Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 7, Issue. 1, pp: 1-15, 2021
URL: https://arpgweb.com/journal/journal/17
DOI: https://doi.org/10.32861/ajams.71.1.15

Academic Research Publishing Group

Original Research | Open Access

The Gompertz Gumbel II Distribution: Properties and Applications

Adebisi Ade Ogunde (Corresponding Author)
Department of Statistics, University of Ibadan, Oyo State, Nigeria
Email: debiz95@yahoo.com

Gbenga Adelekan Olalude
Department of Statistics, Federal Polytechnic Ede, Osun State, Nigeria

Donatus Osaretin Omosigho
Department of Mathematics and Statistics, Federal Polytechnic Ado-Ekiti, Ekiti State, Nigeria

Article History
Received: 30 September 2020
Revised: 28 October, 2020
Accepted: 12 November 2020
Published: 16 November 2020

Abstract
In this paper we introduced Gompertz Gumbel II (GG II) distribution which generalizes the Gumbel II distribution. The new distribution is a flexible exponential type distribution which can be used in modeling real life data with varying degree of asymmetry. Unlike the Gumbel II distribution which exhibits a monotone decreasing failure rate, the new distribution is useful for modeling unimodal (Bathtub-shaped) failure rates which sometimes characterised the real life data. Structural properties of the new distribution namely, density function, hazard function, moments, quantile function, moment generating function, orders statistics, Stochastic Ordering, Renyi entropy were obtained. For the main formulas related to our model, we present numerical studies that illustrate the practicality of computational implementation using statistical software. We also present a Monte Carlo simulation study to evaluate the performance of the maximum likelihood estimators for the GGTT model. Three life data sets were used for applications in order to illustrate the flexibility of the new model.

Keywords: Quantile function; Bathtub-shaped failure rate; Renyl entropy; Orders statistics.

1. Introduction

The Gumbel type-2 distribution plays an important role in Extreme value theory. The distribution can be used for modeling extreme events such as in the field of risk based engineering, flood frequency analysis, Meteorology, structural engineering, software reliability engineering, network engineering and Seismology. The distribution has not gained popularity/prominence in the area of application unlike the Weibull distribution because of its lack of fit. The Gumbel type-2 distribution can only be applied to real life data with monotonic failure rates. On the contrary, in real life situations the hazard rate of many complex phenomena that are often encountered in practice is non-monotone and cannot be modeled by the Gumbel type-2 distribution. To address this limitation [1] proposed the Exponentiated Gumbel (EG) type-2 distribution according to Nadarajah and Kotz [2] version of Gupta, *et al.* [3], Okorie, *et al.* [4] proposed the Kumaraswamy G Exponentiated Gumbel Type-2 Distribution which was obtained by combining Exponentiated Gumbel (EG) and the kumaraswamy distribution [5, 6] proposed and studied the properties of Extended Gumbel type-2 $(EGTT)$ distribution.

Motivated by some of the properties of the generalised distribution with respect to the nature of its hazard function which includes; increasing, decreasing, non-monotone and bathtub shapes as well as the tractability and flexibility of the generalised distribution with improved statistical properties. We propose and study a new distribution called the Gompertz Gumbel type-2 distribution which inherits these desirable properties with improved modeling capabilities most especially in modeling life time data.

The cumulative distribution function (cdf) of the Gumbel type-2 distribution is given by

$$F(x;\theta,\delta) = e^{-\theta x^{-\delta}}, \qquad x > 0;\ \theta,\delta > 0 \tag{1}$$

With the corresponding (pdf) given by

$$f(x;\theta,\delta) = \theta\delta x^{-\delta-1} e^{-\theta x^{-\delta}}, \qquad x > 0;\ \theta,\delta > 0 \tag{2}$$

Where θ is a shape parameter and δ is a scale parameter.

Recently, Alizadeh, *et al.* [7] developed Gompertz-G family of distributions by making use of Transformed-Transformer (T-X) family of distribution, defined as

$$G(x) = \int_a^{J[F(x)]} r(t)dt, \tag{3}$$

The pdf corresponding to equation (3) is given by

$$g(x) = \left\{\frac{d}{dx}J[F(x)]\right\} r\{J[F(x)]\} \tag{4}$$

Setting $W[G(x)] = -log[1 - F(x)]$ and $r(t) = \lambda e^{\alpha t} e^{-\frac{\lambda}{\alpha}(e^{\alpha t}-1)}, t > 0$. The cdf of the $Gompertz - G$ $(Go - G)$ family by

$$G(x) = \int_{0}^{-\log[1-F(x;V)]} \lambda e^{\alpha t} e^{-\frac{\lambda}{\alpha}(e^{\alpha t}-1)} dt = 1 - e^{\frac{\lambda}{\alpha}\{1-[1-F(x;V)]^{-\alpha}\}} \quad (5)$$

Where $F(x; V)$ is the baseline cdf depending on the parameter is vector V and $\alpha, \lambda > 0$ are two additional shape parameters. For a specified baseline G, the $Go - G$ is defined by the cdf (5) which represents a wide family of distributions.

The associated pdf to equation (5) is given by

$$g(x; \alpha, \lambda, V) = \lambda f(x; V)[1 - F(x; V)]^{-\alpha-1} e^{\frac{\lambda}{\alpha}\{1-[1-F(x;V)]^{-\alpha}\}} \quad (6)$$

The new density function is most tractable when $F(x; V)$ and $f(x; V)$ have simple analytical expressions.

Based on the generalization in equation (5) and (6), several flexible distribution have been proposed and studied not limited to the work of Oguntunde, *et al.* [8] studied the properties of Gompertz inverse exponential distribution, Khaleel, *et al.* [9] studied the Gompertz flexible Weibull distribution and its applications etc.

1.1. The Gompertz Gumbel Type-2 ($GGTT$) Distribution

Now, suppose that (1) and (2) are any continuous baseline cumulative distribution function (cdf) and probability density function (pdf) of Gumbel type-2 (GTT) distribution. The Gompertz Gumbel type-2 (GGTT) distribution is obtained by putting equation (1) in (5) given by

$$G(x) = \mathbf{1} - e^{\frac{\lambda}{\alpha}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]^{-\alpha}\right\}} \quad (7)$$

And the corresponding pdf is given by

$$g(x) = \lambda\theta\delta x^{-\delta-1} e^{-\theta x^{-\delta}} \left(1 - e^{-\theta x^{-\delta}}\right)^{-\alpha-1} e^{\frac{\lambda}{\alpha}\left\{1-[1-e^{-\theta x^{-\delta}}]^{-\alpha}\right\}} \quad (8)$$

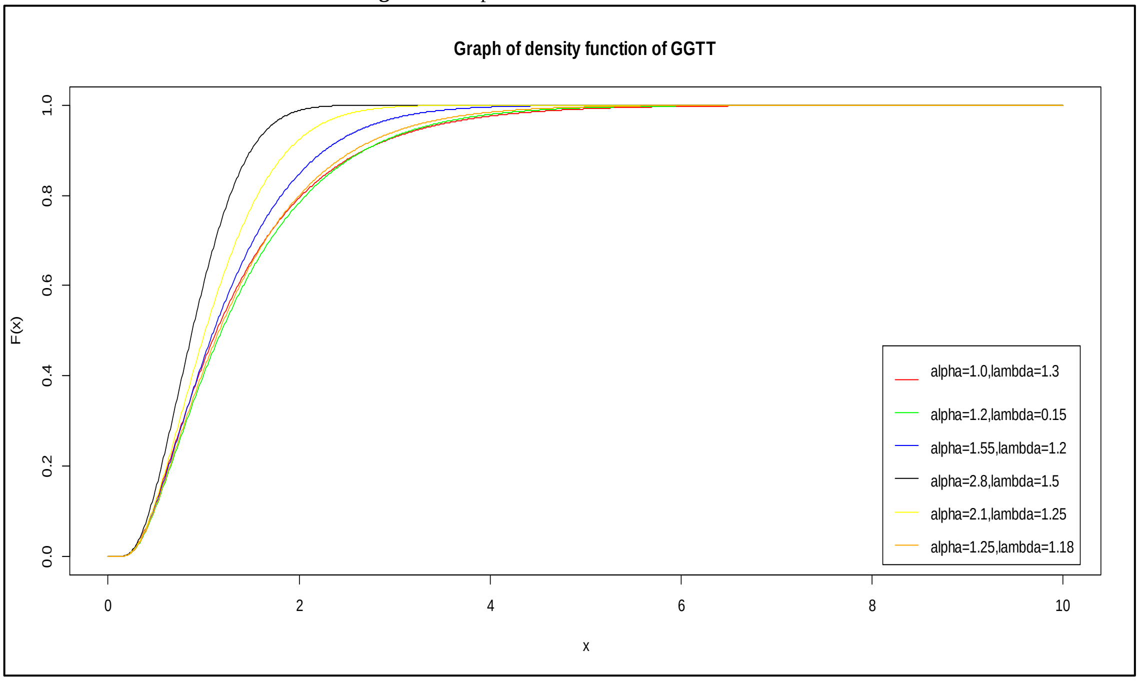
Special sub-models of the GGTT distribution are recorded in Table 1.

Table-1. Sub-models of GGTT distribution

α	β	θ	δ	*Reduced model*	*Authors*
–	–	1	–	*Gompertz Inverse Exponential*	*Oguntunde, et al. [8]*
–	–	–	−1	*Gompertz Exponential*	
1	1	1	–	*Inverse Exponential*	Keller and Kamath [10]
1	1	–	−1	*Exponential*	

The graph of the cdf is drawn below in figure 1. taking the value of parameters $(theta = 0.5\ and\ sigma = 1.2)$ and varying the values of parameters (alpha and lambda).

Figure-1. Graph of the cdf of GGTT distribution



- The figure 1. drawn above indicates the GGTT distribution has a proper pdf.

An expression for the survival function $S(x) = 1 - G(x)$, hazard rate $h(x) = \frac{g(x)}{S(x)}$ and the reversed hazard rate $\psi(x) = \frac{g(x)}{F(x)}$ are respectively given by

$$S(x) = e^{\frac{\lambda}{\alpha}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]^{-\alpha}\right\}} \tag{9}$$

$$h(x) = \lambda\theta\delta x^{-\delta-1} e^{-\theta x^{-\delta}}\left(1-e^{-\theta x^{-\delta}}\right)^{-\alpha-1} \tag{10}$$

and

$$\psi(x) = \frac{\lambda\theta\delta x^{-\delta-1} e^{-\theta x^{-\delta}}\left(1-e^{-\theta x^{-\delta}}\right)^{-\alpha-1} e^{\frac{\lambda}{\alpha}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]-\alpha\right\}}}{\mathbf{1} - e^{\frac{\lambda}{\alpha}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]-\alpha\right\}}} \tag{11}$$

Figure 2. drawn below is the graph of the pdf of GGTT distribution taking $alpha\ = 1.1\ and\ lambda = 0.5$, for diagram I and $theta = 1.5\ , sigma\ = 1.2$ for diagram II. Figure 3. is the graph of the survival function for arbitrary values of the parameters and Figure 4. is the graph of the hazard function of GGTT distribution.

Figure-2. The graph of the pdf of GGTT distribution

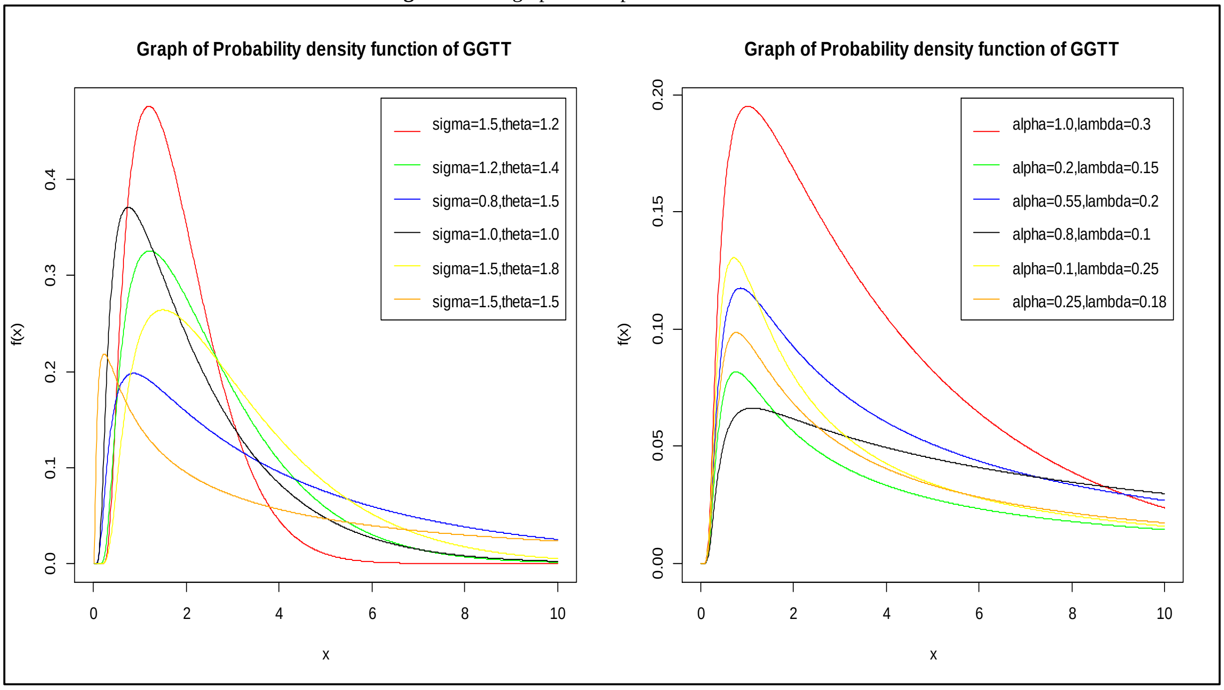


Figure-3. The graph of the Survival function of GGTT distribution

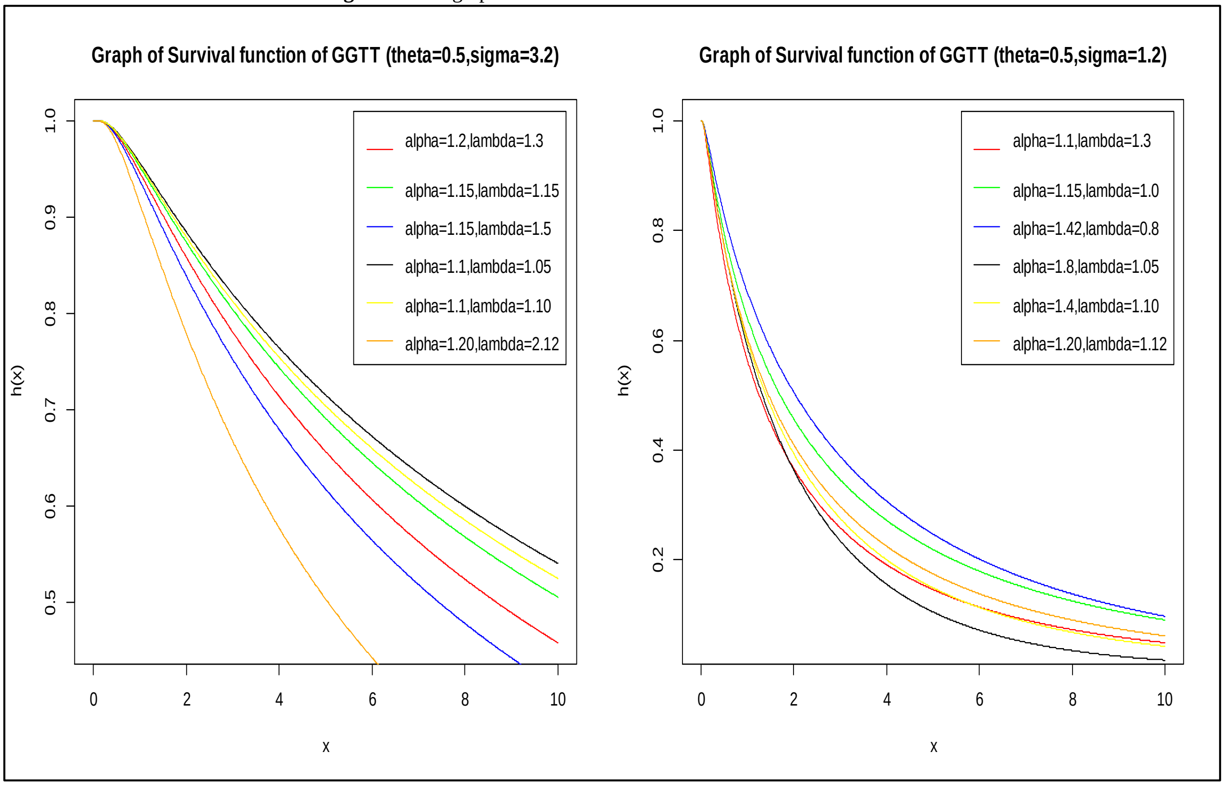
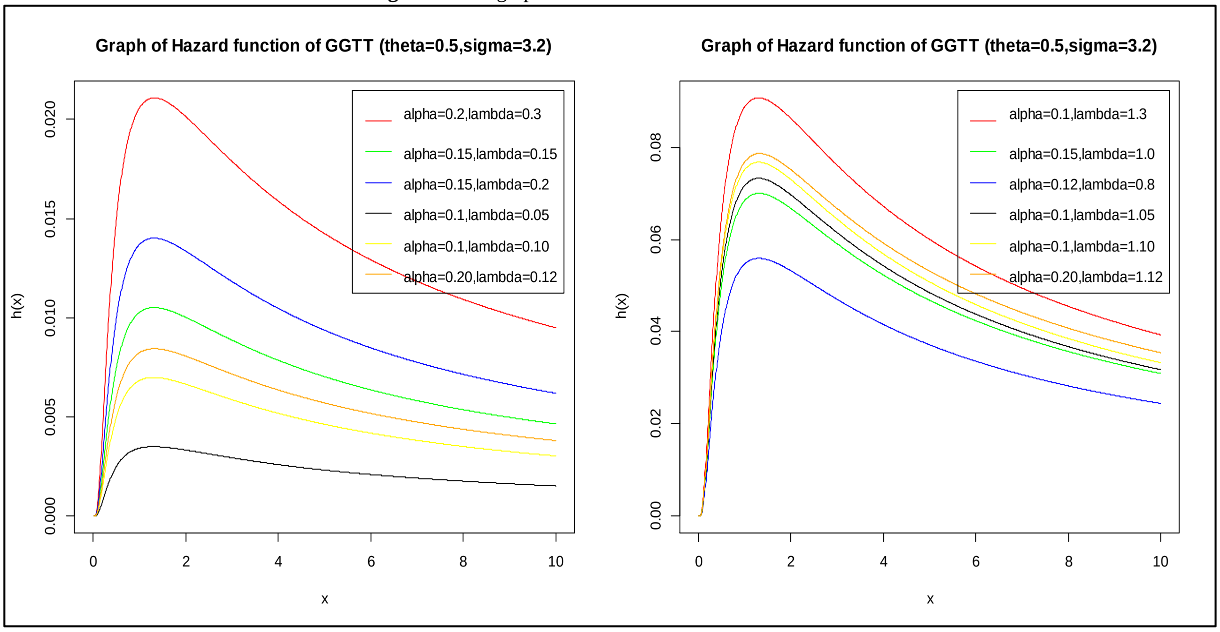


Figure-4. The graph of the hazard of GGTT distribution



1.2. Quantile Function

The quantile function plays a useful role when simulating random variates from a statistical distribution. The quantile function of the $GGTT$ distribution, say $X = Q_u$ is given by:

$$Q_u = \left\{-\frac{1}{\theta}log\left[1-\left(1-\frac{\alpha}{\lambda}log\{1-u\}\right)^{-\frac{1}{\alpha}}\right]\right\}^{-\frac{1}{\delta}}, \qquad 0 < u < 1 \tag{12}$$

In many heavy tailed distributions, the classical measures of skewness and kurtosis may be difficult to obtain as a result of nonexistence of higher moments. In such a situation, the quantile measures could be a better option. The Bowley skewness [11] is one of the earliest measures of skewness based on quartiles of a distribution. It is defined as:

$$\mathfrak{B} = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}} \tag{13}$$

Similarly, the coefficient of kurtosis can be estimated using the Moors' coefficient of kurtosis [12] is obtained based on the octiles of a distribution defined as:

$$\mathcal{M} = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}} \tag{14}$$

It should be noted that the two measures are less sensitive to outliers and they exist for distribution which moment cannot be defined.

Table 1. drawn below gives the various values of Bowley Skewness ($\mathfrak{B}$) and Moor kurtosis ($\mathcal{M}$) for arbitrary values of the parameters taking a fixed value of ($\theta = 2.1, \delta = 0.2$).

Table-2. Skewness and Kurtosis of the $GGTT$ distribution for different values of parameters

α, λ	$Q_{0.25}$	$Q_{0.5}$	$Q_{0.75}$	$Q_{0.125}$	$Q_{0.375}$	$Q_{0.625}$	$Q_{0.875}$	$\mathfrak{B}$	$\mathcal{M}$
2.1, 5.5	0.0942	0.1221	0.1557	0.0781	0.1081	0.1373	0.1824	0.0927	0.2455
4.1, 10.5	0.1195	0.0987	0.0798	0.0681	0.0894	0.1083	0.1349	−0.0479	−0.1335
8.1, 15.5	0.1029	0.0877	0.0729	0.0632	0.0806	0.0950	0.1134	−0.0133	−0.0033
10.1, 20.5	0.0952	0.0819	0.0688	0.0602	0.0757	0.0882	0.1041	−0.0076	−0.0005
15.1, 25.5	0.0886	0.0774	0.0658	0.0579	0.0719	0.0828	0.0960	0.0175	0.0351
20.1, 30.5	0.0841	0.0741	0.0636	0.0562	0.0692	0.0789	0.0905	0.0244	0.0683
25.1, 35.5	0.0807	0.0716	0.0618	0.0549	0.0670	0.0760	0.0864	0.0370	0.0899
50.1, 70.5	0.0694	0.0627	0.0552	0.0496	0.0592	0.0660	0.0736	0.0563	0.1408

- From Table 1. drawn above it can be deduced that the GGTT distribution can be used to model data that are positively or negatively skewed.

1.3. Mathematical Properties of Gompertz Gumbel-Type Distribution

Here, we examined the mathematical properties of $GGTT$ distribution

If $|\kappa| < 1$ and $\gamma > 0$ is a real non-integer, the following expansion exist

$$(1-\kappa)^{-\gamma} = \sum_{i=0}^{\infty}\binom{\gamma+i-1}{i}\kappa^i \tag{15}$$

$$e^k = \sum_{j=0}^{\infty} \frac{k^j}{j!} \tag{16}$$

If γ is an integer, index i in the previous sum, stops at $\gamma - 1$. Using the equations (15) and (16) we can re-write the pdf of $GGTT$ distribution given in (8) as

$$g(x) = \lambda\theta\delta \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty} \left(\frac{\lambda}{\alpha}\right)^i \binom{i}{j} (-1)^j (i!)^{-1} \binom{\alpha(j+1)+k}{k} x^{-\delta-1} e^{-\theta x^{-\delta}(k+1)} \tag{17}$$

Finally the pdf of GGTT distribution can be written as

$$g(x) = \theta\delta \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\, x^{-\delta-1} e^{-\theta x^{-\delta}(k+1)} \tag{18}$$

Where

$$S_{i,j,k}(\alpha,\lambda) = \lambda \sum_{i=0}^{\infty}\sum_{j=0}^{i} \left(\frac{\lambda}{\alpha}\right)^i \binom{i}{j} (-1)^j (i!)^{-1} \binom{\alpha(j+1)+k}{k} \tag{19}$$

The remaining parts of the paper is arranged as follows: section 2 presents the comprehensive study of the moments, variance, skewness, kurtosis and moment generating function of the new model; section 3 presents a study on the Renyi entropy of the new distribution; section 4 presents the Stochastic ordering of the new distribution; section 5 gives a comprehensive review of the r^{th}order statistics of the new distribution which includes the $1^{th}$and n^{th} order; section 6: proposes the maximum likelihood estimation method of estimating the new model; in section 7 we carried out Monte Carlo simulation to validate the maximum likelihood estimation technique that was used to analyse the data; section 8 presents the applications of the new model and section 9 the conclusion.

2. k^{th} Raw Moment

Here we derive an expression for raw moments of GGTT distribution as

$$\mu'_p = E(X^p) = \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta\delta \int_0^{\infty} x^{-\delta-1} e^{-\theta x^{-\delta}(k+1)} \tag{20}$$

By letting $y = \theta x^{-\delta}(k+1)$and substitute in equation (20), we have

$$= \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta\delta[\theta(k+1)]^{p-\delta} \int_0^{\infty} y^{-\frac{p}{\delta}} e^{-y} dy \tag{21}$$

Finally we have

$$\mu'_p = \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta[\theta(k+1)]^{k-\delta}\Gamma\left\{1-\frac{p}{\delta}\right\}, \qquad p < \delta \tag{22}$$

$\delta > p$, Where $\Gamma(w) = \int_0^{\infty} z^{w-1}e^{-z}dz$ the complementary incomplete gamma function.

The first four raw moments for $p = 1, k = 2, p = 3$ and $p = 4$ are respectively,

$$\mu'_1 = \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta[\theta(j+1)]^{1-\delta}\Gamma\left\{1-\frac{1}{\delta}\right\} \tag{23}$$

$$\mu'_2 = \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta[\theta(j+1)]^{2-\delta}\Gamma\left\{1-\frac{2}{\delta}\right\} \tag{24}$$

$$\mu'_3 = \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta[\theta(j+1)]^{3-\delta}\Gamma\left\{1-\frac{3}{\delta}\right\} \tag{25}$$

$$\mu'_4 = \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\theta[\theta(j+1)]^{4-\delta}\Gamma\left\{1-\frac{4}{\delta}\right\} \tag{26}$$

Then, $skewness = \frac{(\mu'_3)^2}{(\mu'_2)^3}$, $kurtosis = \frac{\mu'_4}{(\mu'_2)^2}$, variance$(\mu_2) = \mu'_2 - (\mu'_2)^2$ and coefficient of variation $(CV) = \frac{(\mu_2)^{\frac{1}{2}}}{\mu'_1}$

Table 2. drawn below gives the various values of variance (μ_2) and coefficient of variation (CV) for arbitrary values of the parameters taking a fixed value of $(\theta = 0.1, \delta = 1.15)$

Table-3. Values of (μ_2) and (CV) of GGTT distribution for different values of parameters

α,λ	μ_1'	μ_2'	μ_3'	μ_4'	μ_2	CV
0.5,1.5	0.1622	0.0417	0.0169	0.0102	0.015419	0.7655
1.0,1.0	0.1771	0.0441	0.0145	0.0060	0.012720	0.7011
1.5,2.5	0.0990	0.0118	0.0017	0.0003	0.002001	0.4521
3.5,4.5	0.0712	0.0056	0.0005	$4.5289e-5$	0.000539	0.3262
5.5,7.5	0.0589	0.0038	0.0003	$1.7986e-5$	0.000267	0.2773
8.5,10.5	0.0524	0.0029	0.0002	$1.3890e-5$	0.000165	0.2450
10.5,13.5	0.0489	0.0025	0.0001	$8.0206e-6$	0.000127	0.2305
`15.5,20.5	0.0439	0.0020	$3.6917e-5$	$1.4525e-6$	8.4311e-5	0.2092

2.1. Moment Generating Function

The moment generating function of GGTT distribution is obtained as

$$M_X(t) = E(e^{tX}) \tag{27}$$

Applying relation in equation (16) to equation (27), we have

$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} E(X^p) \tag{28}$$

Substitute for $E(X^k)$ in equation (22), we have

$$M_X(t) = \theta \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda) \frac{t^p}{p!} [\theta(j+1)]^{p-\delta} \Gamma\left\{1-\frac{p}{\delta}\right\} \tag{29}$$

2.2. Incomplete Moment

The incomplete moment has important applications in different fields of study. The first incomplete moment is used in estimation of the Bonferroni and Lorenz curves which are useful in reliability, demography, insurance, seismology and medicine. The k^{th} incomplete moment of the GGTT random variable is:

$$\omega_x(t) = \int_0^t x^k dF_{GGTT}(x) \tag{30}$$

Then

$$\omega_x(t) = \theta\delta \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda) \int_0^t x^{-\delta-1} e^{-\theta x^{-\delta}(j+1)} dx \tag{31}$$

Using the complementary incomplete gamma function, this yields:

$$\omega_x(t) = \theta \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\, [\theta(j+1)]^{p-\delta} \Gamma\left\{1-\frac{p}{\delta}, \theta t^{-\delta}(j+1)\right\} \tag{32}$$

$\delta > k$, Where $\Gamma(p,q) = \int_q^{\infty} z^{p-1} e^{-z} dz$ the complementary incomplete gamma function.

2.3. Mean Residual Life and Mean Inactivity Time

The Mean Residual Life (MRL) or the life expectancy at age t is the expected additional life length for a unit, which is alive at age t. The MRL has several important applications in life time testing of product, life insurance, demography and economics etc. The MRL is given by:

$$m_X(t) = E(X-t/X>t), \quad t>0 \tag{33}$$

Which can also we written as

$$m_X(t) = \frac{\{\mu - \omega_1(t)\}}{S(x)} - 1 \tag{34}$$

Where, $\mu = \mu_1$ and $\omega_1(t)$ is the first incomplete moment and $S(t)$ is the survival function. Thus, the MRL of the $GGTT$ distribution is:

$$m_X(t) = \frac{\left\{\mu - \theta \sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\, [\theta(j+1)]^{p-\delta} \Gamma\left\{1-\frac{p}{\delta}, \theta t^{-\delta}(k+1)\right\}\right\}}{e^{\frac{\lambda}{\alpha}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]^{-\alpha}\right\}}} - 1, \delta > 1 \tag{35}$$

The Mean Inactivity Time (MIT) is the waiting time elapsed since the failure of an item on condition that the failure had occurred in (0,t). The MIT of the $GGTT$ random variable X is defined for $t > 0$ as:

$$\mathcal{Z}_X(t) = E(t - X/X \le t) \tag{36}$$

This can further be expressed as

$$\mathcal{Z}_X(t) = t - \frac{\omega_1(t)}{F(t)} \tag{37}$$

Substituting the first incomplete moment and the CDF of the $GGTT$ random variable yields its MIT as:

$$\mathcal{Z}_X(t) = t - \frac{\theta\sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\,[\theta(j+1)]^{p-\delta}\Gamma\left\{1-\frac{p}{\delta},\theta t^{-\delta}(k+1)\right\}}{1-e^{\frac{\lambda}{\alpha}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]^{-\alpha}\right\}}} \tag{38}$$

3. Renyi Entropy

Entropy has been applied in the field of engineering sciences and information theory as measures of variation of uncertainty. The Renyi entropy [13] of a random variable X having the GGTT distribution is given as:

$$I_R(v) = \frac{1}{1-v}\log\left[\int_0^{\infty} g_{GGTT}^{v}(x)dx\right], \qquad v>0 \text{ and } v\neq 1 \tag{39}$$

Putting equation (19) in (44), we have

$$I_R(v) = \frac{1}{1-v}\log\left[\int_0^{\infty}\left\{\theta\delta\sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\,x^{-\delta-1}e^{-\theta x^{-\delta}(k+1)}\right\}^{v}dx\right] \tag{40}$$

After simple substitution, we have

$$I_R(v) = \frac{1}{1-v}\log\left\{(\theta)^{v}\delta^{v-1}\left(\sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\right)^{v}[\theta v(k+1)]^{\frac{1-(\delta+1)v}{\delta}}\Gamma\left\{\frac{(\delta+1)v-1}{\delta}\right\}\right\} \tag{41}$$

Where $v>0$ and $v\neq 1$. The Renyl entropy converges to the Shannon entropy as v approaches 1. The v entropy, say $H(v)$ of the GGTT random variable is defined by:

$$H(v) = \frac{1}{v-1}\log\{I - I_v(x)\} \tag{42}$$

Where

$$I_v(x) = \left[\int_0^{\infty} f_{GGTT}^{v}(x)dx\right], \qquad v>0 \text{ and } v\neq 1 \tag{43}$$

Hence,

$$H(v) = \frac{1}{v-1}\log\left\{I-(\theta)^{v}\delta^{v-1}\left(\sum_{k=0}^{\infty} S_{i,j,k}(\alpha,\lambda)\right)^{v}[\theta v(k+1)]^{\frac{1-(\delta+1)v}{\delta}}\Gamma\left\{\frac{(\delta+1)v-1}{\delta}\right\}\right\} \tag{44}$$

Table 3. drawn below gives the values of Renyi entropy of GGTT distribution for a fixed value of θ=2.1,δ=3.1 and varying the values of α and λ

Table-4. Values of Renyi Entropy of *GGTT* distribution for different values of parameters

	$I_R(v)$		
α,λ	$v=2$	$v=3$	$v=4$
0.5,0.5	0.6488	0.6059	0.5808
1.5,2.5	0.0877	0.0597	0.0424
2.5,3.5	−0.1442	−0.1705	−0.1871
4.5,5.5	−0.1566	−0.1847	−0.2010
6.5,8.5	−0.2476	−0.2743	−0.2909
10.5,15.5	−0.3523	−0.3797	−0.3967
20.5,30.5	−0.4658	−0.4942	−0.5117

- It should be noted that the higher the value of the Renyi entropy the greater the level of uncertainty in the system.

4. Stochastic Ordering

Stochastic ordering provides the commonest way of describing ordering mechanism in lifetime distributions. Let $X_1 \sim$ GGTT(α_1, λ_1, θ, δ) and $X_2\sim$ GGTT(α_2, λ_2, θ, δ). The random variable X_2 is stochastically greater than X_1 in the

–stochastic order ($X_1 \leq_{st} X_2$) if the associated cdf satisfy: $G_{x_1(x)} \leq G_{x_2(x)}$ for all x.
–hazard rate order ($X_1 \leq_{hr} X_2$) if the associated hazard rate function satisfies: $h_{x_1(x)}\leq h_{x_2(x)}$ for all x.
–likelihood ratio order ($X_1 \leq_{lr} X_2$) if $\frac{g_{x_1(x)}}{g_{x_2(x)}}$ is a decreasing function of x.

Given the pdf of X_1 and X_2

$$\frac{g_{x_1(x)}}{g_{x_2(x)}} = \frac{\lambda_1}{\lambda_2}\left(1-e^{-\theta x^{-\delta}}\right)^{(\alpha_2-\alpha_1)} e^{\left[\frac{\lambda_1}{\alpha_1}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]^{-\alpha_1}\right\}-\frac{\lambda_2}{\alpha_2}\left\{1-\left[1-e^{-\theta x^{-\delta}}\right]^{-\alpha_2}\right\}\right]} \tag{45}$$

Taking the logarithm and differentiating the ratio of the densities gives

$$\frac{d}{dx}\left(\log\left\{\frac{g_{x_1(x)}}{g_{x_2(x)}}\right\}\right) = \theta\delta x^{-\delta}M\left\{\frac{\alpha_2-\alpha_1}{(1-M)} + \left[\lambda_2(1-M)^{-(\alpha_2+1)} - \lambda_1(1-M)^{-(\alpha_1+1)}\right]\right\} < 0 \tag{46}$$

Where $M = e^{-\theta x^{-\delta}}$

If $\alpha_1 < \alpha_2$ and $\lambda_1 < \lambda_2$ $\forall x > 0$. Thus, for $\alpha_1 < \alpha_2$ and $\lambda_1 < \lambda_2, X_1 \leq_{lr} X_2$ $\forall x$. It follows from the implications of stochastic ordering that: $X_1 \leq_{lr} X_2 \Longrightarrow X_1 \leq_{hr} X_2 \Longrightarrow X_1 \leq_{st} X_2$.

5. Order Statistics

In this section, we derive closed form expressions for the pdf of the i^{th} order statistic of the *(EGTT)* distribution. Let $X_1, X_2, \dots, X_n$ be a simple random sample from (GGTT) distribution with cdf and pdf given by (7) and (8), respectively. Let $x(1{:}n) \leq x(2{:}n) \leq \dots \leq x\,(n{:}n)$ denote the order statistics obtained from this sample. The probability density function of $x_{i:n}$ is given by

$$f_{r:n}(x) = W[G(x,\psi)]^{r-1}[1 - G(x,\psi)]^{n-r} g(x,\psi) \tag{47}$$

Where $G(x,\psi)$ and $g(x,\psi)$ are the cdf and the pdf of the GGTT distribution given in equation (7) and (8) respectively, we have

$$f_{r:n}(x) = W\lambda f(x)[1 - F(x)]^{-\alpha-1} e^{\frac{\lambda}{\alpha}\{1-[1-F(x)]^{-\alpha}\}} \left[1 - e^{\frac{\lambda}{\alpha}\{1-[1-F(x)]^{-\alpha}\}} \right]^{r-1} \left[e^{\frac{\lambda}{\alpha}\{1-[1-F(x)]^{-\alpha}\}} \right]^{n-r} \tag{48}$$

Using the series expansion in equation (15) and (16) and also the relation in equation (48), $f_{i:n}(x)$ can be expressed as

$$f_{r:n}(x) = f(x) W \sum_{k=0}^{j} \sum_{i,j=0}^{\infty} (-1)^{i+k} (j!)^{-1} \binom{r-1}{i} \binom{j}{k} \left(\frac{\lambda}{\alpha}\right)^{j} (n - r + i + 1)^{j} [1 - F(x)]^{-[\alpha(k+1)+1]} \tag{49}$$

Applying the series expansion to the last term in equation (49), finally we have

$$f_{r:n}(x) = W \sum_{k=0}^{j} \sum_{i,j,l=0}^{\infty} (-1)^{i+k} (j!)^{-1} \binom{r-1}{i} \binom{j}{k} \left(\frac{1}{\alpha}\right)^{j} (\lambda)^{j+1} (n - r + i + 1)^{j} f(x) F(x)^{l} \tag{50}$$

Where $= \frac{n!}{(n-r)!r!}$, $f(x) and\ F(x)$ are respectively the baseline pdf and cdf respectively.

For $r = 1,2,3, \dots, n$. An expression for the 1^{th} order statistics and the n^{th} order statistics is respectively given in equation (56) and (57).

$$f_{1:n}(x) = W \sum_{k=0}^{j} \sum_{i,j,l=0}^{\infty} (-1)^{i+k} (j!)^{-1} \binom{j}{k} \left(\frac{1}{\alpha}\right)^{j} (\lambda)^{j+1} (n + i)^{j} f(x) F(x)^{l} \tag{51}$$

$$f_{n:n}(x) = W \sum_{k=0}^{j} \sum_{i,j,l=0}^{\infty} (-1)^{i+k} (j!)^{-1} \binom{n-1}{i} \binom{j}{k} \left(\frac{1}{\alpha}\right)^{j} (\lambda)^{j+1} (i + 1)^{j} f(x) F(x)^{l} \tag{52}$$

Where $f(x)$ and $F(x)$ are the baseline pdf and cdf respectively given in equation (1) and (2).

6. Maximum Likelihood Estimation of the Parameters

The likelihood function of GGTT distribution is given by

$$L(x;\alpha,\lambda,\theta,\delta) = (\lambda\theta\delta)^{n} e^{-\theta\sum_{1}^{n} x^{-\delta}} \prod_{i=1}^{n} \left\{ x^{-\delta-1} \left(1 - e^{-\theta x^{-\delta}}\right)^{-\alpha-1} e^{\frac{\lambda}{\alpha}\left\{1-[1-e^{-\theta x^{-\delta}}]^{-\alpha}\right\}} \right\} \tag{53}$$

The log likelihood function is

$$ln\{L(x;\alpha,\lambda,\theta,\delta)\} = nln(\lambda) + nln(\theta) + nln(\delta) - \theta \sum_{i=1}^{n} x_i^{-\delta} + \frac{\lambda}{\alpha} \sum_{i=1}^{n} \left\{1 - [1 - e^{-\theta x^{-\delta}}]^{-\alpha}\right\}$$
$$-(\delta + 1) \sum_{i=1}^{\infty} x_i - (\alpha + 1) \sum_{i=1}^{n} ln\left(1 - e^{-\theta x^{-\delta}}\right) \tag{54}$$

The nonlinear likelihood equations can be obtained for GGTT by differentiating equation (54) with respect to $\alpha, \lambda, \theta\ and\ \delta$. The components of the score vector

$$\mathcal{Z}(\Omega) = \left\{ \frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\alpha}, \frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\lambda}, \frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\theta}, \frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\delta} \right\}$$

Are given by

$$\frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\alpha} = -\sum_{i=1}^{n} ln\left(1 - e^{-\theta x_i^{-\delta}}\right) - \frac{\lambda}{\alpha^2} \sum_{i=1}^{n} ln\left\{1 - [1 - e^{-\theta x_i^{-\delta}}]^{-\alpha}\right\}$$
$$-\frac{\lambda}{\alpha} \sum_{i=1}^{n} \frac{\lambda\theta\delta x_i^{-\delta-1} e^{-\theta x_i^{-\delta}} \left[1 - e^{-\theta x_i^{-\delta}}\right]^{-\alpha-1}}{1 - \left[1 - e^{-\theta x_i^{-\delta}}\right]^{-\alpha}} \tag{55}$$

$$\frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\lambda} = \frac{n}{\lambda} + \frac{1}{\alpha} \sum_{i=1}^{n} \left\{1 - [1 - e^{-\theta x_i^{-\delta}}]^{-\alpha}\right\} \tag{56}$$

$$\frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\theta} = \sum_{i=1}^{n} \frac{\delta x^{-\delta-1} e^{-\theta x^{-\delta}} \left(1 - \theta x_i^{-\delta}\right)}{\theta\delta x_i^{-\delta-1} e^{-\theta x_i^{-\delta}}} + (\alpha + 1) \sum_{i=1}^{n} \frac{\delta x^{-\delta-1} e^{-\theta x^{-\delta}} \left(1 - \theta x_i^{-\delta}\right)}{1 - e^{-\theta x_i^{-\delta}}}$$

$$-\frac{\lambda}{\alpha}\sum_{i=1}^{n}\frac{\delta x^{-\delta-1}e^{-\theta x^{-\delta}}[1-e^{-\theta x_i^{-\delta}}]^{-\alpha-1}}{1-[1-e^{-\theta x_i^{-\delta}}]^{-\alpha}} \tag{57}$$

$$\frac{\partial ln\{L(x;\alpha,\lambda,\theta,\delta)\}}{\partial\delta}=\sum_{i=1}^{n}\frac{\frac{\partial f(x)}{\partial\delta}}{\theta\delta x_i^{-\delta-1}e^{-\theta x_i^{-\delta}}}+(\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial f(x)}{\partial\delta}}{1-e^{-\theta x_i^{-\delta}}}$$

$$\frac{\lambda}{\alpha}\sum_{i=1}^{n}\frac{[1-e^{-\theta x_i^{-\delta}}]^{-\alpha-1}\frac{\partial f(x)}{\partial\delta}}{1-[1-e^{-\theta x_i^{-\delta}}]^{-\alpha}} \tag{58}$$

We can obtain the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (55) - (58) to zero and solve them simultaneously. Therefore, statistical software can be employed in obtaining the numerical solution to the non-linear equations such as R, MATLAB, Maple etc. For the four parameters Gompertz Gumbel type-two pdf, all the second order derivatives can be obtained. Thus the inverse dispersion matrix is given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{\theta}\\ \hat{\delta}\end{pmatrix}\sim N\left[\begin{pmatrix}\alpha\\ \lambda\\ \theta\\ \delta\end{pmatrix}\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\lambda} & \hat{V}_{\alpha\theta} & \hat{V}_{\alpha\delta}\\ \hat{V}_{\lambda,\alpha} & \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\theta} & \hat{V}_{\lambda\delta}\\ \hat{V}_{\alpha} & \hat{V}_{\theta\lambda} & \hat{V}_{\theta\theta} & \hat{V}_{\theta\delta}\\ \hat{V}_{\delta\alpha} & \hat{V}_{\delta\lambda} & \hat{V}_{\delta\theta} & \hat{V}_{\delta\delta}\end{pmatrix}\right],$$

$$V^{-1}=-E\begin{pmatrix}V_{\alpha\alpha} & V_{\alpha\lambda} & V_{\alpha\theta} & V_{\alpha\delta}\\ V_{\lambda,\alpha} & V_{\lambda\lambda} & V_{\lambda\theta} & V_{\lambda\delta}\\ V_{\theta\alpha} & V_{\theta\lambda} & V_{\theta\theta} & V_{\theta\delta}\\ V_{\delta\alpha} & V_{\delta\lambda} & V_{\delta\theta} & V_{\delta\delta}\end{pmatrix} \tag{59}$$

Where $\widehat{U}_{ij}=U_{ij}\big|_{w=w_i}=(\alpha,\lambda,\theta,\delta)$ with $\{J_{ij}\}=[-l_{ij}]^{-1}=\left[\frac{\partial^2 l}{\partial w_i\partial w_j}\right]$. This gives the approximate variance covariance matrix. By solving for the inverse of the dispersion matrix, the solution will give the asymptotic variance and covariance of the MLs for $\hat{\alpha},,\hat{\lambda},\hat{\theta}$ and $\hat{\delta}$. The approximate $100(1-v)\%$ confidence intervals for $\alpha,\lambda,\theta, and\ \delta$ can be obtained respectively as

$$\hat{a}\pm z_{\frac{\eta}{2}}\sqrt{\widehat{U}_{\alpha\alpha}},\quad \hat{b}\pm z_{\frac{\eta}{2}}\sqrt{\widehat{U}_{\lambda\lambda}},\quad \hat{\alpha}\pm z_{\frac{\eta}{2}}\sqrt{\widehat{U}_{\theta\theta}}\ and\ \hat{v}\pm z_{\frac{\eta}{2}}\sqrt{\widehat{U}_{\delta\delta}},$$

7. Simulation Study

In this sub-section, we carried out a simulation study to examine the performance of maximum likelihood estimators of Gompertz Gumbel type-2 distribution. In this context, we carried out the simulation study using the Monte Carlo simulation, as explained in the following works: Lemonte [14], Cordeiro and Lemonte [15] and De Andrade, *et al.* [16]. We investigated the behavior of the MLEs for the parameters of the GGTT model by generating from (12) samples sizes n = 50,100,300 and 500 with selected values for values for α,λ,δ and θ. We consider 5,000 Monte Carlo replications. The simulation process was performed in the R software using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) maximization method in the optimum script. To ensure the reproducibility of the experiment, we use the seed for the random number generator: set.seed (90)

The results of the simulations are presented in Table 4. 4.1, and 4.2, including the means, Absolute Bias (AB), Standard Error (SE) and the Mean Square Error (MSE). We observed that the estimated values of the parameters are very close to the true values and also the MSE consistently decreases as the sample size increases. This is a desirable property to show the adequacy of the estimation technique.

Table-5. Means, AB, SE and MSE of $\hat{\alpha},\hat{\lambda},\hat{\delta}\ and\ \hat{\theta}$ for the GGTT model ($\alpha=0.5,\lambda=0.5,\delta=0.5,\theta=0.5\ as\ a\ true\ parameter\ values$)

	Parameter	*Mean*	*AB*	*SE*	*MSE*
$n=50$	α	0.2550	0.2450	0.1570	0.0846
	λ	0.5039	0.0039	1.0605	1.1247
	δ	0.6035	0.1035	0.5691	0.3346
	θ	0.3434	0.1566	0.9161	0.8638
$n=100$	α	0.2183	0.2817	0.0818	0.1619
	λ	0.0778	0.4222	0.0461	0.1804
	δ	1.2790	0.7970	0.4791	0.8647
	θ	0.0115	0.4885	0.0241	0.2392
$n=300$	α	0.2968	0.2033	0.1041	0.0522
	λ	0.1002	0.3988	0.0744	0.1646
	δ	1.0772	0.5772	0.4124	0.1631
	θ	0.0732	0.4268	0.0115	0.1823
$n=500$	α	0.4371	0.0629	0.1020	0.0144
	λ	0.3388	0.1612	0.2536	0.0903
	δ	0.5884	0.0884	0.1882	0.0432
	θ	0.3206	0.1794	0.3114	0.1291

Table-5.1. Means, AB, SE and MSE of $\hat{\alpha}, \hat{\lambda}, \hat{\delta}$ *and* $\hat{\theta}$ for the GGTT model ($\alpha = 1.5, \lambda = 1.75, \delta = 1.5, \theta = 1.0$ *as a true parameter values*)

	Parameter	Mean	AB	SE	MSE
$n = 50$	α	0.6587	0.8413	0.6043	1.0729
	λ	0.9451	0.8049	4.0407	16.9753
	δ	2.402	0.9024	3.5460	13.3886
	θ	0.5234	0.4767	2.0768	4.5404
$n = 100$	α	0.286	1.0714	0.1289	1.2768
	λ	0.0716	1.6784	0.0380	2.8183
	δ	5.5966	4.0966	1.5102	19.0625
	θ	0.0124	0.9876	0.0182	0.9755
$n = 300$	α	0.5414	0.9586	0.1496	0.9412
	λ	0.0817	1.6683	0.0495	2.7856
	δ	4.7457	3.2457	1.3375	12.3231
	θ	0.0274	0.9726	0.0396	0.9476
$n = 500$	α	1.2299	0.2701	0.6008	0.4339
	λ	0.9206	0.5794	1.2971	2.0181
	δ	2.0242	0.5242	0.8826	1.0537
	θ	0.7689	0.2311	0.0274	0.0541

Table-5.2. Means, AB, SE and MSE of $\hat{\alpha}, \hat{\lambda}, \hat{\delta}$ *and* $\hat{\theta}$ for the *GGTT* model ($\alpha = 0.3, \lambda = 0.5, \delta = 0.55, \theta = 0.35$ *as a true parameter values*)

	Parameter	Mean	AB	SE	MSE
$n = 50$	α	0.1263	0.1738	0.1459	0.0515
	λ	0.5368	0.0368	0.8309	0.6917
	δ	0.6045	0.0545	0.4515	0.2068
	θ	0.2187	0.1314	0.5230	0.2908
$n = 100$	α	0.2288	0.0712	0.1242	0.0205
	λ	0.1233	0.3767	0.1352	0.1602
	δ	1.0364	0.4864	0.6743	0.6912
	θ	0.0245	0.3255	0.0813	0.1125
$n = 300$	α	0.1907	0.1094	0.1347	0.0301
	λ	1.0696	0.5696	0.8471	1.0419
	δ	0.4191	0.1309	0.1320	0.1491
	θ	0.7055	0.3555	0.4991	0.3755
$n = 500$	α	0.2813	0.0187	0.0587	0.0037
	λ	0.6038	0.1038	0.3699	0.1476
	δ	0.4916	0.0584	0.1282	0.0198
	θ	0.4176	0.0675	0.2927	0.0903

8. Applications to Real Life Data

In this section, we present three examples that demonstrate the flexibility and the applicability of the GGTT distribution in modelling real world data. We fit the density functions of the GGTT distribution and compare its fits with that of Exponentiated Gumbel Type-two (EGT), Extended Gumbel type-two (EGTT) distribution and its sub-models Gumbel Type-two (GT) distribution. For all the fitted models, we compute the MLEs of the model parameters (with their corresponding standard errors in parentheses) and also the values of the Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnoff (KS) statistic, Anderson Darling statistic (A^*) and the probability value (PV) used as methods of comparing fits of distributions to data. In general, it is considered that the smaller the values of AIC, BIC, HQIC, CAIC and (A^*) and the larger the PV the better the model fit to the data.

8.1. Pig Data Set

The first data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [17]. The starting point of the iterative processes for the guinea pigs data set is (1:0; 0:009; 10:0; 0:1; 0:1). Survival Times (in days) of Guinea Pigs Infected with Virulent Tubercle Bacilli. The data is give as: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08,1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24,1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63,1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02,2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53,2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32,4.58, 5.55. Table 6. gives the MLEs and Table 7.0 gives the selection, criteria statistics for the pig data. Figure 5 gives the graph of the Total Time on Test plot and the graph of the kernel density of the pig data which clearly shows that the data exhibits an increasing failure rate and positively skewed (unimodal) also Figure 6 gives the fitted densities of the pig data.

Figure-5. Graph of TTT plot and the Kernel density function of the cancer data

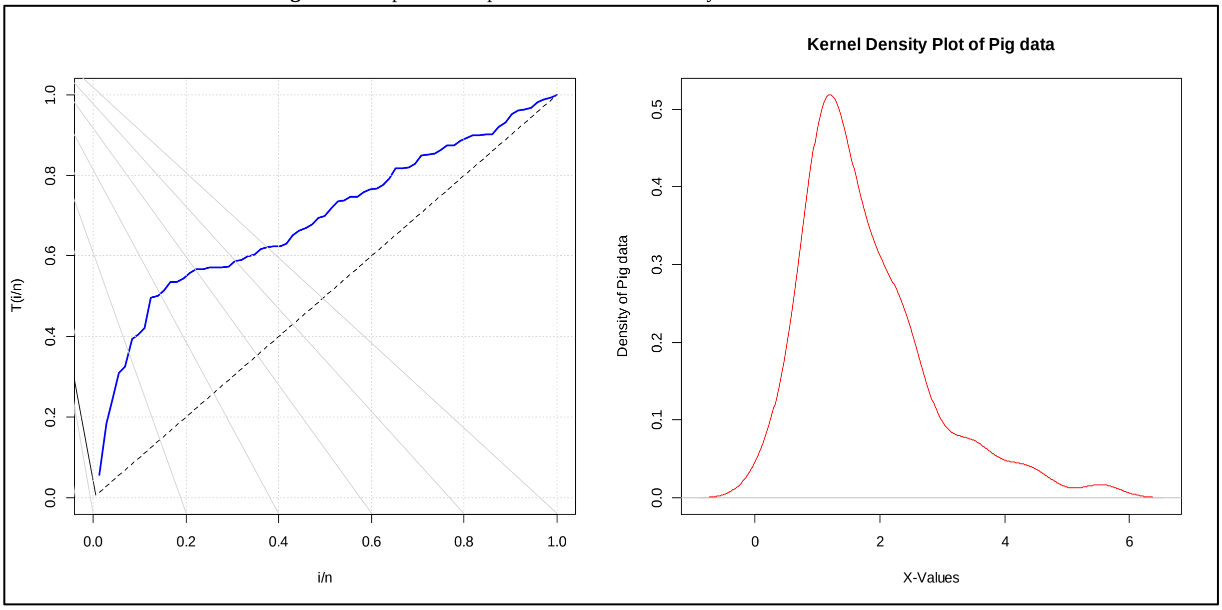


Figure-6. Fitted densities for pig data

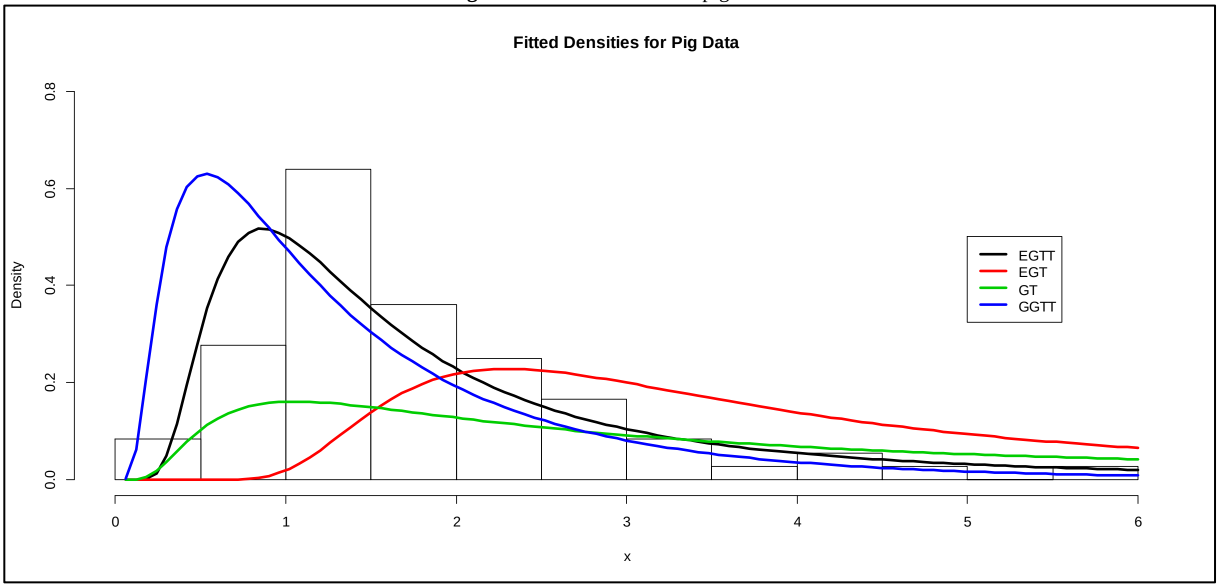


Table-6. Parameter estimate, standard error (parenthesis)

Distribution	Estimates			
GGTT	1.3863 (0.6439)	0.0140 (0.0149)	1.3030 (0.5516)	0.0899 (0.1110)
EGTT	14.0571 (8.7149)	1.1494 (0.7270)	0.5629 (0.1486)	3.6017 (1.0458)
EGT	1.3020 (1.8073)	1.1747 (0.0844)	0.8173 (1.1345)	– (–)
GT	1.1760 (0.0842)	1.0709 (0.1328)	– (–)	– (–)

Table-7. Selection criteria statistics for pig data

Distribution	$-l$	*AIC*	*BIC*	*CAIC*	*HQIC*	A^*	*KS*	*P value*
GGTT	95.853	199.707	208.813	200.304	203.332	0.7710	0.1120	0.3267
EGTT	98.063	204.126	213.331	204.723	207.758	0.8151	0.1192	0.2581
EGT	118.167	242.333	249.163	242.686	245.052	3.3560	0.1981	0.0070
GT	118.167	240.334	244.887	240.508	242.147	3.3656	0.1958	0.0080

8.2. Cancer Remission Time Data

The second data set consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang [18]. The starting point of the iterative processes for the cancer patients data set is (1:0; 0:009; 10:0; 0:1; 0:1). Given as: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. Table 8. gives the MLEs and Table 9. gives the selection, criteria statistics for the cancer data. Figure 7 gives the graph of the Total Time on Test plot and the graph of the kernel density of the cancer data which clearly shows the cancer remission data exhibits a unimodal failure rate and positively skewed, also Figure 8 gives the fitted densities to the cancer data.

Fig-7. Graph of TTT plot and the Kernel density function of the cancer data

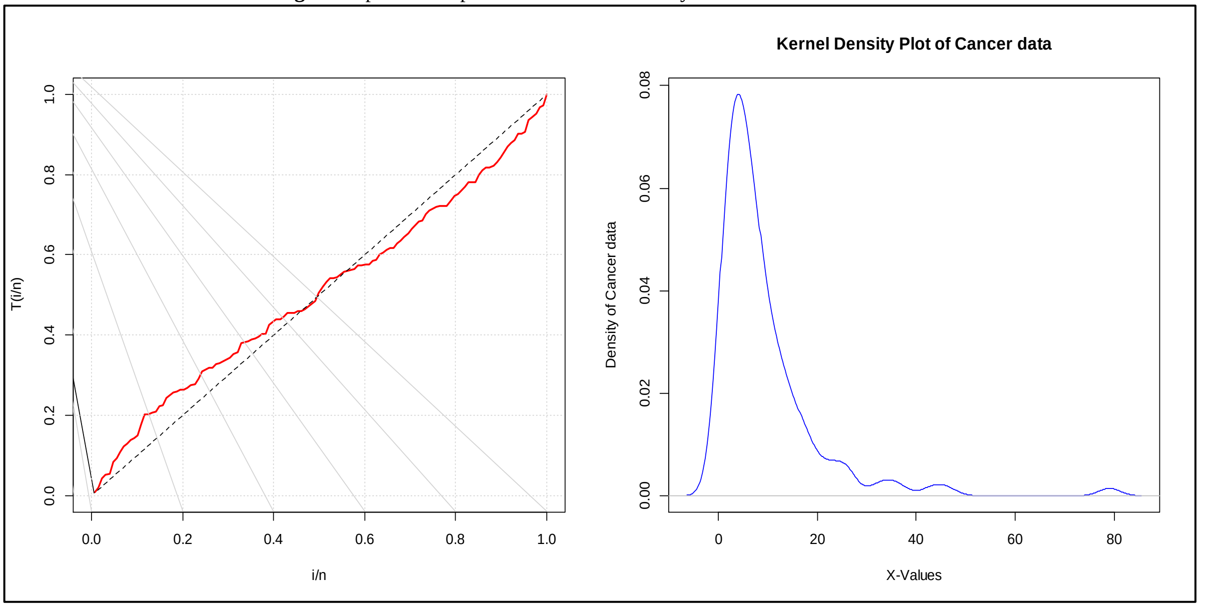


Fig-8. Fitted densities of the cancer data

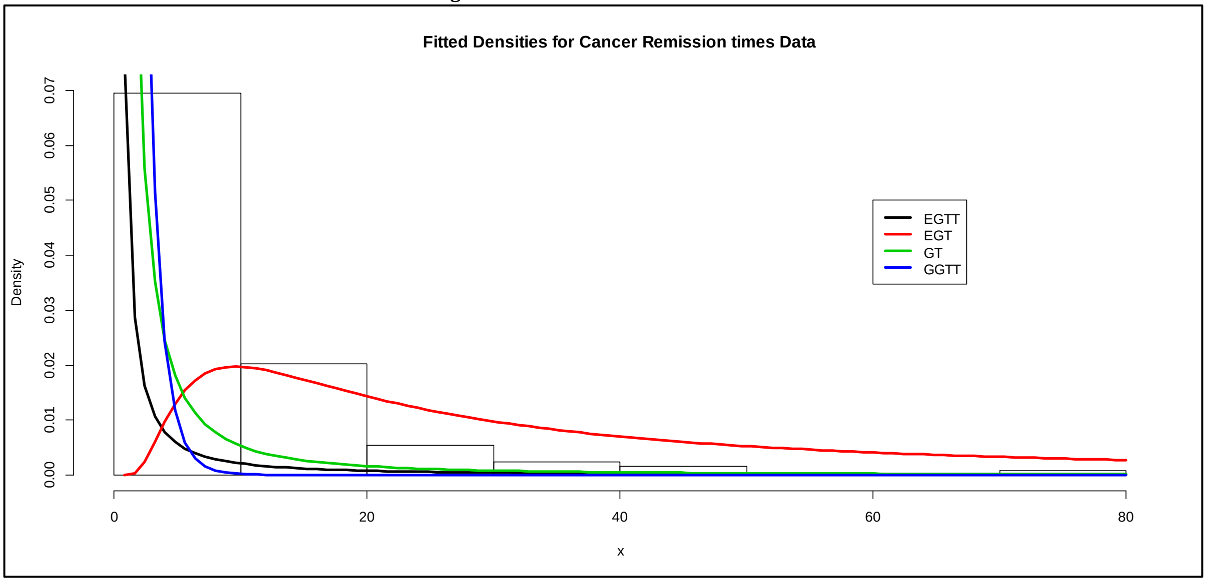


Table-8. Parameter estimate, standard error (parenthesis) for cancer data

Distribution	**MLE estimates**			
GGGT	0.9373 (0.4151)	0.0163 (0.0147)	1.0855 (0.4508)	0.1495 (0.1835)
EGGT	11.7689 (6.4231)	0.8918 (0.2526)	0.3674 (0.0641)	5.5657 (1.0114)
EGT	2.2915 (1.9970)	0.7512 (0.0425)	1.0551 (0.9195)	– (–)
GT	0.7528 (0.0424)	2.4389 (0.2200)	– (–)	– (–)

Table-9. Selection criteria statistics for cancer data

Distribution	$-l$	AIC	BIC	$CAIC$	$HQIC$	A^*	KS	P value
$GGTT$	413.580	835.161	846.569	835.486	839.796	0.7451	0.0722	0.5166
$EGTT$	415.856	839.712	851.120	840.037	844.347	0.8731	0.0828	0.3434
EGT	444.003	894.005	902.561	894.199	897.482	4.4534	0.1404	0.0129
GT	118.167	892.003	897.707	892.099	894.321	4.5548	0.1410	0.0124

8.3. Yarn Specimen Data Set

The third dataset consists of data on the number of cycles of failure for 25 specimens of 100 cm specimens of yarn, tested at a particular strain level by Lawless [19]. The starting points for the iterative processes in the yarn specimens' data are (109:35; 1:2920; 3:6125; 1:0; 1:0): 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, and 653. Table 10 gives the MLEs and Table 11 gives the selection, criteria statistics for the pig data. Figure 9.0 gives the graph of the Total Time on Test plot and the graph of the kernel density of the pig data which clearly shows that the yarn specimen data exhibits an increasing failure rate and positively skewed, also Figure 10. Gives the fitted densities of Yarn specimen data

Figure-9. Graph of TTT plot and the Kernel density function of Yarn specimen data

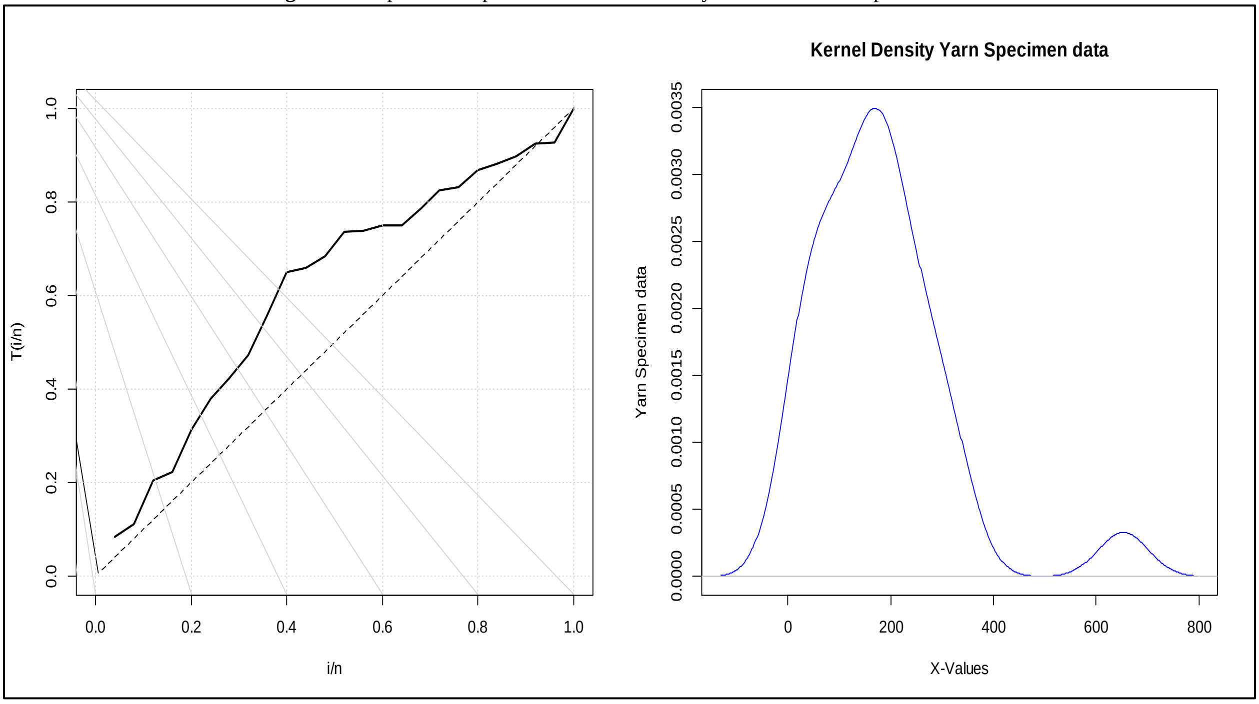


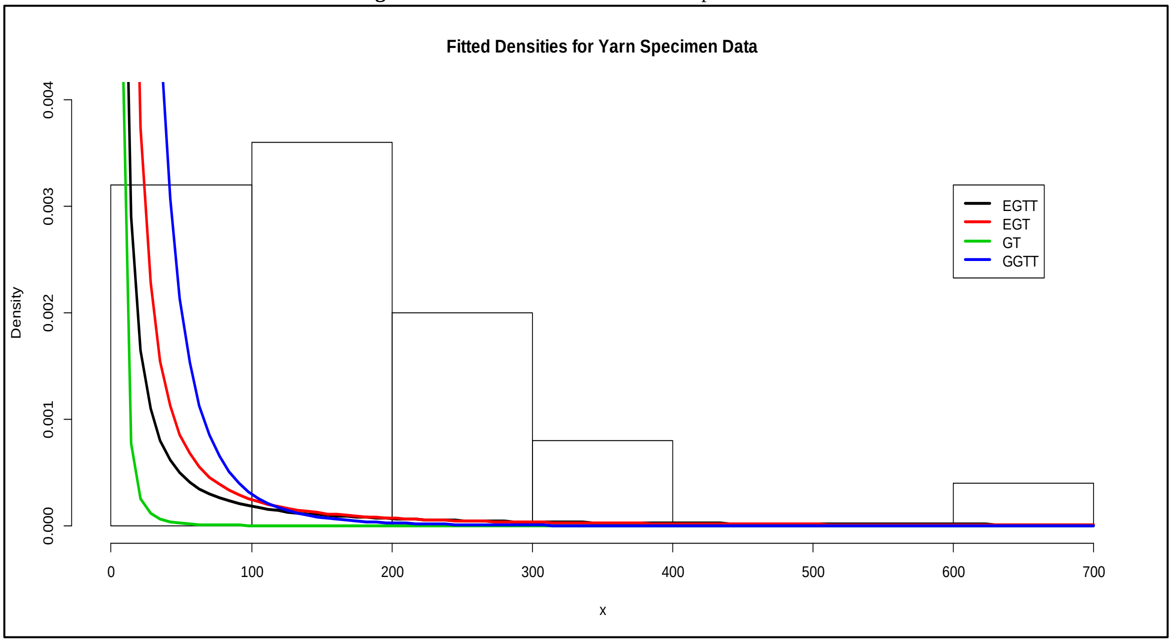
Table-10. Parameter estimate, standard error (parenthesis) for Yarn Specimen Data

Distribution	MLE estimates			
$GGTT$	1.5075 (NaN)	0.0025 (0.0007)	0.8236 (NaN)	1.0476 (0.5556)
$EGGT$	7.6847 (5.7316)	2.2426 (1.5938)	0.3739 (0.1050)	11.2555 (5.9351)
EGT	18.1251 (12.3175)	1.0109 (0.1408)	4.7572 (16.3559)	$-$ $(-)$
GT	0.6870 (0.0883)	20.0564 (7.6266)	$-$ $(-)$	$-$ $(-)$

Table-11. Selection criteria statistics for Yarn Specimen Data

Distribution	$-l$	AIC	BIC	$CAIC$	$HQIC$	A^*	KS	P value
$GGTT$	152.853	313.706	318.582	315.706	315.054	0.3394	0.1743	0.4333
$EGTT$	155.313	318.626	323.501	320.626	319.878	0.9916	0.1977	0.2825
EGT	**158.579**	323.158	326.815	324.301	315.058	1.5861	0.2115	0.2132
GT	161.860	327.719	330.157	328.265	328.394	1.2950	0.2742	0.0467

Figure-10. Fitted densities of the Yarn specimen data



9. Conclusion

In this study, a four-parameter model called GGTT distribution is proposed and its statistical properties are derived. We discussed the maximum likelihood method to estimate the model parameters and presented a Monte Carlo simulation study to evaluate the performance of the maximum likelihood estimators for the GGTT model. Finally, three applications illustrate the potential of the GGTT distribution for fitting survival data. The performance of the GGTT distribution with regards to providing good fit to the data sets is assessed by comparing it with other models including its sub-model. The results show that the GGTT model provides a more reasonable parametric fit to the data sets.

9.1. Data Availability Statement

The data used for this research are commonly and predominantly use data in our area of research. There is absolutely no conflict of interest between the authors and producers of the data because we do not intend to use these data as an avenue for any litigation but for the advancement of knowledge.

9.2. Funding Statement

The research was funded by effort or the authors.

References

[1] Okorie, I. E., Akpanta, A. C., and Ohakwe, J., 2016. "The exponentiated gumbel type- 2 distribution: Properties and application." *International Journal of Mathematics and Mathematical Sciences,* Available: https://www.hindawi.com/journals/ijmms/2016/5898356/

[2] Nadarajah, S. and Kotz, S., 2006. "The exponentiated type distributions." *Acta Applicandae Mathematica,* vol. 92, pp. 97-111.

[3] Gupta, R. C., Gupta, P. L., and Gupta, R. D., 1998. "Modeling failure time data by Lehman alternatives." *Communications in Statistics-Theory and Methods,* vol. 27, pp. 887-904.

[4] Okorie, I. E., Akpanta, A. C., Ohakwe, C., Chikezie, and Obi, O. E., 2017. "The kumaraswamy g exponentiated gumbel type-2 distribution." *Afrika Statistika,* vol. 12, p. 1367. Available: http://dx.doi.org/10.16929/as/2017.1367.109

[5] Kumaraswamy, P., 1980. "A generalized probability density function for double-bounded random processes." *Journal of Hydrology,* vol. 46, pp. 79-88.

[6] Ogunde, A. A., 2020. *Extended gumbel type two distribution: Properties and applications*. In press.

[7] Alizadeh, M., Cordeiro, G. M., Bastos Pinho, L. G., and Ghosh, I., 2017. "The Gompertz-g family of distributions." *Journal of Statistical Theory and Practice,* vol. 11, pp. 179-207.

[8] Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O., Okagbue, H. I., Opanuga, A. A., and Owolabi, F. O., 2018. "The gompertz inverse exponential (goie) distribution with applications." *Cogent Mathematics and Statistics,* vol. 5, p. 1507122.

[9] Khaleel, M. A., Oguntunde, P. E., Ahmed, M. T., Ibrahim, N. A., and Loh, Y. F., 2020. "Gompertz flexible Weibull distribution and it applications." *Malaysian Journal of Mathematical Sciences,* vol. 14, pp. 169-190.

[10] Keller, A. Z. and Kamath, A. R., 1982. "Reliability analysis ofCNC machine tools." *Reliability Engineering,* vol. 3, pp. 449-473.

[11] Kenney, J. F. and Keeping, E. S., 1962. *Mathematics of statistics*. 3rd ed. Princeton, NJ: Chapman and Hall. pp. 101-102.
[12] Moors, J. J., 1988. "A quantile alternative for kurtosis." *J. Royal Statist. Soc. D*, vol. 37, pp. 25-32.
[13] Renyi, A., 1960. "On measures of information and entropy." In *Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability*. pp. 547-561.
[14] Lemonte, A. J., 2013. "A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function." *Computational Statistics and Data Analysis,* vol. 62, pp. 149-170.
[15] Cordeiro, G. M. and Lemonte, A. J., 2014. "The exponentiated generalized Birnbaum-Saunders distribution." *Applied Mathematics and Computation,* vol. 247, pp. 762-779.
[16] De Andrade, T. A. N., Rodrigues, H., Bourguignon, M., and Cordeiro, G. M., 2015. "The exponentiated generalized Gumbel distribution." *Revista Colombiana de Estadstica,* vol. 38, pp. 123-143.
[17] Bjerkedal, T., 1960. "Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli." *American Journal of Hygiene,* vol. 72, pp. 130-148.
[18] Lee, E. T. and Wang, J., 2003. *Statistical methods for survival data analysis* vol. 476. John Wiley and Sons.
[19] Lawless, B. J., 1999. *Needle punch nonwoven component for refastenable fastening device*. US Patent 5, 891,547.