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On The Ternary Quadratic Equation $x^2+y^2=z^2+141$

A. Vijayasankar
Assistant Professor, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamil Nadu, India

Sharadha Kumar (Corresponding Author)
Research Scholar, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamil Nadu, India
Email: sharadhak12@gmail.com

M. A. Gopalan
Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India

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Abstract

This article concerns with the method of determining different solutions in integers to $x^2+y^2=z^2+141$ by reducing it to $\beta^2=D\alpha^2+141$ $(D>0 \text{ and square-free})$ through employing transformations. A special case has been illustrated along with the corresponding properties. Also, given an integer solution, a process of obtaining sequence of integer solutions based on its given solution is exhibited.

Keywords: Ternary quadratic; Non-homogeneous quadratic; Integer solutions; Pell equation.

1. Introduction

It is well known that ternary second degree diophantine equations are rich in variety [1-15]. This paper presents different solutions in integers to the ternary quadratic equation $x^2+y^2=z^2+141$. Some properties among the solutions are exhibited. A generation formula for exhibiting solutions in integers is presented.

2. Method of Analysis

Consider the second degree equation with three variables

$$x^2+y^2=z^2+141 \tag{1}$$

The introduction of the transformations

$$\left.\begin{aligned} x&=(2k^2+20k-21)\alpha,\\ z&=(2k^2+20k-20)\alpha, k>0, \alpha\neq 0\end{aligned}\right\} \tag{2}$$

in (1) gives

$$y^2=(4k^2+40k-41)\alpha^2+141 \tag{3}$$

which represents the positive pell equation. The initial positive integer solution to (3) is $\alpha_0=1, y_0=2k+10$

To obtain the other integer solutions to (3), consider the corresponding pell equation

$$y^2=(4k^2+40k-41)\alpha^2+1 \tag{4}$$

whose least positive integer solution is $(\tilde{\alpha}_0, \tilde{y}_0)$.

The general solution $(\tilde{\alpha}_n, \tilde{y}_n)$ of (4) is given by

$$\tilde{y}_n=\frac{1}{2}f_n \tag{5}$$

$$\tilde{\alpha}_n=\frac{1}{2\sqrt{4k^2+40k-41}}g_n \tag{6}$$

where

$$f_n=\left(\tilde{y}_0+\sqrt{4k^2+40k-41}\,\tilde{\alpha}_0\right)^{n+1}+\left(\tilde{y}_0-\sqrt{4k^2+40k-41}\,\tilde{\alpha}_0\right)^{n+1}, n=-1,0,1,2.......$$

$$g_n = \left(\tilde{y}_0 + \sqrt{4k^2 + 40k - 41}\ \tilde{\alpha}_0\right)^{n+1} - \left(\tilde{y}_0 - \sqrt{4k^2 + 40k - 41}\ \tilde{\alpha}_0\right)^{n+1}, n = -1,0,1,2.......$$

Employing the lemma of Brahmagupta between the solutions (α_0, y_0) and $(\tilde{\alpha}_n, \tilde{y}_n)$, the other solutions to (3) are represented by

$$\alpha_{n+1} = \alpha_0\tilde{y}_n + y_0\tilde{\alpha}_n\ ,\ n = -1,0,1,2..... \quad (7)$$

$$y_{n+1} = y_0\tilde{y}_n + (4k^2 + 40k - 41)\alpha_0\tilde{\alpha}_n\ , n = -1,0,1,2,..... \quad (8)$$

To study the properties among the solutions, one has to go for particular values of k. For simplicity and brevity the choice $k = 1$ in (3), (4), (5) and (6) correspondingly leads to

$$y^2 = 3\alpha^2 + 141,\ \alpha_0 = 1\ , y_0 = 12$$

$$y^2 = 3\alpha^2 + 1,\ \tilde{\alpha}_0 = 1\ , \tilde{y}_0 = 2$$

$$\tilde{y}_n = \frac{1}{2} f_n\ , f_n = \left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right]$$

$$\tilde{\alpha}_n = \frac{1}{2\sqrt{3}} g_n\ , g_n = \left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right]\ , n = -1,0,1........$$

$$\alpha_{n+1} = \frac{1}{2} f_n + 2\sqrt{3}\, g_n \quad (9)$$

$$y_{n+1} = 6 f_n + \frac{1}{2}\sqrt{3}\, g_n \quad (10)$$

Substituting $k = 1$ in (2) and using (9), we get

$$x_{n+1} = \frac{1}{2} f_n + 2\sqrt{3}\, g_n \quad (11)$$

$$z_{n+1} = f_n + 4\sqrt{3}\, g_n \quad (12)$$

Thus, (10), (11) and (12) represent different positive solution in integers to (1).

A few numerical examples are given in the following table 1 below:

Table-1. Numerical Examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	1	12	2
0	14	27	28
1	55	96	110
2	206	357	412
3	769	1332	1538
4	2870	4971	5740
5	10711	18552	21422
6	39974	69237	79948

From the above table, one may generate Ramanujan numbers of second order from suitable values of x, y and z

2.1. Illustration

$$y_2 = 96 = 2*48 = 4*24 = 6*16 = 8*12$$

$$= 25^2 - 23^2 = 14^2 - 10^2 = 11^2 - 5^2 = 10^2 - 2^2$$

$$25^2 - 23^2 = 14^2 - 10^2 \Rightarrow 25^2 + 10^2 = 23^2 + 14^2 = 725$$

$$25^2 - 23^2 = 11^2 - 5^2 \Rightarrow 25^2 + 5^2 = 23^2 + 11^2 = 650$$

$$25^2 - 23^2 = 10^2 - 2^2 \Rightarrow 25^2 + 2^2 = 23^2 + 10^2 = 629$$

$$14^2 - 10^2 = 11^2 - 5^2 \Rightarrow 14^2 + 5^2 = 10^2 + 11^2 = 221$$

$$11^2 - 5^2 = 10^2 - 2^2 \Rightarrow 11^2 + 2^2 = 5^2 + 10^2 = 125$$

Thus, $725, 650, 629, 221, 125$ are Ramanujan numbers of second order.

Recurrence relations for x, y and z are:

$$x_{n+3} - 4x_{n+2} + x_{n+1} = 0, n = -1, 0, 1.....$$

$$y_{n+3} - 4y_{n+2} + y_{n+1} = 0, n = -1, 0, 1.....$$

$$z_{n+3} - 4z_{n+2} + z_{n+1} = 0, n = -1, 0, 1.....$$

Some combinations between the solutions are given below:

i. $y_{n+1} - x_{n+2} + 2x_{n+1} = 0$

ii. $y_{n+3} - 7x_{n+2} + 2x_{n+1} = 0$

iii. $4y_{n+1} - x_{n+3} + 7x_{n+1} = 0$

iv. $2y_{n+2} - x_{n+3} + x_{n+1} = 0$

Cubical integer:

i. $\frac{1}{47}\left[(8x_{3n+4} - 18x_{3n+3}) + 3(8x_{n+2} - 18x_{n+1})\right]$

ii. $\frac{1}{188}\left[(8x_{3n+5} - 64x_{3n+3}) + 3(8x_{n+3} - 64x_{n+1})\right]$

iii. $\frac{1}{47}\left[(8y_{3n+3} - 2x_{3n+3}) + 3(8y_{n+1} - 2x_{n+1})\right]$

iv. $\frac{1}{94}\left[(8y_{3n+4} - 28y_{3n+3}) + 3(8y_{n+2} - 28x_{n+1})\right]$

Bi-quadratic integer:

i. $\frac{1}{47^2}\left[(376x_{4n+5} - 846x_{4n+4}) + 4(8x_{n+2} - 18x_{n+1})^2 - 4418\right]$

ii. $\frac{1}{188^2}\left[(1504x_{4n+6} - 12032x_{4n+4}) + 4(8x_{n+3} - 64x_{n+1})^2 - 70688\right]$

iii. $\frac{1}{47^2}\left[(376y_{4n+4} - 94x_{4n+4}) + 4(8y_{n+1} - 2x_{n+1})^2 - 4418\right]$

iv. $\frac{1}{94^2}\left[(752y_{4n+5} - 2632x_{4n+4}) + 4(8y_{n+2} - 28x_{n+1})^2 - 17672\right]$

Nasty number:

i. $\frac{1}{47}\left[564 + 48x_{2n+3} - 108x_{2n+2}\right]$

ii. $\frac{1}{188}\left[2256 + 48x_{2n+4} - 384x_{2n+2}\right]$

iii. $\frac{1}{47}\left[564 + 48y_{2n+2} - 12x_{2n+2}\right]$

iv. $\frac{1}{94}\left[1128 + 48y_{2n+3} - 168x_{2n+2}\right]$

2.2. Remarkable Observations

I. Choices of hyperbola with their solutions generated through the known solutions are in Table 2 below:

Table-2. Hyperbola

Sl.no	Hyperbola	(X_n, Y_n)
1	$3X_n^2 - Y_n^2 = 26508$	$[(8x_{n+2} - 18x_{n+1}), (28x_{n+1} - 2x_{n+2})]$
2	$3X_n^2 - Y_n^2 = 424128$	$[(8x_{n+3} - 64x_{n+1}), (110x_{n+1} - 2x_{n+3})]$
3	$3X_n^2 - Y_n^2 = 26508$	$[(8y_{n+1} - 2x_{n+1}), (24x_{n+1} - 2y_{n+1})]$
4	$3X_n^2 - Y_n^2 = 106032$	$[(8y_{n+2} - 28x_{n+1}), (54x_{n+1} - 2y_{n+2})]$

II. Employing linear combination among the solutions other choices of parabola are presented in Table 3 below:

Table-3. Parabola

Sl.no	Parabola	(X_n, Y_n)
1	$141X_n - Y_n^2 = 26508$	$[(94+8x_{2n+3} - 18x_{2n+2}),(28x_{n+1} - 2x_{n+2})]$
2	$564X_n - Y_n^2 = 424128$	$[(376+8x_{2n+4} - 64x_{2n+2}),(110x_{n+1} - 2x_{n+3})]$
3	$141X_n - Y_n^2 = 26508$	$[(94+8y_{2n+2} - 2x_{2n+2}),(24x_{n+1} - 2y_{n+1})]$
4	$282X_n - Y_n^2 = 106032$	$[(188+8y_{2n+3} - 28x_{2n+2}),(54x_{n+1} - 2y_{n+2})]$

2.3. Generation of Solutions

Let (x_0, y_0, z_0) be a known solution of (1).

Consider the second solution (x_1, y_1, z_1) of (1) to be

$$x_1 = h - x_0, y_1 = h - y_0, z_1 = h + z_0 \tag{13}$$

where h is a non-zero integer to be determined.

Substituting (13) in (1) and simplifying, we get

$$h = 2(x_0 + y_0 + z_0) \tag{14}$$

Using (14) in (13), the second solution of (1) is represented in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where $M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ and t is the transpose

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) in the matrix form as

$$(x_{n+1}, y_{n+1}, z_{n+1})^t = \tilde{M}(x_0, y_0, z_0)^t, \tag{15}$$

where

$$\tilde{M} = \begin{pmatrix} \frac{Y_n - (-1)^n}{2} & \frac{Y_n + (-1)^n}{2} & X_n \\ \frac{Y_n + (-1)^n}{2} & \frac{Y_n - (-1)^n}{2} & X_n \\ X_n & X_n & Y_n \end{pmatrix}, n = 0,1,2......$$

in which (x_n, y_n) represents the general solution of the pell equation $Y^2 = 2X^2 + 1$.

Thus, given an integer solution (x_0, y_0, z_0), one may generate sequence of integer solutions to the given equation based on the known solution through employing (15).

2.4. Remark

In addition to (2), one may introduce the transformations

$$x = \frac{1}{2}(k^2 + 11k - 6)\alpha \ , \ z = \frac{1}{2}(k^2 + 11k - 4)\alpha$$

in (1) leading to

$$y^2 = (k^2 + 11k - 5)\alpha^2 + 141, \ \alpha_0 = 2, y_0 = 2k + 11$$

Following the procedure presented above, another set of integer solutions to (1) are obtained.

3. Conclusion

This paper presents a set of integer solutions to the second order equation with three variables $x^2 + y^2 = z^2 + 141$. However, there may be other sets of solutions to (1) which is left as an exercise for the readers.

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