Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 6, Issue. 7, pp: 93-99, 2020
URL: https://arpgweb.com/journal/journal/17
DOI: https://doi.org/10.32861/ajams.67.93.99

Academic Research Publishing Group

Original Research **Open Access**

Positive Rational Number of the Form $\varphi(km^a)/\varphi(ln^b)$

Hongjian Li (Corresponding Author)
Is with School of Mathematics, South China Normal University, Guangzhou 510631, China
Email: yuanpz@scnu.edu.cn

Pingzhi Yuan
Is with School of Mathematics, South China Normal University, Guangzhou 510631, China

Hairong Bai
Is with School of Mathematical Science, South China Normal University, Guangzhou 510631, China

Article History
Received: June 30,2020
Revised: July 20, 2020
Accepted: July 25,2020
Published: July 28,2020

Abstract

Let k,l,a and b be positive integers with $\max\{a,b\}\ge 2$. In this paper, we show that every positive rational number can be written as the form $\varphi(km^a)/\varphi(ln^b)$, where $m,n\in N$ if and only if $\gcd(a,b)=1$ or $(a,b,k,l)=(2,2,1,1)$. Moreover, if $\gcd(a,b)>1$, then the proper representation of such representation is unique.

Keywords: Representation of positive rational numbers; Euler's totient function.

1. Introduction

Let N be the set of all positive integers and φ Euler's totient function, which is defined as $\phi(n) = \#\{r: r\in N, 0<r\le n, gcd(r,n)=1\}$, the number of integers in the set $1,2,\dots,n$ that are relatively prime to n. Let

$$n=\prod_{i=1}^{s} p_i^{\alpha_i},\ p_1<p_2<\cdots p_s$$ are prime, $\alpha_i\in N$

be the standard factorization of a positive integer n, it is well-known that

$$\varphi(n)=\varphi\left(\prod_{i=1}^{s} p_i^{\alpha_i}\right)=\prod_{i=1}^{s}(p_i-1)p_i^{\alpha_i-1} \tag{1}$$

(See. e.g., [1], page 20). Sun [2], proposed many challenging conjectures on representations of positive rational numbers. Recently, Krachun and Sun [3] proved that:any positive rational number can be written as the form $\varphi(m^2)/\varphi(n^2)$,where $m,n\in N$.

For given positive integers a,b,k and l with $\max\{a,b\}\ge 2$, in this paper, we consider a more general problem when every positive rational number can be written as the form $\varphi(km^a)/\varphi(ln^b)$, where $m,n\in N$.

Note that if p is a prime with either $p|\gcd(m,n)$ or $\gcd(p,mn)=1$, then $\frac{\varphi(m)}{\varphi(m)}=\frac{\varphi(mp)}{\varphi(mp)}$. From this we can easily derive that

$$\frac{\varphi(km^a d_1^a)}{\varphi(ln^b d_2^b)}=\frac{\varphi(km^a)}{\varphi(ln^b)} \tag{2}$$

whenever d_1 and d_2 are positive integers with $d_1^a=d_2^b$, and for each prime $p|d_1d_2$, either $p|\gcd(km,ln)$ or $\gcd(p,klmn)=1$. Hence we have the following definition.

Definition 1: Let k,l,a and b be positive integers with $\max\{a,b\}\ge 2$ and r a positive rational numbers. A representation of $\frac{\varphi(km^a)}{\varphi(ln^b)}$ is called a *proper* representation if there are no positive integers $d_1>1$ and d_2

such that $m=m_1d_1$, $n=n_1d_2$, $d_1^a=d_2^b$, and for each prime $p|d_1d_2$, either $p|\gcd(km_1,ln_1)$ or $\gcd(p,klm_1n_1)=1$.

For example,when $(a,b,k,l)=(2,2,1,1)$,then a representation of $r=\frac{\varphi(m^2)}{\varphi(n^2)}$ is called a *proper* representation if there are no positive integers $d>1$ such that $m=m_1d$, $n=n_1d$, and for each prime $p|d$, either $p|\gcd(m_1,n_1)$ or $\gcd(p,m_1n_1)=1$.

The main purpose of this note is to show the following result.

Theorem 1 Let k,l,a and b be positive integers with $\max\{a,b\}\ge 2$. Then any positive rational number can be written as the form $\varphi(km^a)/\varphi(ln^b)$, where $m,n\in N$ if and only if $\gcd(a,b)=1$ or $(a,b,k,l)=(2,2,1,1)$. Moreover, if $\gcd(a,b)>1$, then the *proper* representation of such representation of a positive rational number is unique.

2. Proof of Theorem 1

Proof. For a positive rational number r with $r\ne 1$, let

$$r=\prod_{i=1}^{s}p_i^{\alpha_i},\ p_1<p_2<\cdots p_s,\alpha_i\in \mathbb{Z}\setminus\{0\}$$

be the standard factorization of r, where $p_1<p_2<\cdots p_s$ are primes. Let $P(r)=p_s$ denote the maximal prime factor of r, $v_{p_s}(r)=\alpha_s$,the p_s-valuation of r,and we let $P(1)=1$.

We first prove that if $\gcd(a,b)=1$ or $(a,b)=(2,2)$ and $(k,l)=(1,1)$, then any positive rational number can be written as the form $\varphi(km^a)/\varphi(ln^b)$, where m, n ∈ N. Recall that Krachun and Sun [3] have proved the case of $(a,b,k,l)=(2,2,1,1)$.It suffices to show the statement holds for a and b with $\gcd(a,b)=1$.For any integer c,it is well-known that there are positive integers x and y such that $ax-by=c$ since $\gcd(a,b)=1$ .Hence for each prime p with $v_p(r)\ne 0$ or $p|kl$,there are positive integers x_p and y_p such that

$$v_p(r)=v_p(k)+ax(p)-v_p(l)-by(p).$$

Let

$$m=\prod_{p|kl,or\ v_p(r)\ne 0}p^{x(p)},n=\prod_{p|kl,or\ v_p(r)\ne 0}p^{y(p)}.$$

Then it is easy to check that

$$r=\frac{\varphi(km^a)}{\varphi(ln^b)}$$

since for any prime p with $v_p(r)\ne 0$ or $p|kl$,we have $p|\gcd(m,n)$ and hence

$$v_p(r)=v_p(k)+av_p(m)-1-(v_p(l)+bv_p(n)-1)=v_p(k)+ax(p)-v_p(l)-by(p).$$

Therefore we have proved that any positive rational number can be written as the form $\frac{\varphi(km^a)}{\varphi(\ln^b)},m,n\in N$ when $\gcd(a,b)=1$ or $(a,b)=(2,2)$ and $(k,l)=(1,1)$.

Next, we will show that if $\gcd(a,b)\ge 2$ and $(a,b,k,l)\ne(2,2,1,1)$, then there exists a positive rational number r such that r cannot be written as the form

$$\frac{\varphi(km^a)}{\varphi(ln^b)},m,n\in\mathbb{N}.$$

If $\gcd(a,b)=\mu>2$, take t to be a positive integer with $t\equiv 1(\bmod \mu)$ and p a prime with $p>P(kl)$, we will show that p^t cannot be written as the form $\frac{\varphi(km^a)}{\varphi(ln^b)}, m,n\in\square$.

Suppose that there are positive integers m and n such that

$$p^t=\frac{\varphi(km^a)}{\varphi(ln^b)}.$$

Without loss of generality, we may assume that the above representation is proper,so $P(klmn)=p$,and hence $P(mn)=p$ since $p>P(kl)$.Now we have

$$t=\begin{cases} v_p\left(\varphi\left(p^{av_p(m)}\right)/\varphi\left(p^{bv_p(n)}\right)\right)=av_p(m)-bv_p(n)\equiv 0(\bmod \mu), & if \quad v_p(n)\neq 0, \\ v_p\left(\varphi\left(p^{av_p(m)}\right)\right)=av_p(m)-1, & if \quad v_p(n)=0, \end{cases}$$

Which implies that $t\equiv 0(\bmod \mu)$ or $t\equiv -1(\bmod \mu)$,a contradiction to $t\equiv 1(\bmod \mu)$ and $\mu\geq 3$.Hence p^t can not be written as

$$\frac{\varphi(km^a)}{\varphi(ln^b)}, m,n\in\square .$$

For the case of $\gcd(a,b)=\mu=2$ and $(a,b)\neq(2,2)$,we have that $\max\{a,b\}>2$.We only consider the case where $a>2$ (the argument for the case of $b>2$ is similar).We have, for a prime p with $p>P(kl)$,

$$v_p\left(\frac{\varphi(km^a)}{\varphi(ln^b)}\right)=av_p(m)-bv_p(n)\equiv 0(\bmod 2)\equiv 0(\bmod 2),$$

or $av_p(m)-1\geq 3$ when $v_p(n)=0$, or $1-bv_p(n)<0$ when $v_p(m)=0$,so p can not be written as

$$\frac{\varphi(km^a)}{\varphi(ln^b)}, m,n\in\square .$$

Now we consider the case where $(a,b)=(2,2)$ and $(k,l)\neq(1,1)$,then $\max\{k,l\}\geq 2$.Without loss of generality,we may assume that $p=P(kl)=P(k)$ (the argument for the case of $p=P(kl)=P(l)$ is similar). If $p|\gcd(k,l)$, then we have

$$v_p\left(\frac{\varphi(km^a)}{\varphi(ln^b)}\right)=v_p(k)-v_p(l)+2\left(v_p(m)-v_p(n)\right)\equiv v_p(k)-v_p(l)(\bmod 2),$$

so $p^{|v_p(k)-v_p(l)|+1}$ can not be written as $\frac{\varphi(km^a)}{\varphi(ln^b)}, m,n\in\square$. .If p Œ(the case where p Œ is similar, and we omit the detail), then we have

$$v_p\left(\frac{\varphi(km^a)}{\varphi(ln^b)}\right)=v_p(k)+2\left(v_p(m)-v_p(n)\right)\equiv v_p(k)(\bmod 2),$$

or $v_p(k)-1+2v_p(m)\geq 0$ when $v_p(n)=0$,so $p^{-v_p(k)-1}$ can not be written as $\frac{\varphi(km^a)}{\varphi(ln^b)}, m,n\in\square$.

This proves the first statement.

To prove the last statement, we use a double induction on $P(kl)$.We first prove that the statement holds for $P(kl)=1$ by induction on $P(r)$,then we prove that the statement holds for any positive integers k,l by induction on $P(kl)$.

Let $d=\gcd(a,b)>1$ and r a positive rational number.Suppose that the representation of

$$r=\frac{\varphi(km^a)}{\varphi(ln^b)},m,n\in\mathbb{N}. \tag{3}$$

is proper. We will show that m and n are uniquely determined by r, k and l,in other words,the proper representation (3)of r is unique. We prove this by a double induction on $P(kl)$.To begin with,we show that for the proper representation(3)of r is unique when $P(kl)=1$, i.e., $k=l=1$. In this case,we use induction on $P(r)$.For $P(r)=1$,let $1=\frac{\varphi(m^a)}{\varphi(n^b)}$ be a proper representation of 1,and let $p=P(mn)$. If $p>1$,then p is a prime and we have

$$0=v_p(1)=av_p(m)-bv_p(n).$$

Hence $1=\frac{\varphi(m_1^a)}{\varphi(n_1^b)}$,where $m_1=m/p^{v_p(m)}$, $n_1=n/p^{v_p(n)}$,and $av_p(m)=bv_p(n)$, which implies that the representation of $1=\frac{\varphi(m^a)}{\varphi(n^b)}$ is not proper, a contradiction. Hence $p=1$, and 1 has only the proper representation of $1=\frac{\varphi(1^a)}{\varphi(1^b)}$.For a proper representation (3)of r with $P(r)>1$,we claim that

$$P(mn)=P(r). \tag{4}$$

Let $q=P(mn)$.Obviously,we have $q\geq P(r)$.Note that

$$v_q\left(\frac{\varphi(m^a)}{\varphi(n^b)}\right)=\begin{cases}av_q(m)-bv_q(n), if\ \ v_q(m)\neq 0\ \ and\ \ v_q(n)\neq 0,\\ av_q(m)-1, & if\ \ v_q(n)=0,\\ -bv_q(n)+1, & if\ \ v_q(m)=0.\end{cases} \tag{5}$$

If $q>P(r)$, then we have $0=v_q(r)=v_q\left(\frac{\varphi(m^a)}{\varphi(n^b)}\right)$,so $v_q(m)\neq 0$ and $v_q(n)\neq 0$. Hence $0=av_q(m)-bv_q(n)$, which implies that $av_q(m)=bv_q(n)$. Therefore, we have

$$r=\frac{\varphi(m^a)}{\varphi(n^b)}=\frac{\varphi(m_1^a)}{\varphi(n_1^b)},$$

where $m_1=m/q^{v_q(m)}$ and $n_1=n/q^{v_q(n)}$,i.e.,the representation of $r=\frac{\varphi(m^a)}{\varphi(n^b)}$ is not proper by definition, a contradiction. Hence $P(mn)=P(r)$ when the representation (3) of r is proper.

For $P(r)=2$,by (4),for any nonzero integer c and any proper representation of

$$2^c=\frac{\varphi(m^a)}{\varphi(n^b)},m,n\in\mathbb{N}$$

we have that $P(mn)=2$, so there are non-negative integers x and y such that $m=2^x$ and $n=2^y$.For simplicity, we assume that $c>0$(the case where $c<0$ is similar).If $d=\gcd(a,b)|c$,then it follows from (5) that $c=ax-by,x>0,y>0$.Let $(x,y)=(x_0,y_0)$, $x_0>0$, $y_0>0$ be the least positive integer solution of the linear equation

$$ax-by=c,x>0,y>0.$$

It is well-known that all positive integer solution (X,Y) of the equation $aX-bY=c$ are given by

$$X=x_0+\frac{b}{\gcd(a,b)}t,Y=y_0+\frac{a}{\gcd(a,b)}t,t\in\mathbb{N}\cup\{0\}. \tag{6}$$

We claim that $x=x_0$ and $y=y_0$.Otherwise, there exists a positive integer t such that

$$x = x_0 + \frac{b}{\gcd(a,b)} t, y = y_0 + \frac{a}{\gcd(a,b)} t.$$

Hence we have

$$2^c = \frac{\varphi(2^{ax} d_1^a)}{\varphi(2^{by} d_2^b)} = \frac{\varphi(2^{ax_0})}{\varphi(2^{by_0})}, d_1^a = d_2^b = 2^{\frac{abt}{\gcd(a,b)}},$$

Which contradicts with the proper representation of 2^c. Therefore $x = x_0$ and $y = y_0$. Hence the proper representation of 2^c is unique. If $\gcd(a,b)/|c$, then it follows from (5) that $c = ax - 1$ and $y = 0$, so $x = (c+1)/a$, the proper representation of 2^c is also unique. This proves the result for $P(r) = 2$.

Now let q be an odd prime and assume that the result holds for $P(r) < q$. Let r be a positive rational number with $P(r) = q$, and the representation (3) of r is proper. By (4), we have $P(mn) = q$. If $v_q(n) = 0$, then $v_q(r) > 0$ and it follows from (5) that $v_q(r) = av_q(m) - 1$ and $v_q(n) = 0$, so $v_q(m) = (v_q(r) + 1)/a$.Let $r_0 = r/q^{v_q(r)}(q-1)$. Then we have

$$r_0 = \frac{\varphi(m_1{}^a)}{\varphi(n_1{}^b)},$$

where $m_1 = m/q^{v_q(m)}$ and $n_1 = n$.Note that the representation of $r_0 = \varphi(m_1{}^a)/\varphi(n_1{}^b)$ is proper, so by the induction hypothesis, m_1 and n_1 are uniquely determined by r_0 since $P(r_0) < P(r) = q$. Therefore the proper representation of r is unique.

If $v_q(m) = 0$ and $v_q(n) > 0$, then $v_q(r) < 0$ and it follows from (5) that $v_q(r) = -bv_q(n) + 1$ and $v_q(m) = 0$, so $v_q(n) = (-v_q(r) + 1)/b$. Let $r_0 = r(q-1)/q^{v_q(r)}$. Then we have

$$r_0 = \frac{\varphi(m_1{}^a)}{\varphi(n_1{}^b)},$$

where $m_1 = m$ and $n_1 = n/q^{v_q(n)}$. Now the representation of $r_0 = \varphi(m_1{}^a)/\varphi(n_1{}^b)$ is proper, so by the induction hypothesis m_1 and n_1 is uniquely determined by r_0 since $P(r_0) < P(r) = q$. Therefore the proper representation of r is unique.

If $v_q(m) > 0$ and $v_q(n) > 0$, then it follows from (5) that $v_q(r) = av_q(m) - bv_q(n)$ since $v_q(m) > 0$ and $v_q(n) > 0$.By the similar argument as above, we have $(v_q(m),\ v_q(n)) = (x,\ y)$ is the least positive integer solution of the linear equation $ax - by = v_q(r)$. Let $r_0 = r/q^{v_q(r)}$. Then we have

$$r_0 = \frac{\varphi(m_1{}^a)}{\varphi(n_1{}^b)},$$

where $m_1 = m/q^{v_q(m)}$ and $n_1 = n/q^{v_q(n)}$. It is easy to verify that the representation of $r_0 = \varphi(m_1{}^a)/\varphi(n_1{}^b)$ is proper. Since $P(r_0) < P(r) = q$, by the induction hypothesis, m_1 and n_1 are uniquely determined by r_0. Therefore the proper representation of r is unique.

In view of the above, we have proved that the representation (3) of r is unique. This completes the proof of the case $(k, l) = (1,1)$.

Now we assume that the statement holds for $P(kl) < p$, where p is a prime.
That is, for any positive integers k, l with $P(kl) < p$ and any proper representation of

$$r = \frac{\varphi(km^a)}{\varphi(ln^b)}, \qquad m, n \in N,$$

of a rational number number r, m, n are uniquely determined by r, k, l. We prove that the statement holds for $P(kl) = p$. For the first step, we let r_0 be a positive rational number with least $P(r_0)$ such that

$$r_0 = \frac{\varphi(km^a)}{\varphi(ln^b)}, \qquad m, n \in N,$$

is a proper representation of r_0. We show that m, n are uniquely determined by r_0, k, l. Obviously, $q = P(klmn) \geq P(r_0)$, then we have

$$0 = v_q\left(\frac{\varphi(km^a)}{\varphi(ln^b)}\right) = \begin{cases} av_q(m) + v_q(k) - bv_q(n) - v_q(l), & if\, v_q(km) \cdot v_q(ln) \neq 0 \\ av_q(m) + v_q(k) - 1 & , if\, v_q(ln) = 0 \\ -bv_q(n) - v_q(l) + 1 & , if\, v_q(km) = 0 \end{cases}$$

If $v_q(ln) = 0$(resp. $v_q(km) = 0$), then $v_q(n) = 0$ and $v_q(m)$ is uniquely determined by k (resp. $v_q(m) = 0$ and $v_q(n)$ is uniquely determined by l).

If $v_q(km) \neq 0$, then we have $av_q(m) + v_q(k) - bv_q(n) - v_q(l) = 0$. By the same argument as before, we see that $(v_q(m), v_q(m)) = (x, y)$ is the least non-negative (resp. positive) solution of the linear equation $ax - by = v_q(l) - v_q(k)$ when $v_q(k) > 0$ and $v_q(l) > 0$ (resp. when $v_q(k) = 0$ or $v_q(l) = 0$). Hence $v_q(k)$ and $v_q(l)$ are uniquely determined by k, l. We have

$$r_1 = r = \frac{\varphi(k_1 {m_1}^a)}{\varphi(l_1 {n_1}^b)}$$

where $k_1 = k/q^{v_q(k)}$, $l_1 = l/q^{v_q(l)}$, $m_1 = m/q^{v_q(m)}$ and $n_1 = n/q^{v_q(n)}$, and the above representation of r_1 is proper. Further, if q> p = P(kl), then

(k, l) = (k_1, l_1) and $av_q(m) = bv_q(n)$, so the representation of $r_0 = \varphi(km^a)/\varphi(ln^b)$, $m, n \in N$ is not proper. Hence q= p = P(kl), and therefore P($k_1 l_1$) < P(kl) = p. By the induction hypothesis on P(kl), we know that the representation of $r_0 = \varphi(k_1 {m_1}^a)/\varphi(l_1 {n_1}^b)$ is unique, so m_1, n_1 are uniquely determined by r_0, k_1, l_1, and hence m, n are uniquely determined by r_0, k, l.

If q = P(klmn) = P(r_0), then we have $v_q(r_0)$ =

$$v_q\left(\frac{\varphi(km^a)}{\varphi(ln^b)}\right) = \begin{cases} av_q(m) + v_q(k) - bv_q(n) - v_q(l), & if\, v_q(km) \cdot v_q(ln) \neq 0 \\ av_q(m) + v_q(k) - 1, & if\, v_q(ln) = 0 \\ -bv_q(n) - v_q(l) + 1, & if\, v_q(km) = 0 \end{cases}$$

By the same argument as in the case of $P(klmn) > P(r_0)$, we get $v_q(m)$, $v_q(n)$ are uniquely determined by r_0, k, l and we have

$$r_1 = r_0/q^{v_q(r_0)} = \frac{\varphi(k_1 {m_1}^a)}{\varphi(l_1 {n_1}^b)},$$

where $k_1 = k/q^{v_q(k)}$, $l_1 = l/q^{v_q(l)}$, $m_1 = m/q^{v_q(m)}$ and $n_1 = n/q^{v_q(n)}$, and the above representation of r_1 is proper. Further, we have $P(r_1) < P(r_0)$

and $P(k_1 l_1) \leq P(kl) = p$. If $P(k_1 l_1) = P(kl) = p$, then (k,l)= $(k_1,\ l_1)$,

$$r_1 = \frac{\varphi(k{m_1}^a)}{\varphi(l{n_1}^b)},$$

and $P(r_1) < P(r_0)$, which is impossible by the definition of r_0. If $P(k_1 l_1) < P(kl) = p$, then by the induction hypothesis on P(kl), we know that the representation of $r_1 = \varphi(k_1 {m_1}^a)/\varphi(l_1 {n_1}^b)$ is unique, so m_1, n_1 are uniquely determined by r_1, k_1, l_1, and hence m, n are uniquely determined by r_0, k, l.

Next we assume that the statement holds for all positive rational number r with $P(r_0) \leq P(r) = q$, q is an odd prime. We show that the statement holds for all positive rational number r with $P(r) = q$. Let

$$r = \frac{\varphi(km^a)}{\varphi(ln^b)}, \qquad m, n \in N,$$

be a proper representation of r. Obviously, $q = P(klmn) \geq P(r)$. If $q = P(klmn) > P(r)$, then we have 0 =

$$v_q\left(\frac{\varphi(km^a)}{\varphi(ln^b)}\right) = \begin{cases} av_q(m) + v_q(k) - bv_q(n) - v_q(l), & if\, v_q(km) \cdot v_q(ln) \neq 0 \\ av_q(m) + v_q(k) - 1, & if\, v_q(ln) = 0 \\ -bv_q(n) - v_q(l) + 1, & if\, v_q(km) = 0 \end{cases}$$

By the same argument as in the case of $r = r_0$, $v_q(m)$, $v_q(n)$ are uniquely determined by k, l. And we hav

$$r_1 = r = \frac{\varphi(k_1 {m_1}^a)}{\varphi(l_1 {n_1}^b)},$$

where $k_1 = k/q^{v_q(k)}$, $l_1 = l/q^{v_q(l)}$, $m_1 = m/q^{v_q(m)}$ and $n_1 = n/q^{v_q(n)}$, and the above representation of r_1 is proper. Similarly, by the induction hypothesis on P(kl), we know that the representation is unique, so m_1, n_1 are uniquely determined by r, k_1, l_1, and hence m, n are uniquely determined by r, k, l.

The argument of the case where $P(klmn) = P(r)$ is the same as in the case of $r = r_0$. Now we have $v_q(m)$, $v_q(n)$ are uniquely determined by r, k, l and

$$r_1 = r/q^{v_q(r)} = \frac{\varphi(k_1 {m_1}^a)}{\varphi(l_1 {n_1}^b)}$$

where $k_1 = k/q^{v_q(k)}$, $l_1 = l/q^{v_q(l)}$, $m_1 = m/q^{v_q(m)}$ and $n_1 = n/q^{v_q(n)}$, and the above representation of r_1 is proper. Further, we have $P(r_1) < P(r)$

and $P(k_1 l_1) \leq P(kl) = p$. If $P(k_1 l_1) = P(kl) = p$, then $(k,\ l) = (k_1,\ l_1)$,

$$r_1 = \frac{\varphi(k{m_1}^a)}{\varphi(l{n_1}^b)},$$

and $P(r_1) < P(r_0)$, by the induction hypothesis on P(r), we obtain that m_1, n_1 are uniquely determined by r_1, k_1, l_1, and hence m, n are uniquely determined by r_0, k, l. If $P(k_1 l_1) < P(kl) = p$, then by the induction hypothesis on P(kl), we know that the representation of $r_1 = \varphi(k_1 {m_1}^a)/\varphi(l_1 {n_1}^b)$ is unique, so m_1, n_1 are uniquely determined by r_1, k_1, l_1, and hence m, n are uniquely determined by r, k, l.

This completes the proof.

Two examples: (1) For $(a,\ b) = (2, 3)$ and $r = 5/11$, we have

$$\frac{5}{11} = \frac{\varphi(5^4 11^2)}{\varphi(5^3 11^3)} = \frac{\varphi(5^2 11^2)}{\varphi(11^3)} = \frac{\varphi(55^2)}{\varphi(22^3)}$$

Both $\varphi(55^2)/\varphi(22^3)$ and $\varphi(55^2)/\varphi(22^3)$ are proper representation of $\frac{5}{11}$.

(2) For $(a,\ b) = (2, 2)$ and $r = 19/47$, we have

$$\frac{19}{47} = \frac{19 \times 46}{\varphi(47^2)} = \frac{19\varphi(23^2)}{11\varphi(47^2)} = \frac{5\varphi(19^2 \times 23^2)}{9\varphi(11^2 \times 47^2)} = \frac{\varphi(3^2 \times 5^2 \times 19^2 \times 23^2)}{4\varphi(3^4 \times 11^2 \times 47^2)}$$

$$= \frac{\varphi(2^2 \times 3^2 \times 5^2 \times 19^2 \times 23^2)}{\varphi(2^4 \times 3^4 \times 11^2 \times 47^2)} = \frac{\varphi(13110^2)}{\varphi(18612^2)} = \frac{\varphi(39330^2)}{\varphi(55836^2)}$$

Here $\varphi(13110^2)/\varphi(18612^2)$ is the proper representation of $r = 19/47$, while the representation $\varphi(39330^2)/\varphi(55836^2)$ in [3] is not proper.

Remarks: (1) It follows from the proof of Theorem 1 that if a, b, m, n are positive integers such that $min\{a, b\} > 1$ and $\varphi(m^a) = \varphi(n^b)$, then $m^a = n^b$.

(2) For $(a,\ b) = (2, 2)$, by the main theorem of [3] and Theorem 1, we have that any positive rational number can be written as the form $\varphi(m^a)/\varphi(n^b)$, where $m, n \in N$. Moreover, the proper representation of any positive rational number of the form $\varphi(m^a)/\varphi(n^b)$ is unique.

(3) Let p be a prime and a, b, c positive integers with $\gcd(a, b) = 1$ and $c = au - 1, u \in N$. Let $1/p - 1 = \varphi(m^a)/\varphi(n^b)$ be a proper representation of $1/p - 1$ and x, y be the least positive integer solution of the linear equation $ax - by = au - 1$. Then

$$p^c = \frac{\varphi(p^{xa})}{\varphi(p^{yb})} = \frac{\varphi(p^{ua} m^a)}{\varphi(n^b)}$$

And both representations above of pc are proper. The first equality of the above equation does not hold when $\gcd(a, b) > 1$ since the related linear equation $ax - by = au - 1$ has no integer solution (x, y).

Acknowledgments

This work is supported by the National Natural Science Foundation of China (grants no.11671153).

References

[1] Ireland, K. and Rosen, M., 1990. *A classical introduction to modern num-ber theory*. 2nd ed. vol. 84. New York: Springer.

[2] Sun, Z.-W., 2017. "Conjectures on representations involving primes." In *Nathanson. M. ed. Combinatorial and Additive Number Theory II. Springer proceedings in mathematics and Statistics. Cham: Springer.* pp. 279-310.

[3] Krachun, D. and Sun, Z.-W., 2020. "Each positive rational number has the form (equation)." *Amer. Math*