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A Peer Search on Integer Solutions to Quadratic Diophantine Equation with Three Unknowns $8(x^2+y^2)-15xy+2(x+y)+4=47z^2$

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Abstract

The non- homogeneous ternary quadratic diophantine equation $8(x^2+y^2)-15xy+2(x+y)+4=47z^2$ is analyzed for its patterns of non-zero distinct integral solutions. Various interesting relations between the solutions and special numbers namely polygonal, Pronic and Gnomonic numbers are exhibited.

Keywords: Ternary quadratic; Non-homogeneous quadratic; Integer solutions.

1. Introduction

It is well known that ternary quadratic diophantine equations are rich in variety [1-17]. This paper concerns with another interesting quadratic diophantine equation with three unknowns $8(x^2+y^2)-15xy+2(x+y)+4=47z^2$ for determining its infinitely many non-zero integral solutions. Various interesting relations between the solutions and special numbers namely polygonal, Pronic and Gnomonic numbers are exhibited.

1.1. Notations

- $t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]=$ Polygonal number of rank n with sides m
- $Pr_n=n(n+1)=$ Pronic number of rank n
- $GN(n)=2n+1=$ Gnomonic number

2. Method of Analysis

The Ternary quadratic equation to be solved for its non-zero distinct integral solution is

$$8(x^2+y^2)-15xy+2(x+y)+4=47z^2 \tag{1}$$

Introduction of the linear transformations

$$x=u+v,\ y=u-v \quad ,\ (u\neq v\neq 0) \tag{2}$$

in (1) leads to

$$U^2+31v^2=47z^2 \tag{3}$$

where

$$U=u+2 \tag{4}$$

Solving (3) through different methods, different sets of integer solutions to (1) are obtained.

2.1. Method 1

(3) is written in the form of ratio as

$$\frac{(U+4z)}{(z-v)}=\frac{31(z+v)}{(U-4z)}=\frac{\alpha}{\beta},\beta\neq 0 \tag{5}$$

which is equivalent to the system of equations

$$U\beta+v\alpha+z(4\beta-\alpha)=0$$
$$-U\alpha+31v\beta+z(4\alpha+31\beta)=0$$

$$\left.\begin{aligned} u &= 4\alpha^2 + 62\alpha\beta - 124\beta^2 - 2 \\ v &= \alpha^2 - 8\alpha\beta - 31\beta^2 \\ z &= \alpha^2 + 31\beta^2 \end{aligned}\right\} \quad (6)$$

Substituting the values of $u \,\&\, v$ in (2), we get

$$\left.\begin{aligned} x &= x(\alpha,\beta) = 5\alpha^2 + 54\alpha\beta - 155\beta^2 - 2 \\ y &= y(\alpha,\beta) = 3\alpha^2 + 70\alpha\beta - 93\beta^2 - 2 \end{aligned}\right\} \quad (7)$$

Thus (6) & (7) represent the non-zero distinct integral solutions of (1).
A few numerical examples are presented in Table 1 below:

Table-1. Numerical examples

α	β	x	y	z
1	1	-98	-22	32
2	1	-29	57	35
3	1	50	142	40
4	1	139	212	47

2.2. Properties

- $y(1,\beta) + 93pr_\beta - GN(\beta) - 322t_{3,\beta} + 161t_{4,\beta} = 0$
- $x(\alpha,1) - t_{12,\alpha} + 1 \equiv 0 \pmod 2$
- $x(\alpha,1) + z(\alpha,1) - 54pr_\alpha + 54t_{4,\alpha} + 126$ is a nasty number.

From the values of x ,y and z in Table 1,one may obtain second order Ramanujan numbers. For illustration, consider

$$x_1 = -98$$
$$= 98*(-1) = 49*(-2) = 14*(-7)$$

Now,

$$98*(-1) = 49*(-2) \Rightarrow 97^2 + 51^2 = 99^2 + 47^2 \Rightarrow 12010$$
$$49*(-2) = 14*(-7) \Rightarrow 47^2 + 21^2 = 51^2 + 7^2 \Rightarrow 2650$$

Thus 12010, 2560 are second order Ramanujan number whose base numbers are Real integers.

Note:

(3) is also written in the form of ratio as

$$\frac{(U-4z)}{31(z-v)} = \frac{(z+v)}{(U+4z)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Following the procedure as above, the corresponding integer solutions to (1) are given by

$$x = x(\alpha,\beta) = -93\alpha^2 + 70\alpha\beta + 3\beta^2 - 2$$
$$y = y(\alpha,\beta) = -155\alpha^2 + 54\alpha\beta + 5\beta^2 - 2$$
$$z = z(\alpha,\beta) = 31\alpha^2 + \beta^2$$

A few numerical examples are presented in Table 2 below:

Table-2. Numerical examples

α	β	x	y	z
1	1	202	158	32
2	1	721	507	125
3	1	1550	1042	280
4	1	2689	1763	497

2.3. Properties

- $x(\alpha,1) - y(\alpha,1) - 46pr_\alpha - 108t_{3,\alpha} + 54t_{4,\alpha} + 17$ is a perfect square.
- $x(1,\beta) + 5pr_\beta - GN(\beta) \equiv 152 \pmod{57}$
- $y(\alpha,1) - z(\alpha,1) - 75pr_\alpha + 75t_{4,\alpha} - t_{18,\alpha} - GN(\alpha) + 7$ is a nasty number.

2.4. Method 2

Introducing the linear transformations

$$z = X + 31T\ ,\ v = X + 47T\ ,\ U = 4W \quad (8)$$

in (3),we have

$$X^2 - W^2 = 1457T^2 \tag{9}$$

which is satisfied by

$$T = 2rs,\ W = 1457r^2 - s^2,\ X = 1457r^2 + s^2$$

In view of (8), (4) and (2),the corresponding integer solutions to (1) are given by

$$x(r,s) = 7285r^2 + 94rs - 3s^2 - 2$$

$$y(r,s) = 4371r^2 - 94rs - 5s^2 - 2$$

$$z(r,s) = 1457r^2 + 62rs + s^2$$

A few numerical examples are presented in Table 3 below:

Table-3. Numerical examples

r	s	x	y	z
1	1	7374	4270	1520
1	2	7459	4161	1585
1	3	7538	4042	1652
1	4	7611	3913	1721

2.5. Properties

- $x(1,s) + 3pr_s \equiv 7283 \pmod{97}$
- $x(r,1) - z(r,1) - 5828pr_r + 11538t_{3,r} - 5796t_{4,r} + 6 = 0$
- $z(1,s) - 496$ is a perfect square

Also, (9) can be expressed as the system of double equations as presented below in Table 4:

Table-4. System of Double Equations

System	1	2	3	4	5
$X+W$	T^2	$31T^2$	$47T^2$	$1457T$	$47T$
$X-W$	1457	47	31	T	$31T$

Solving each of the above system of equations, the values of X, W and T are obtained.

In view of (8), (4) and (2), the corresponding integer solutions to (1) are obtained. For simplicity, we present below the corresponding solutions to (1) with a few properties.

Solutions from system 1:

$$x = 10k^2 + 104k - 2138$$

$$y = 6k^2 - 88k - 3690$$

$$z = 2k^2 + 64k + 760$$

A few numerical examples are presented in Table 5 below:

Table-5. Numerical examples

k	x	y	z
1	-2024	-3772	826
2	-1890	-3842	896
3	-1736	-3900	970
4	-1562	-3946	1048

2.6. Properties

- $x - y - 4pr_k \equiv 0 \pmod{2}$
- $z - GN(k^2) + 64pr_k + 64t_{4,k} - 759 = 0$
- $5z - y + t_{34,k} + 247pr_k + 9t_{4,k} + 110$ is a perfect square

Solutions from system 2:

$$x = 310k^2 + 404k + 52$$

$$y = 186k^2 + 92k - 120$$

$$z = 62k^2 + 124k + 70$$

A few numerical examples are presented in Table 6 below:

Table-6. Numerical examples

k	x	y	z
1	766	158	256

2	2100	808	558
3	4054	1830	988
4	6628	3224	1542

2.7. Properties

- $x-y-124pr_k \equiv 0(\mathrm{mod}\,2)$
- $y+z-98pr_k - GN(k) - 224t_{3,k} + 112t_{4,k} + 51$ is a nasty number
- $y - t_{36,k} - 108pr_k + 108t_{4,k} + 120$ is a perfect square

Solutions from system 3:

$$x = 470k^2 + 564k + 116$$

$$y = 282k^2 + 188k - 56$$

$$z = 94k^2 + 156k + 70$$

A few numerical examples are presented in Table 7 below:

Table-7. Numerical examples

k	x	y	z
1	1150	414	320
2	3124	1448	758
3	6038	3046	1384
4	9892	5208	2198

2.8. Properties

- $z - 188t_{3,k} - GN(k) \equiv 0(\mathrm{mod}\,3)$
- $y - 188pr_k + 195t_{4,k} + 56$ is a perfect square
- $x - z - t_{754,k} - 803pr_k + 803t_{4,k} - 46 = 0$

Solutions from system 4:

$$x = 3688T - 2$$

$$y = 2136T - 2$$

$$z = 760T$$

A few numerical examples are presented in Table 8 below:

Table-8. Numerical examples

T	x	y	z
1	3686	2134	760
2	7374	4270	1520
3	11062	6406	2280
4	14750	8542	3040

2.9. Properties

- $z - x - 2928pr_T - 24T^2 - 2$ is a nasty number
- $y - 2x + 5240pr_T - 56t_{4,T} - 2$ is a perfect square
- $z - 760pr_T + 760t_{4,T} = 0$

Solutions from system 5:

$$x = 118T - 2$$

$$y = -54T - 2$$

$$z = 70T$$

A few numerical examples are presented in Table 9 below:

Table-9. Numerical examples

T	x	y	z
1	116	-56	70

2	234	-110	140
3	352	-164	210
4	470	-218	280

2.10. Properties

- $y - x + 172pr_T - 3t_{4,T}$ is a perfect square
- $2y - z + 178pr_T - 28t_{4,T} + 4$ is a nasty number
- $x - 236t_{3,T} + t_{238,T} + 117pr_T - 117t_{4,T} + 2 = 0$

2.11. Method 3:

47 is written as

$$47 = (4 + i\sqrt{31})(4 - i\sqrt{31}) \tag{10}$$

Also $z = \alpha^2 + 31\beta^2$ (11)

Substituting (11) and (10) in (3) and employing the factorization method, define

$$(U + i\sqrt{31}v) = (4 + i\sqrt{31})(\alpha + i\sqrt{31}\beta)^2$$

On equating real and imaginary parts and using (4), we have

$$v = \alpha^2 + 8\alpha\beta - 31\beta^2$$

$$u = 4\alpha^2 - 62\alpha\beta - 124\beta^2 - 2$$

In view of (2), we have

$$\left.\begin{aligned} x &= 5\alpha^2 - 62\alpha\beta - 124\beta^2 - 2 \\ y &= 3\alpha^2 - 70\alpha\beta - 93\beta^2 - 2 \end{aligned}\right\} \tag{12}$$

Thus (11) and (12) represent the non-zero distinct integral solutions of (1).
A few numerical examples are presented in Table 10 below:

Table-10. Numerical examples

α	β	x	y	z
1	1	-206	-162	32
2	1	-245	-223	35
3	1	-274	-278	40
4	1	-293	-327	47

- $x(\alpha,\beta) - t_{12,\alpha} + 50pr_\alpha - 14t_{4,\alpha} + 157$ is a perfect square
- $y(\alpha,\beta) + z(\alpha,\beta) - t_{10,\alpha} + 67pr_\alpha - 13t_{4,\alpha} + 64$ is a nasty number
- $y(1,\beta) - 42t_{3,\beta} + 93pr_\beta + 21t_{4,\beta} - GN(\beta) = 0$

2.12. Method 4:

One may write (3) as

$$U^2 + 31v^2 = 47z^2 * 1 \tag{13}$$

Write 1 as

$$1 = \frac{(15 + i\sqrt{31})(15 - i\sqrt{31})}{256} \tag{14}$$

Substituting (10), (11) and (14) in (13) and employing the factorization method, define

$$(U + i\sqrt{31}v) = (4 + i\sqrt{31})(\alpha + i\sqrt{31}\beta)^2 \frac{(15 + i\sqrt{31})}{16}$$

Following the procedure as in Method 3, the corresponding integer solutions to (1)
Are found to be

$$z = 256\chi^2 + 7936\delta^2$$

$$\left.\begin{aligned} x &= 768\chi^2 - 17920\chi\delta - 23808\delta^2 - 2 \\ y &= 160\chi^2 - 19776\chi\delta - 4960\delta^2 - 2 \end{aligned}\right\}$$

A few numerical examples are presented in Table 11 below:

Table-11. Numerical examples

χ	δ	x	y	z
1	1	-40962	-24578	8192
2	1	-56578	-43874	8960
3	1	-70658	-62850	10240
4	1	-83202	-81506	12032

.

2.13. Properties

- $x(\chi,1)-t_{86,\chi}+GN(\chi)+t_{35756,\chi}-17878t_{4,\chi}+23809$ is a nasty number
- $x(1,\delta)-y(1,\delta)+18848pr_{\delta}\equiv 0(\bmod 2)$
- $x(1,\delta)+z(1,\delta)+15872pr_{\delta}\equiv 0(\bmod 2)$

Note: It is noted that 1 may also be expressed as

- $1=\dfrac{(2k^2+2k-15)^2+31(2k+1)^2}{(2k^2+2k+16)^2}$
- $1=\dfrac{(31r^2-s^2)^2+31(2rs)^2}{(31r^2+s^2)^2}$

Employing (10) and (11) along with the above values of 1 respectively on the R.H.S of (13), two more sets of integer solutions to (1) are obtained.

3. Conclusion

In this paper, an attempt has been made to obtain many non-zero distinct integers solutions to the quadratic equation with three unknowns represented by $8(x^2+y^2)-15xy+2(x+y)+4=47z^2$ through employing linear transformations and method of factorization. To conclude, one may search for integer solutions to quadratic equations with four or more variables along with suitable relations among the solutions.

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