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Increasing the Accuracy of the Difference Scheme Using the Richardson Extrapolation Based on the Movable Node Method

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Abstract

A one-dimensional convective-diffusion problem is considered. To improve the quality of difference schemes, the method of moving nodes is used in combination with Richardson interpolation. Approximate analytical solutions and improved schemes are obtained. Numerical experiments carried out.

Keywords: Movable nodes method; Difference method; Convection-diffusion equation.

1. Introduction

In mathematical modeling of various physical phenomena, initial and boundary-value problems arise for differential equations with small parameters at higher derivatives [1].

Due to the importance of such problems, the construction of various schemes of the convective-diffusion problem is the subject of the work of many authors [2-14]. The choice of the optimal sampling scheme for convective flows is one of the main problems in modeling flows.

The construction of discrete analogues of the convective-diffusion equation plays an essential role for transport processes. This is especially true when discrete analogues of the Navier-Stokes equation are constructed for large Reynolds numbers. In this regard, the movable nodes method (MNM) allows in many cases to design higher-quality discrete analogs of differential equations.

MNM arose in connection with the solution of differential equations by numerical methods [15, 16]. When approximating derivatives (ordinary or partial) in a differential equation by difference relations, or by the finite volume method, we obtain a discrete equation. MMN for simple cases allows you to get an analytical representation of the solution between the nodal points of the boundary value problem. Based on this representation, it is possible to construct a higher-quality discrete scheme. In the case of a coarse mesh (one nodal point inside the region), an approximate analytical solution of the boundary value problem can be obtained. In the simplest cases, this solution is accurate. To refine the solution, you can increase the number of moved nodes.

Using MNM, it is possible to improve the quality of the difference scheme. An increase in the accuracy of various schemes of the convective-diffusion problem using extrapolation of Richardson is given. Based on the developed algorithm, numerical calculations were performed.

1.1. Problem Statement

Let's consider a boundary value problem

$$\frac{d\Phi}{dx}=\frac{1}{Pe}\frac{d^2\Phi}{dx^2}+S(x),\quad W<x<E \tag{1}$$

$$\Phi(W)=\Phi_W,\quad \Phi(E)=\Phi_E \tag{2}$$

where Pe ($Pe=\rho vL/\Gamma$) -Peclet number, (v - a velocity, ρ – a denseness, *L*-scale of length, Γ - a diffusivity, *x*-dimensionless co-ordinate, *S(x)*- a source.

Exponential character of a solution and presence of narrow areas with the big gradients at values $Pe>>1$ are characteristic for this equation.

For a difference solution (1) there are various schemes. Here is an improvement of the difference scheme for (1) using Richardson extrapolation in combination with the method of moving nodes.

In [15, 16] two aspects of the application of the method of movable nodes are given. On the one hand, this method can be used to obtain an approximate analytical solution, and on the other, to obtain improved schemes. Here, the movable knot method is applied to improve the quality of the scheme using the Richardson method.

2. The Method of Movable Nodes for a One-Dimensional Convective-Diffusion Problem

Let inside the segment $x \in (W,E)$ take an arbitrary one node. Consider a difference analogue of Eq. (1), in which the convective term is approximated by upwind difference scheme.

Then the upwind difference scheme has the form

$$Pe\frac{U^1-U_W^1}{x-W}=\frac{2}{(E-W)}\left(\frac{U_E^1-U^1}{E-x}-\frac{U^1-U_W^1}{x-W}\right)+Pe\cdot S(x). \quad (3)$$

This scheme can be rewritten as follows:

$$a_P^1\,U^1=a_E^1U_E^1+a_W^1U_W^1+F^1(x),$$

Here

$$a_E^1=\frac{2}{(E-W)(E-x)},\; a_W^1=\frac{Pe}{(x-W)}+\frac{2}{(E-W)(x-W)},\; a_P^1=a_E^1+a_W^1,\; F^1(x)=Pe\cdot S(x)$$

From here we have

$$U^1=\frac{2(x-W)U_E^1+(E-x)\big(2+Pe(E-W)\big)U_W^1}{(E-W)\big(2+Pe(E-x)\big)}+\frac{(x-W)(E-x)}{2+Pe(E-x)}Pe\cdot S(x) \quad (4)$$

When changes $x\in(W,E)$ the position (we will make its moved in an interval (W,E), on the basis of (4) we will receive values of unknown function in each position. In other words, U^1 received by means of (4), will give us problem approximate solution. We will notice that in this case $U_W^1=\Phi(W),\,U_E^1=\Phi(E).$ The Superscript corresponds to an amount of moved grids.

When $x\in(W,E)$ changes its position (let's make it moveable within the interval (W,E)), based on (4) we get the values of the unknown function at each position. In other words, U^1 obtained with the help of (4), will give us an approximate solution to the problem. Note that in this case the Superscript corresponds to the number of movable nodes.

Add additional moved nodes: $x_1=\dfrac{x+W}{2},\; x_2=\dfrac{x+E}{2}.$

Now we have three moved nods $x,\,x_1,\,x_2.$ We will note that if x changes the positions x_1 and x_2 also change the positions.

The scheme of type (3) for a segment $[W,x]$ has the form:

$$Pe\frac{U_1^3-U_W^3}{(x-W)/2}=\frac{2}{(x-W)}\left(\frac{U^3-U_1^3}{x-x_1}-\frac{U_1^3-U_W^3}{x_1-W}\right)+Pe\cdot S(x_1). \quad (5)$$

Here $U_1^3=U^3(x_1)$. The scheme of type (3) for a segment $[x,E]$ has the form:

$$Pe\frac{U_2^3-U^3}{(E-x)/2}=\frac{2}{(E-x)}\left(\frac{U_E^3-U_2^3}{E-x_2}-\frac{U_2^3-U^3}{x_2-x}\right)+Pe\cdot S(x_2). \quad (6)$$

The upwind scheme for a segment $[x_1,x_2]$:

$$Pe\frac{U^3-U_1^3}{x-x_1}=\frac{2}{(x_2-x_1)}\left(\frac{U_2^3-U^3}{x_2-x}-\frac{U^3-U_1^3}{x-x_1}\right)+Pe\cdot S(x). \quad (7)$$

Here $U_2^3=U^3(x_2)$

In (7) we exclude $U_1^3,\,U_2^3$ using (5) and (6). Then we get the following scheme:

$$Pe\frac{U^3-U_W^3}{\frac{(x-W)}{2}\cdot(1+\tau_1)}=\frac{4}{(E-W)}\left(\frac{U_E^3-U^3}{\frac{E-x}{2}\cdot(1+\gamma_1)}-\frac{U^3-U_W^3}{\frac{x-W}{2}\cdot(1+\tau_1)}\right)+F^3(x) \quad (8)$$

The notation is introduced here $\tau_1 = 2/(2+\sigma),\ \gamma_1 = (2+\theta)/2$, $\sigma = Pe(x-W),$

$\theta = Pe(E-x),$

$$F^3(x) = Pe\cdot S(x) + \frac{4+Pe\cdot(E-W)}{E-W}\cdot\frac{1-\tau_1}{1+\tau_1}\cdot S(x_1) + \frac{4}{E-W}\cdot\frac{\gamma_1-1}{\gamma_1+1}\cdot S(x_2).$$

Where $U_W^3 = \Phi(W),\ U_E^3 = \Phi(E).$

(8) can be rewritten as follows:

$$a_P^3\ U^3 = a_E^3 U_E^3 + a_W^3 U_W^3 + F^3(x), \tag{9}$$

where
$$a_E^3 = \frac{8}{(E-W)(E-x)(1+\gamma_1)},\ a_W^3 = \frac{2Pe}{(x-W)(1+\tau_1)} + \frac{8}{(E-W)(x-W)(1+\tau_1)},$$

$$a_P^3 = a_W^3 + a_E^3.$$

Increase the number of moveable nodes: $x_1^- = \frac{x_1+W}{2} = \frac{x+3W}{4},$

$$x_1^+ = \frac{x_1+x}{2} = \frac{3x+W}{4},\quad x_2^- = \frac{x_2+x}{2} = \frac{3x+E}{4},\quad x_2^+ = \frac{x_2+E}{2} = \frac{x+3E}{4}.$$

In the difference scheme (9), the unknown function appears in three nodes: W, x, E.

Function S is calculated in points $x_1,\ x,\ x_2.$ We will write the scheme of type (9) for each of segments $[W,x].\ [x,W]$ and $[x_1,x_2].$

The scheme like (9) for a segment $[W,x]$ has the form:

$$a_{x_1}^3\ U_{x_1}^3 = a_x^3 U_x^3 + a_{W^-}^3 U_W^3 + F_-^3(x_1), \tag{10}$$

where
$$a_x^3 = \frac{8}{(x-W)(x-x_1)(1+\gamma_1^-)},\ a_{W^-}^3 = \frac{2Pe}{(x_1-W)(1+\tau_1^-)} + \frac{8}{(x-W)(x_1-W)(1+\tau_1^-)},$$

$$a_{x_1}^3 = a_x^3 + a_{W^-}^3,$$

$$F_-^3(x_1) = Pe\cdot S(x_1) + \frac{4+Pe\cdot(x-W)}{x-W}\cdot\frac{1-\tau_1^-}{1+\tau_1^-}\cdot S(x_1^-) + \frac{4}{x-W}\cdot\frac{\gamma_1^- -1}{\gamma_1^- +1}\cdot S(x_1^+),$$

$$\tau_1^- = 2/(2+\sigma^-),\ \gamma_1^- = (2+\theta^-)/2,\quad \sigma^- = Pe(x_1-W),\quad \theta^- = Pe(x-x_1).$$

Similarly, scheme of type (10) we will write for segments $[x,W]$ and $[x_1,x_2].$ Excluding in the received three sets of equations $U_{x_1}^3$ and $U_{x_2}^3$ we will receive the scheme with seven moved grids:

$$a_P^7\ U^7 = a_E^7 U_E^7 + a_W^7 U_W^7 + F^7(x), \tag{11}$$

where
$$a_E^7 = \frac{2^5(1-\gamma_2)}{(E-W)(E-x)(1-\gamma_2^4)},\ a_W^7 = \frac{4Pe(1-\tau_2)}{(x-W)(1-\tau_2^4)} + \frac{2^5(1-\tau_2)}{(E-W)(x-W)(1-\tau_2^4)},$$

$$a_P^7 = a_W^7 + a_E^7.\quad \tau_2 = 4/(4+\sigma),\ \gamma_2 = (4+\theta)/4$$

$$F^7(x) = Pe\cdot S(x) + \frac{8+Pe\cdot(x-W)}{x-W}\cdot\frac{(1-\tau_2)^2}{1-\tau_2^4}\cdot\sum_{j=1}^{3}\sum_{i=1}^{j}\tau_2^{i-1}S\left(W + j\frac{x-W}{4}\right) -$$

$$\frac{8}{E-W}\cdot\frac{(1-\gamma_2)^2}{1-\gamma_2^4}\cdot\sum_{j=1}^{3}\sum_{i=1}^{j}\gamma_2^{i-1}S\left(x+(4-j)\frac{E-x}{4}\right).$$

Continuing thus, we can receive the scheme with 2^k-1 moved grids

$$a_P^{(2^k-1)}\ U^{(2^k-1)} = a_E^{(2^k-1)} U_E^{(2^k-1)} + a_W^{(2^k-1)} U_W^{(2^k-1)} + F^{(2^k-1)}(x), \tag{12}$$

where

$$a_E^{(2^k-1)} = \frac{2^{2k+1}(1-\gamma_k)}{(E-W)(E-x)(1-\gamma_k^{2^k})},\ a_W^{(2^k-1)} = \frac{2^{2k+1}Pe(1-\tau_k)}{(x-W)(1-\tau_k^{2^k})} + \frac{2^{2k+1}(1-\tau_k)}{(E-W)(x-W)(1-\tau_k^{2^k})},$$

$$a_P^{(2^n-1)} = a_W^{(2^n-1)} + a_E^{(2^n-1)}. \ \tau_k = 2^k / (2^k + \sigma), \gamma_k = (2^k + \theta) / 2^k,$$

$$F^{(2^k-1)}(x) = Pe \cdot S(x) + \frac{2^{k+1} + Pe \cdot (E - W)}{E - W} \frac{(1-\tau_k)^2}{1-\tau_k^{2^k}} \sum_{j=1}^{2^k-1} \sum_{i=1}^{j} \tau_k^{i-1} \cdot S\left(x + j\frac{x-W}{2^k}\right) -$$

$$\frac{2^{k+1}}{E-W} \frac{(1-\gamma_k)^2}{1-\gamma_k^{2^k}} \sum_{j=1}^{2^k-1} \sum_{i=1}^{j} \gamma_k^{i-1} \cdot S\left(x + (2^k - j)\frac{E-x}{2^k}\right).$$

Fig. 1 shows the graphs of approximate solutions to problem (1), (2) obtained by (12) for with different movable nodes.

Fig-1. Pe=20, $\Phi_W = 0, \ \Phi_E = 1, \ S(x) = 0.$

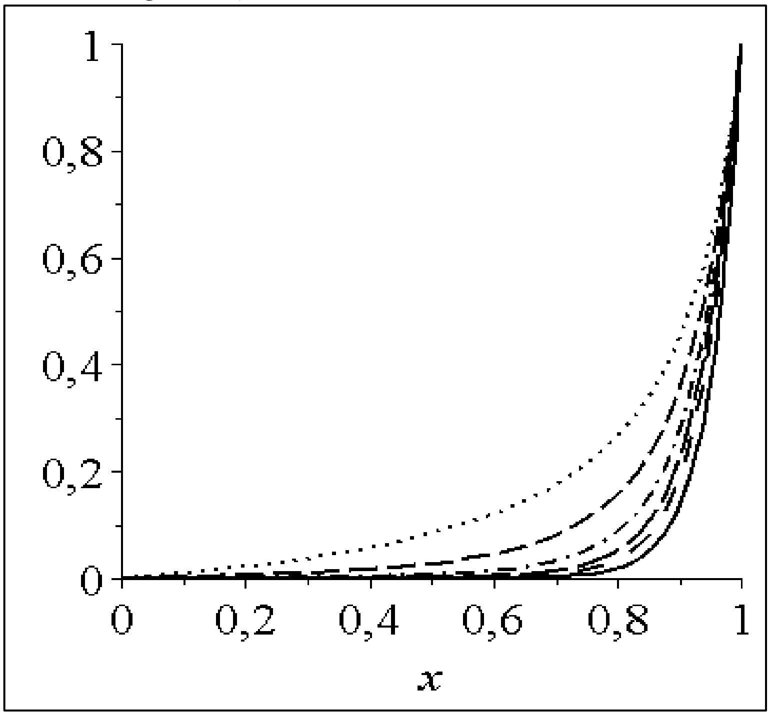
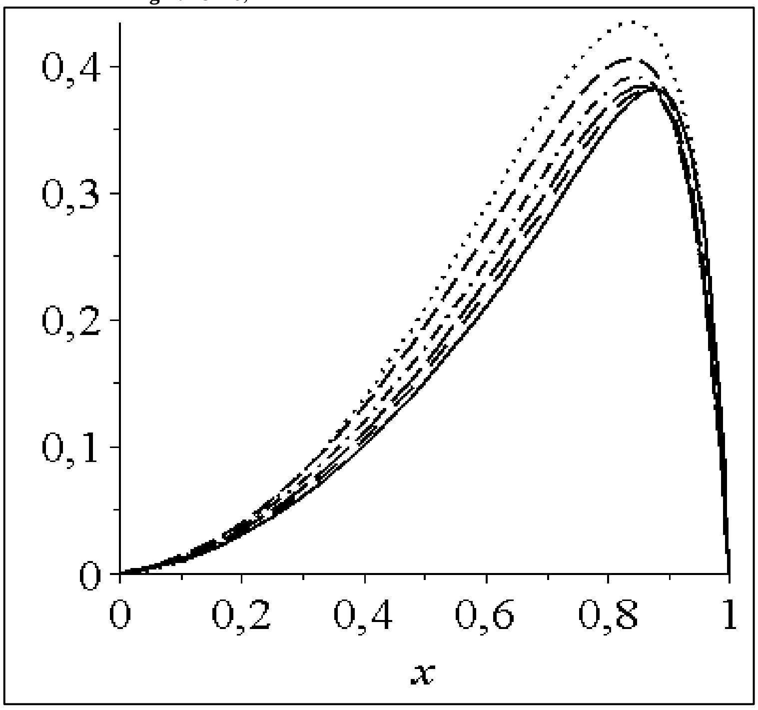


Fig-2. Pe=20, $\Phi_W = 0, \ \Phi_E = 0, \ S(x) = x,$



Approximate solutions of problem. Pointwise - at k=1, dashed - k=2, is pointwise-dotted - k=3, long dashed - k=4, seldom dashed - k=5. The solid line is exact solution.

The graphs show that approximate solutions give good results.

3. Improving Accuracy with Richardson Extrapolation

Using the method described in Marchuk and Shaidurov [16], we can improve the accuracy of approximate solutions to the problem. Linear combination $Q^3(x) = -\frac{1}{3}U^1(x) + \frac{4}{3}U^3(x)$ more closely approximates the solution. With a linear combination $U^1(x), U^3(x)$ and $U^7(x)$ in the form $Q^7(x) = \frac{1}{45}U^1(x) - \frac{4}{9}U^3(x) + \frac{64}{45}U^7(x)$ we get a more refined solution to the problem.

Figure 3 shows the graphs of approximate solutions of problem (1), (2) obtained by (12) by Richardson extrapolation at $W = 0,\ E = 1$. The solid line in Fig. 3 the exact solution.

Fig-3. $\Phi_W = 0,\ \Phi_E = 1,\ S(x) = 0,$

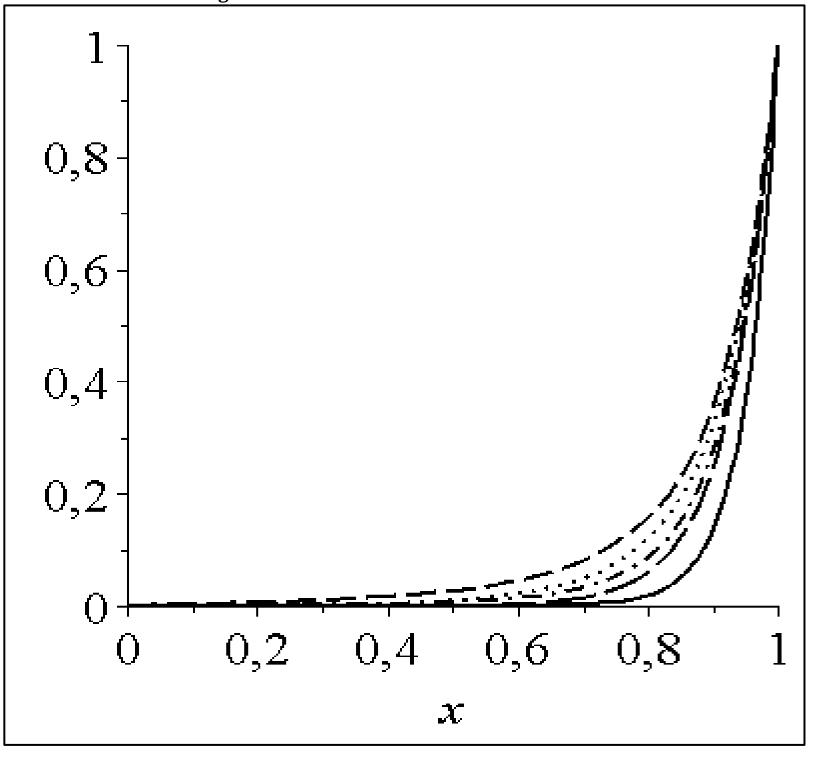
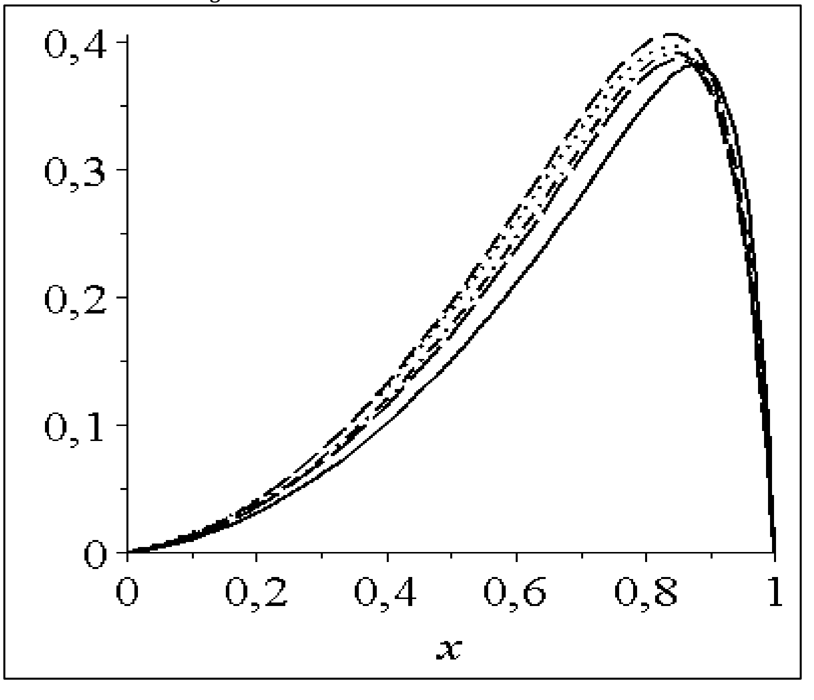


Fig-4. $\Phi_W = 0,\ \Phi_E = 0,\ S(x) = x,$



$Pe = 20.$ Comparisons of solutions. A dashed line - $U^3(x)$, pointwise - $Q^3(x)$, pointwise-dashed - $U^7(x)$, long dashed- $Q^7(x)$

Fig-5. $\Phi_W = 0,\ \Phi_E = 1,\ S(x) = 0,$

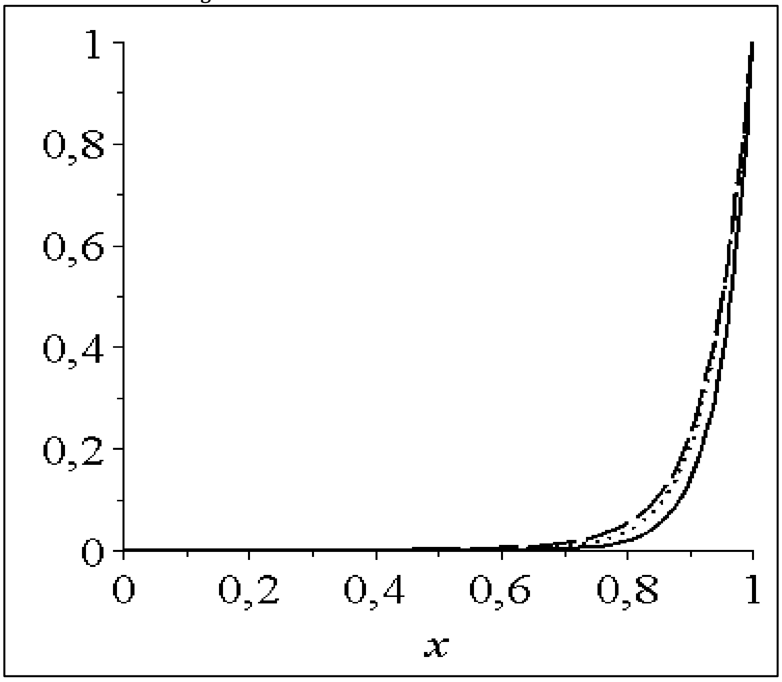
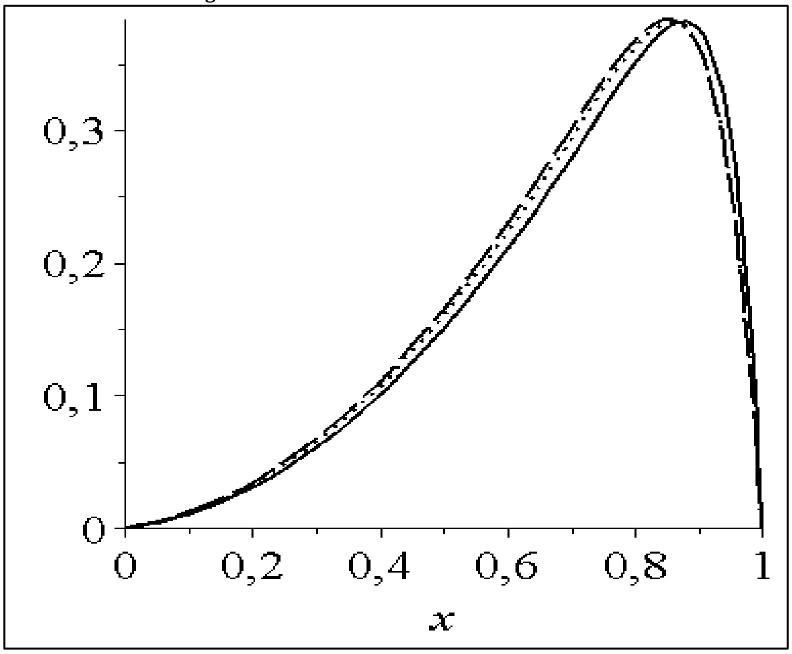


Fig-6. $\Phi_W = 0,\ \Phi_E = 0,\ S(x) = x,$



$Pe = 20.$ Comparisons of solutions. A dashed line - $U^{15}(x)$, pointwise – $Q^{15}(x)$

4. Numerical Experiments

Simulation of the approximate solution to the problem given above can be used to construct difference schemes. Suppose we put $W = 0, \quad E = 1.$ Introduce on [0,1] a non-uniform grid

$$\Omega = \{x_i, i = 0,1,2,...,N,\ 0 = x_0 < x_1 < ... < x_{i-1} < x_i < x_{i+1} < ... < x_N = 1\}$$

Let put $W = 0,\ E = 1.$. We introduce a non-uniform grid on [0,1]

$$\Omega = \{x_i, i = 0,1,2,...,N,\ 0 = x_0 < x_1 < ... < x_{i-1} < x_i < x_{i+1} < ... < x_N = 1\}$$

If we replace $W \to x_{i-1},\ x \to x_i,\ E \to x_{i+1}$ in (12), we obtain a difference scheme approximating equation (1) in the node $x_i\ (i = 1, 2, \ldots, N-1)$.

The accuracy of the scheme (3), with a uniform arrangement of grid nodes, is. $O(h)$

The accuracy of the scheme (3), with a uniform arrangement of grid nodes, is $O(h)$. Scheme (9) has the order $O(h/2)$. For the linear combination $Q^3(x_i) = -\frac{1}{3}U^1(x_i) + \frac{4}{3}U^3(x_i)$, we obtain an approximation error $O(h^2)$ on a uniform grid. The linear combination $U^1(x_i), U^3(x_i)$ and $U^7(x_i)$ in the form $Q^7(x_i) = \frac{1}{45}U^1(x_i) - \frac{4}{9}U^3(x_i) + \frac{64}{45}U^7(x_i)$ has an approximation order $O(h^4)$.

Consider $S(x) = x^2$, $Pe = 30$. Table 1 shows the absolute difference between the exact and approximate solutions according to the schemes.

Table-1.

x	**0,1**	**0,2**	**0,3**	**0,4**	**0,5**	**0,6**	**0,7**	**0,8**	**0,9**
Схема (3)	0,001	0,004	0,007	0,011	0,017	0,025	0,039	0,073	0,160
$U^3(x_i)$	0,001	0,002	0,004	0,006	0,008	0,012	0,018	0,034	0,089
$U^7(x_i)$	0,000	0,001	0,002	0,003	0,005	0,007	0,010	0,019	0,046
$Q^3(x_i)$	0,000	0,001	0,002	0,004	0,006	0,007	0,010	0,021	0,065
$Q^7(x_i)$	0,000	0,001	0,001	0,002	0,003	0,005	0,007	0,014	0,030

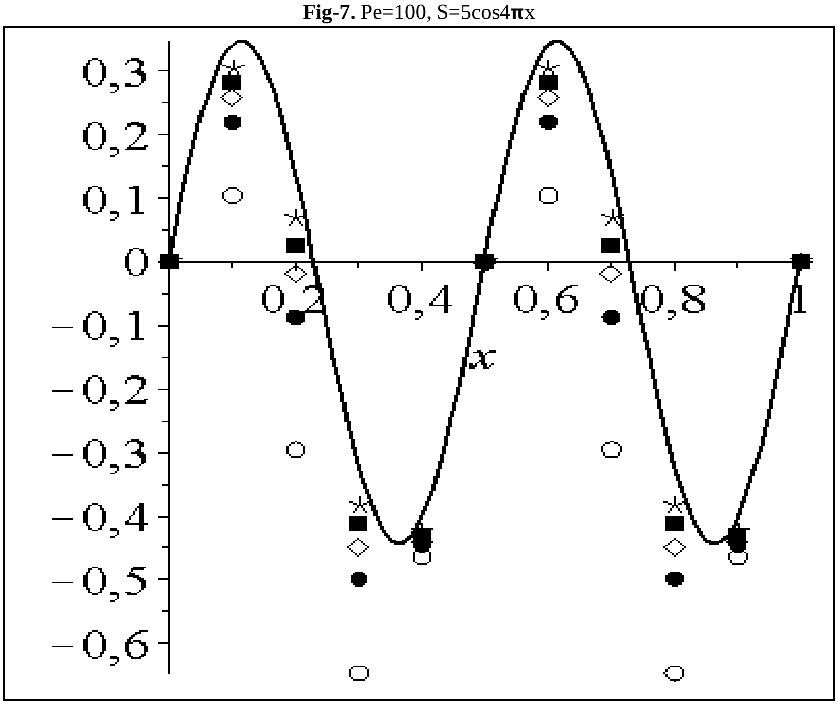
Table 2 shows the standard error $\sigma = \sqrt{\sum_1^N (\Phi(x_i) - U_i)^2 / N}$ of the considered schemes. $\Phi(x_i)$ the exact solution at the nodal points, U_i is the numerical solution obtained by the considered schemes.

Table-2.

Scheme	**(3)**	$U^3(x)$	$U^7(x)$	$Q^3(x)$	$Q^7(x)$
$S=x^2$, Pe=50, $\Phi_W=0$, $\Phi_E=1$	0,047	0,023	0,011	0,015	0,006
S=10, Pe=50, $\Phi_W=0$, $\Phi_E=1$	0,033	0,017	0,008	0,011	0,005
$S=x^2$, Pe=100, $\Phi_W=0$, $\Phi_E=1$	0,034	0,014	0,006	0,008	0,003
S=5cos(4**π**x), Pe=50, $\Phi_W=0$, $\Phi_E=1$.	0,213	0,120	0,061	0,090	0,038

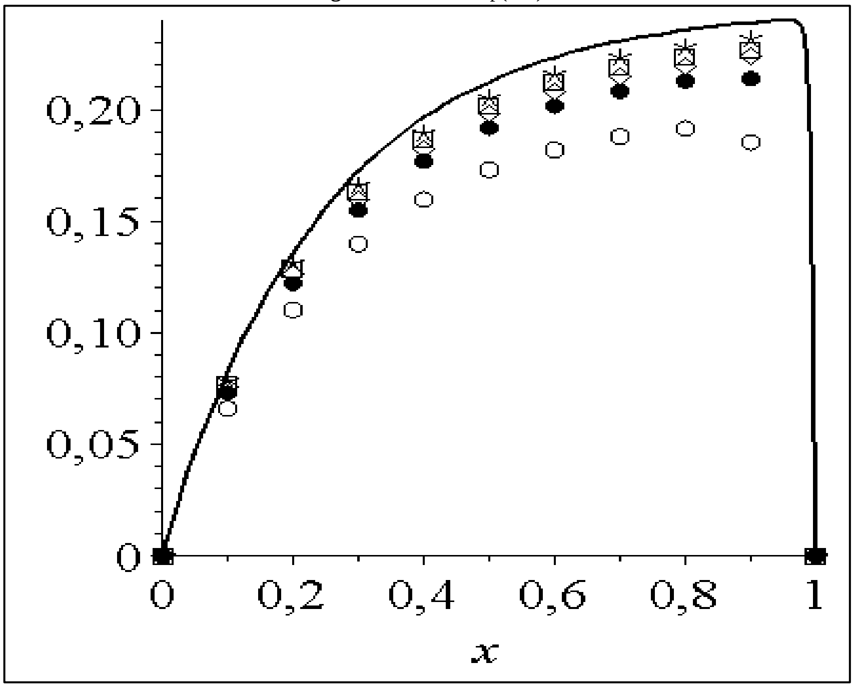
Figure 7, 8 shows the numerical solutions for $\Phi_W = 0$, $\Phi_E = 0$.

Fig-7. Pe=100, S=5cos4**π**x



The solid curve is the exact solution, the circle obtained according to scheme (3), the circle according to U^3, the solid rectangle according to U^7, the diamond Q^3, the star according to Q^7.

Fig-8. Pe=200, S=exp(-4x)



The solid curve is the exact solution, the circle obtained according to scheme (3), the circle according to U^3, the solid rectangle according to U^7, the diamond Q^3, the star according to Q^7.

From the graphs in Fig. 7, 8 and from Tables 1, 2 it is clear that the linear combination according to Richardson gives a more improved scheme.

5. Conclusions

With the help of the method of movable nodes and the method of the Richardson extrapolation, it is possible to construct a better scheme. The approach presented here can be successfully applied to other boundary value problems.

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