary

In this paper we define a new generalization for Pell sequence called bi-periodic Pell sequences in a way that is similar to the bi-periodic Fibonacci and Lucas sequences. We define two recurrence relations with respect to the parity of terms. In the recurrence relation we use two different real variables. We gave some properties of these new sequence such as Binet formula, generating function, Catalan and the D’ocagne’s identities and sum formulas.

References

[1] Edson, M. and Yayenie, O., 2009. "A new generalization of Fibonacci sequences and the extended Binet’s formula, INTEGERS Electron." *J. Comb. Number Theor.*, vol. 9, pp. 639-654.

[2] Yayenie, O., 2011. "A note on generalized Fibonacci sequence." *Appl. Math. Com- put.*, vol. 217, pp. 5603-5611.

[3] Jun, S. P. and Choi, K. H., 2016. "Some properties of the generalized fibonacci sequence by matrix methods." *Korean J. Math*, vol. 24, pp. 681-691.

[4] Bilgici, G., 2014. "Two generalizations of Lucas sequence." *Applied Mathematics and Computation*, vol. 245, pp. 526-538.

[5] Coskun, A. and Taskara, N., 2016. "The matrix sequence in terms of bi-periodic Fi¬bonacci numbers, arXiv:1603.07487v2 [math.NT]."

[6] Coskun, A., Yilmaz, N., and Taskara, N., 2016. "A note on the bi-periodic Fibonacci and Lucas matrix sequences, arXiv:1604.00766v1 [math.NT]."

[7] Uygun, S. and Owusu, E., 2016. "A new generalization of jacobsthal numbers (bi¬periodic jacobsthal sequences)." *Journal of Mathematical Analysis*, vol. 7, pp. 28-39.

[8] Uygun, S. and Owusu, E., 2017. "Matrix Representation of Bi-Periodic Jacobsthal, arXiv:1702.00604v1 [math.CO]."

[9] Horadam, A. F., 1971. "Pell Identities." *Fibonacci Quarterly*, vol. 9, pp. 245-252.