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On the Positive Pell Equation $y^2 = 11x^2 + 22$

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Abstract

The binary quadratic equation $y^2 = 11x^2 + 22$ representing the hyperbola is studied for its non-zero distinct integer solutions. A few interesting properties among the solutions are presented. Employing the integer solutions of the equation under consideration, integer solutions for special straight lines, hyperbolas and parabolas are exhibited.

Keywords: Binary quadratic; Hyperbola; Parabola; Integer solutions; Pell equation.

1. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-11]. In this communication, yet another interesting hyperbola given by $y^2 = 11x^2 + 22$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. Method of Analysis

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 11x^2 + 22 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 3,\ y_0 = 11$$

To obtain the other solution of (1), consider the Pell equation

$$y^2 = 11x^2 + 1$$

The initial solution of Pell equation is

$$\tilde{x}_0 = 3,\ \tilde{y}_0 = 10$$

whose general solution is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{11}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where,
$$f_n = (10 + 3\sqrt{11})^{n+1} + (10 - 3\sqrt{11})^{n+1}$$
$$g_n = (10 + 3\sqrt{11})^{n+1} - (10 - 3\sqrt{11})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2\sqrt{11}}[3\sqrt{11} f_n + 11 g_n] \qquad (2)$$

$$y_{n+1} = \frac{1}{2\sqrt{11}}[11\sqrt{11} f_n + 33 g_n] \qquad (3)$$

where n= -1,0,2,…

Thus (2) and (3) represent non- zero distinct integral solutions of (1) which represents a hyperbola. The recurrence relation satisfied by x and y are given by,

$$x_{n+3} - 20x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 20y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the following Table (1) below:

Table-1. Numerical Values

n	x_{n+1}	y_{n+1}
-1	3	11
0	63	209
1	1257	4169
2	25077	83171
3	500283	1659251

From the above table, we observe some interesting relations among the solutions which are presented below.

- x_{n+1} values are odd and $x_{n+1} \equiv 0 (\mathrm{mod}\, 3)$
- y_{n+1} values are odd and $y_{n+1} \equiv 0 (\mathrm{mod}\, 11)$

2.1. Each of the following Expression is a Nasty Number

- $6[y_{2n+2} - 3x_{2n+2} + 2]$
- $[2x_{2n+3} - 38x_{2n+3} + 12]$
- $\frac{1}{10}[x_{2n+4} - 379x_{2n+2} + 120]$
- $\frac{6}{10}[y_{2n+3} - 63x_{2n+2} + 20]$
- $\frac{6}{119}[y_{2n+4} - 1257x_{2n+2} + 398]$
- $\frac{1}{220}[2508y_{2n+2} - 396x_{2n+3} + 2640]$
- $\frac{1}{2189}[25014y_{2n+2} - 198x_{2n+4} + 26268]$
- $\frac{1}{33}[378y_{2n+2} - 18y_{2n+3} + 132]$
- $\frac{1}{660}[7542y_{2n+2} - 18y_{2n+4} + 7920]$
- $\frac{1}{33}[1254x_{2n+4} - 25014x_{2n+3} + 396]$
- $\frac{1}{11}[1254y_{2n+3} - 4158x_{2n+3} + 132]$
- $\frac{1}{110}[1254y_{2n+4} - 82962x_{2n+3} + 1320]$
- $\frac{1}{110}[25014y_{2n+3} - 4158x_{2n+4} + 1320]$
- $\frac{1}{11}[25014y_{2n+4} - 82962x_{2n+4} + 132]$
- $\frac{1}{363}[82962y_{2n+3} - 4158y_{2n+4} + 4356]$

2.2. Each of the following Expressions is a Cubical Integer

- $\left[y_{3n+3}-3x_{3n+3}+3y_{n+1}-9x_{n+1}\right]$
- $\frac{1}{3}\left[x_{3n+4}-19x_{3n+3}-57x_{n+1}+3x_{n+2}\right]$
- $\frac{1}{60}\left[x_{3n+5}-379x_{3n+3}-1137x_{n+1}+3x_{n+3}\right]$
- $\frac{1}{10}\left[y_{3n+4}-63x_{3n+3}-189x_{n+1}+3y_{n+2}\right]$
- $\frac{1}{199}\left[3y_{n+3}-3771x_{n+1}-1257x_{3n+3}+y_{3n+5}\right]$
- $\frac{1}{220}\left[1254y_{n+1}-198x_{n+2}-66x_{3n+4}+418y_{3n+3}\right]$
- $\frac{1}{2189}\left[12507y_{n+1}-99x_{n+3}-33x_{3n+5}+4169y_{3n+3}\right]$
- $\frac{1}{33}\left[189y_{n+1}-9y_{n+2}-3y_{3n+4}+63y_{3n+3}\right]$
- $\frac{1}{660}\left[3771y_{n+1}-9y_{n+3}-3y_{3n+5}+1257y_{3n+3}\right]$
- $\frac{1}{33}\left[627x_{n+3}-12507x_{n+2}-4169x_{3n+4}+209x_{3n+5}\right]$
- $\frac{1}{11}\left[627y_{n+2}-2079x_{n+2}-693x_{3n+4}+209y_{3n+4}\right]$
- $\frac{1}{110}\left[627y_{n+3}-41481x_{n+2}-13827x_{3n+5}+209y_{3n+5}\right]$
- $\frac{1}{110}\left[12507y_{n+2\backslash}-2079x_{n+3}+4169y_{3n+4}-693x_{3n+5}\right]$
- $\frac{1}{11}\left[12507y_{n+3\backslash}-41481x_{n+3}+4169y_{3n+5}-13827x_{3n+5}\right]$
- $\frac{1}{363}\left[41481y_{n+2\backslash}-2079y_{n+3}+13827y_{3n+4}-693y_{3n+5}\right]$

2.3. Each of the following Expressions is a Biquadratic Integer

- $\left[y_{4n+4}-3x_{4n+4}-12x_{2n+2}+4y_{2n+2}+6\right]$
- $\frac{1}{3}\left[x_{4n+5}-19x_{4n+4}-76x_{2n+2}+4x_{2n+3}+18\right]$
- $\frac{1}{60}\left[x_{4n+6}-379x_{4n+4}-1516x_{2n+2}+4x_{2n+4}+360\right]$
- $\frac{1}{10}\left[y_{4n+5}-63x_{4n+4}-252x_{2n+2}+4y_{2n+3}+60\right]$
- $\frac{1}{199}\left[y_{4n+6}-1257x_{4n+4}-5028x_{2n+2}+4y_{2n+4}+1194\right]$
- $\frac{1}{220}\left[418y_{4n+4}-66x_{4n+5}-264x_{2n+3}+1672y_{2n+2}+1320\right]$
- $\frac{1}{2189}\left[4169y_{4n+4}-33x_{4n+6}-132x_{2n+4}+16676y_{2n+2}+13134\right]$
- $\frac{1}{33}\left[63y_{4n+4}-3x_{4n+5}-12y_{2n+3}+252y_{2n+3}+198\right]$

- $\frac{1}{660}\left[1257y_{4n+4}-3y_{4n+6}-12y_{2n+4}+5028y_{2n+2}+3960\right]$
- $\frac{1}{33}\left[209x_{4n+6}-4169x_{4n+5}-16676x_{2n+3}+836x_{2n+4}+198\right]$
- $\frac{1}{11}\left[209y_{4n+5}-693x_{4n+5}-2772x_{2n+3}+836y_{2n+3}+66\right]$
- $\frac{1}{110}\left[209y_{4n+6}-13827x_{4n+5}-55308x_{2n+3}+836y_{2n+4}+660\right]$
- $\frac{1}{110}\left[4169y_{4n+5}-693x_{4n+6}+16676y_{2n+3}-2772x_{2n+4}+660\right]$
- $\frac{1}{11}\left[4169y_{4n+6}-13827x_{4n+6}+16676y_{2n+4}-55308y_{2n+4}+66\right]$
- $\frac{1}{363}\left[13827y_{4n+5}-693y_{4n+6}+55308y_{2n+3}-2772y_{2n+4}+2178\right]$

2.4. Each of the following Expression is a Quintic integer

- $\left[y_{5n+5}-3x_{5n+5}+5y_{3n+3}-15x_{3n+3}+10y_{n+1}-30x_{n+1}\right]$
- $\frac{1}{60}\left[x_{5n+7}-379x_{5n+5}+5x_{3n+5}-1895x_{3n+3}-3790x_{n+1}+10x_{n+3}\right]$
- $\frac{1}{10}\left[y_{5n+6}-63x_{5n+5}+5y_{3n+4}-315x_{3n+3}-630x_{n+1}+10y_{n+2}\right]$
- $\frac{1}{199}\left[y_{5n+7}-1257x_{5n+5}+10y_{n+3}-12570x_{n+1}-6285x_{3n+3}+5y_{3n+5}\right]$
- $\frac{1}{220}\begin{bmatrix}418y_{5n+5}-66x_{5n+6}+4180y_{n+1}-660x_{n+2}-330x_{3n+4}\\+2090y_{3n+3}\end{bmatrix}$
- $\frac{1}{2189}\begin{bmatrix}4169y_{5n+5}-33x_{5n+7}+41690y_{n+1}-330x_{n+3}-165x_{3n+5}\\+20845y_{3n+3}\end{bmatrix}$
- $\frac{1}{33}\left[63y_{5n+5}-3y_{5n+6}+630y_{n+1}-30y_{n+2}-15y_{3n+4}+315y_{3n+3}\right]$
- $\frac{1}{660}\begin{bmatrix}1257y_{5n+5}-3y_{5n+7}+12570y_{n+1}-30y_{n+3}-15y_{3n+5}\\+6285y_{3n+3}\end{bmatrix}$
- $\frac{1}{33}\begin{bmatrix}209x_{5n+7}-4169x_{5n+6}+2090x_{n+3}-41690x_{n+2}-20845x_{3n+4}\\+1045x_{3n+5}\end{bmatrix}$
- $\frac{1}{11}\begin{bmatrix}209y_{5n+6}-693x_{5n+6}+2090y_{n+2}-6930x_{n+2}-3465x_{3n+4}\\+1045y_{3n+4}\end{bmatrix}$
- $\frac{1}{110}\begin{bmatrix}209y_{5n+7}-13827x_{5n+6}+2090y_{n+3}-138270x_{n+2}\\-69135x_{3n+5}\end{bmatrix}$
- $\frac{1}{110}\begin{bmatrix}4169y_{5n+6}-693x_{5n+7}+41690y_{n+2}-6930x_{n+3}\\+20845y_{3n+4}-3465x_{3n+5}\end{bmatrix}$
- $\frac{1}{11}\begin{bmatrix}4169y_{5n+7}-13827x_{5n+7}+41690y_{n+3\backslash}-138270x_{n+3}\\+20845y_{3n+5}-69135x_{3n+5}\end{bmatrix}$
- $\frac{1}{363}\begin{bmatrix}13827y_{5n+6}-693y_{5n+7}+138270y_{n+2}-6930y_{n+3}\\+69135y_{3n+4}-3465y_{3n+5}\end{bmatrix}$
- $\left[y_{5n+5}-3x_{5n+5}+5y_{3n+3}-15x_{3n+3}+10y_{n+1}-30x_{n+1}\right]$

2.5. Relations among the Solutions are Given Below

- $3y_{n+1} + 10x_{n+1} - x_{n+2} = 0$
- $60y_{n+1} + 199x_{n+1} - x_{n+3} = 0$
- $10y_{n+1} + 33x_{n+1} - y_{n+2} = 0$
- $199y_{n+1} + 660x_{n+1} - y_{n+3} = 0$
- $x_{n+3} + x_{n+1} - 20x_{n+2} = 0$
- $3y_{n+2} + x_{n+1} - 10x_{n+2} = 0$
- $3y_{n+3} + 10x_{n+1} - 199x_{n+2} = 0$
- $6y_{n+2} + x_{n+1} - x_{n+3} = 0$
- $199y_{n+2} + 33x_{n+1} - 10y_{n+3} = 0$
- $60y_{n+3} + x_{n+1} - 199x_{n+3} = 0$
- $3y_{n+1} + 199x_{n+2} - 10x_{n+3} = 0$
- $y_{n+1} + 66x_{n+2} - 10y_{n+2} = 0$
- $y_{n+1} + 66x_{n+2} - y_{n+3} = 0$
- $199x_{n+2} + 33y_{n+1} - 10x_{n+3} = 0$
- $199y_{n+2} - 33x_{n+3} - 10y_{n+1} = 0$
- $y_{n+1} + 660x_{n+3} - 199y_{n+3} = 0$
- $y_{n+1} + 33x_{n+3} - 10y_{n+2} = 0$
- $y_{n+3} + y_{n+1} - 20y_{n+2} = 0$
- $199y_{n+3} + 660x_{n+1} - y_{n+1} = 0$
- $33y_{n+2} + 10x_{n+2} - x_{n+3} = 0$
- $33y_{n+3} + x_{n+2} - 10x_{n+3} = 0$
- $33y_{n+2} + x_{n+1} - 10x_{n+2} = 0$
- $y_{n+1} + 33x_{n+2} - 10y_{n+2} = 0$
- $10y_{n+2} + 33x_{n+2} - y_{n+3} = 0$
- $33y_{n+3} + 10x_{n+1} - 199x_{n+2} = 0$
- $33x_{n+3} + y_{n+2} - 10y_{n+3} = 0$
- $33x_{n+2} + 10y_{n+2} - y_{n+3} = 0$

3. Remarkable Observations

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of straight line which are presented in the table 2 below:

Table-2. Straight Line

S.No:	Straight Line	(X,Y)
1.	$20X = Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = x_{n+3} - 379x_{n+1})$
2.	$10X = 3Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = y_{n+2} - 63x_{n+1})$
3.	$199X = 3Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = y_{n+3} - 1257x_{n+1})$
4.	$220X = 3Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = 418y_{n+1} - 66x_{n+2})$
5.	$2189X = 3Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = 4169y_{n+1} - 33x_{n+2})$
6.	$11X = Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = 63y_{n+1} - 3y_{n+2})$
7.	$220X = Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = 1257y_{n+1} - 3y_{n+3})$
8.	$110X = 3Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = 209y_{n+3} - 13827x_{n+2})$
9.	$11X = 3Y$	$(X = x_{n+2} - 19x_{n+1},\ Y = 4169y_{n+3} - 13827x_{n+3})$

10.	$6X = Y$	$(X = y_{n+2} - 63x_{n+1},\ Y = x_{n+3} - 379x_{n+1})$
11.	$199X = 60Y$	$(X = x_{n+3} - 379x_{n+1},\ Y = y_{n+3} - 1257x_{n+1})$
12.	$11X = 3Y$	$(X = x_{n+3} - 379x_{n+1},\ Y = 418y_{n+1} - 66x_{n+2})$
13.	$2189X = 60Y$	$(X = x_{n+3} - 379x_{n+1},\ Y = 4169y_{n+1} - 33x_{n+3})$
14.	$10X = Y$	$(X = 209y_{n+2} - 693x_{n+2},\ Y = 209y_{n+3} - 13827x_{n+2})$
15.	$X = 3Y$	$(X = 209x_{n+3} - 4169x_{n+2},\ Y = 209y_{n+2} - 693x_{n+2})$
16.	$11X = Y$	$(X = y_{n+3} - 1257x_{n+1},\ Y = 4169y_{n+1} - 33x_{n+3})$
17.	$3X = Y$	$(X = 418y_{n+1} - 66x_{n+2},\ Y = 1257y_{n+1} - 3y_{n+3})$
18.	$X = 3Y$	$(X = 63y_{n+1} - 3y_{n+2},\ Y = 209y_{n+2} - 693x_{n+2})$
19.	$X = Y$	$(X = 209y_{n+3} - 13827x_{n+2},\ Y = 4169y_{n+2} - 693x_{n+3})$
20.	$X = 10Y$	$(X = 4169y_{n+2} - 693x_{n+3},\ Y = 4169y_{n+3} - 13827x_{n+3})$
21.	$33X = Y$	$(X = 4169y_{n+3} - 13827x_{n+3}, Y = 13827y_{n+2} - 693y_{n+3})$
22.	$X = 3Y$	$(X = x_{n+2} - 19x_{n+1}, Y = y_{n+1} - 3x_{n+1})$
23.	$X = 10Y$	$(X = y_{n+2} - 63x_{n+1}, Y = y_{n+1} - 3x_{n+1})$
24.	$X = Y$	$(X = 63y_{n+1} - 3y_{n+2}, Y = 209x_{n+3} - 4169x_{n+2})$
25.	$X = 11Y$	$(X = 4169y_{n+3} - 13827x_{n+3}, Y = y_{n+1} - 3x_{n+1})$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 3 below.

Table-3. Hyperbola

S.No	Hyperbola	(X_n, Y_n)
1	$11Y_n^2 - X_n^2 = 44$	$(11x_{n+1} - 3y_{n+1},\ y_{n+1} - 3x_{n+1})$
2	$11Y_n^2 - 9X_n^2 = 396$	$(21x_{n+1} - x_{n+2},\ x_{n+2} - 19x_{n+1})$
3	$121Y_n^2 - 9X_n^2 = 1742400$	$(419x_{n+1} - x_{n+3},\ x_{n+3} - 379x_{n+1})$
4	$11Y_n^2 - X_n^2 = 4400$	$(209x_{n+1} - 3y_{n+2},\ y_{n+2} - 63x_{n+1})$
5	$11Y_n^2 - X_n^2 = 1742444$	$(4169x_{n+1} - 3y_{n+3},\ y_{n+3} - 1257x_{n+1})$
6	$Y_n^2 - 11X_n^2 = 193600$	$(22x_{n+2} - 126y_{n+1},\ 418y_{n+1} - 66x_{n+2})$
7	$Y_n^2 - 11X_n^2 = 19166884$	$(11x_{n+3} - 1257y_{n+1},\ 4169y_{n+1} - 33x_{n+3})$
8	$Y_n^2 - 11X_n^2 = 4356$	$(y_{n+2} - 19y_{n+1},\ 63y_{n+1} - 3y_{n+2})$
9	$Y_n^2 - 11X_n^2 = 1742400$	$(y_{n+3} - 379y_{n+1},\ 1257y_{n+1} - 3y_{n+3})$
10	$Y_n^2 - 11X_n^2 = 4356$	$(1257x_{n+2} - 63x_{n+3},\ 209x_{n+3} - 4169x_{n+2})$
11	$Y_n^2 - 11X_n^2 = 484$	$(209x_{n+2} - 63y_{n+2},\ 209y_{n+2} - 693x_{n+2})$
12	$Y_n^2 - 11X_n^2 = 48400$	$(4169x_{n+2} - 63y_{n+3},\ 209y_{n+3} - 13827x_{n+2})$
13	$Y_n^2 - 11X_n^2 = 48400$	$(209x_{n+3} - 1257y_{n+2},\ 4169y_{n+2} - 693x_{n+3})$

14	$Y_n^2 - 11X_n^2 = 484$	$\begin{pmatrix} 4169x_{n+3} - 1257y_{n+3}, \\ 4169y_{n+3} - 13827x_{n+3} \end{pmatrix}$
15	$Y_n^2 - 11X_n^2 = 527076$	$\begin{pmatrix} 209y_{n+3} - 4169y_{n+2}, \\ 13827y_{n+2} - 693y_{n+3} \end{pmatrix}$

Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 4 below

Table-4. Parabola

S.No	Parabola	(X_n, Y_n)
1	$11Y_n - X_n^2 = 22$	$(11x_{n+1} - 3y_{n+1}, y_{2n+2} - 3x_{2n+2})$
2	$11Y_n - 3X_n^2 = 66$	$\begin{pmatrix} 21x_{n+1} - x_{n+2}, \\ x_{2n+3} - 19x_{2n+2} \end{pmatrix}$
3	$Y_n - 660X_n^2 = 5808000$	$\begin{pmatrix} 419x_{n+1} - x_{n+3}, \\ x_{n+3} - 379x_{n+1} \end{pmatrix}$
4	$1210Y_n - 11X_n^2 = 24200$	$\begin{pmatrix} 209x_{n+1} - 3y_{n+2}, \\ y_{2n+3} - 63x_{2n+2} \end{pmatrix}$
5	$435611Y_n - X_n^2 = 173373178$	$\begin{pmatrix} 4169x_{n+1} - 3y_{n+3}, \\ y_{2n+4} - 1257x_{2n+2} \end{pmatrix}$
6	$20Y_n - X_n^2 = 8800$	$\begin{pmatrix} 22x_{n+2} - 126y_{n+1}, \\ 418y_{2n+2} - 66x_{2n+3} \end{pmatrix}$
7	$199Y_n - X_n^2 = 871222$	$\begin{pmatrix} 11x_{n+3} - 1257y_{n+1}, \\ 4169y_{2n+2} - 33x_{2n+4} \end{pmatrix}$
8	$3Y_n - X_n^2 = 198$	$\begin{pmatrix} y_{n+2} - 19y_{n+1}, \\ 63y_{2n+2} - 3y_{2n+3} \end{pmatrix}$
9	$6Y_n - X_n^2 = 79200$	$\begin{pmatrix} y_{n+3} - 379y_{n+1}, \\ 1257y_{2n+3} - 3y_{2n+4} \end{pmatrix}$
10	$Y_n - 11X_n^2 = 2178$	$\begin{pmatrix} 1257x_{n+2} - 63x_{n+3}, \\ 209x_{2n+4} - 4169x_{2n+3} \end{pmatrix}$
11	$Y_n - 11X_n^2 = 242$	$\begin{pmatrix} 209x_{n+2} - 63y_{n+2}, \\ 209y_{2n+3} - 693x_{2n+3} \end{pmatrix}$
12	$Y_n - 11X_n^2 = 24200$	$\begin{pmatrix} 4169x_{n+2} - 63y_{n+3}, \\ 209y_{2n+4} - 13827x_{2n+3} \end{pmatrix}$
13	$Y_n - 11X_n^2 = 24200$	$\begin{pmatrix} 209x_{n+3} - 1257y_{n+2}, \\ 4169y_{2n+3} - 693x_{2n+4} \end{pmatrix}$
14	$Y_n - 11X_n^2 = 242$	$\begin{pmatrix} 4169x_{n+3} - 1257y_{n+3}, \\ 4169y_{2n+4} - 13827x_{2n+4} \end{pmatrix}$
15	$Y_n - 11X_n^2 = 263538$	$\begin{pmatrix} 209y_{n+3} - 4169y_{n+2}, \\ 13827y_{2n+3} - 693y_{2n+4} \end{pmatrix}$

3.1. Generation of Pythagorean Triangle

Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}.$ Note that $p > q > 0.$ Treat p, q as the generators of the Pythagorean triangle $T(X, Y, Z)$ where $X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2.$

Let A,P represents the area and perimeter of Pythagorean triangle. Then the following results are observed.

a) $X - 11Y + 10Z + 22 = 0.$

b) $\frac{2A}{P} = x_{n+1}y_{n+1}$.

c) $3(Z-Y)$ is a nasty number.

d) $3(X-\frac{4A}{P})$ is a nasty number.

e) $X-\frac{4A}{P}+Y$ is written as the sum of two squares.

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the Diophantine equations represented by the hyperbola $y^2 = 11x^2 + 22$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their solutions with the suitable properties.

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