Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 7, Issue. 2, pp: 106-112, 2021
URL: https://arpgweb.com/journal/journal/17
DOI: https://doi.org/10.32861/ajams.72.106.112
Academic Research Publishing Group
Original Research
Open Access

Bayesian Methods and Maximum Likelihood Estimations of Exponential Censored Time Distribution with Cure Fraction

Dr. Al Omari Mohammed Ahmed
Department of Mathematics, Faculty of Arts and Science in Qilwah, AlBaha University, Baha, Saudi Arabia
Email: alomari1050@hotmail.com

Article History
Received: February 2, 2021
Revised: February 27, 2021
Accepted: March 2, 2021
Published: March 6, 2021

Abstract
This paper is focused on estimating the parameter of Exponential distribution under right-censored data with cure fraction. The maximum likelihood estimation and Bayesian approach were used. The Bayesian method is implemented using gamma, Jeffreys, and extension of Jeffreys priors with two loss functions, which are; squared error loss function and Linear Exponential Loss Function (LINEX). The methods of the Bayesian approach are compared to maximum likelihood counterparts and the comparisons are made with respect to the Mean Square Error (MSE) to determine the best for estimating the parameter of Exponential distribution under right-censored data with cure fraction. The results show that the Bayesian with gamma prior under LINEX loss function is a better estimation of the parameter of Exponential distribution with cure fraction based on right-censored data.
Keywords: Exponential distribution; Maximum likelihood estimation; Bayesian method; Right censored data; Cure fraction.

1. Introduction
Majority of the survival models presume that all individuals are susceptible to a predetermined event with sufficient follow-up. Yet, this assumption could be overturned by a group of individuals who will experience the particular event. Such a group is usually described in related literature as non-susceptible or cured. Several decades ago, the survival models started to include the cured percentage in the analysis, and a new group of models has since been developed. They are called in brief, cure models and are broadly used in analyzing data from disease clinical trials.

The leading cure model was advanced by Boag [1] and a few years later was reworked by Berkson and Gage [2]. This type of model is mixture cure rate model and as given below:

$$S(t) = \pi + (1-\pi)S_u(t)$$

where $S(t)$ and $S_u(t)$ are the survival functions for the entire population and the uncured patients respectively, and π is the cured fraction. The failure time function in this model of the uncured patients will be estimated parametrically, and the parametric estimation of cure models was assuming the distribution such as the Exponential distribution for the failure time of uncured patients. The theory and applications of mixture models, that is noticed in [3-10], etc.

Despite that fact that the mixture model is used to a large extent in survival analysis, Chen, *et al.* [11] discussed some drawbacks to the mixture model and proposed as an alternative the Bounded Cumulative Hazard model. This model has been applied in medical research. Aljawadi, *et al.* [12], used Maximum likelihood estimation to estimate the cure fraction in cancer trials under interval censored data and the estimation procedure was handled via the parametric and nonparametric techniques.

In the Bayesian method, Upadhyay and Gupta [13], Soliman, *et al.* [14] and Al Omari [15] discussed some estimation for complete samples and independent priors for the parameters.

The objective of this paper is to estimate the parameter of the Exponential distribution based on right censored data with cure fraction, by using maximum likelihood estimator and Bayesian approach using gamma, Jeffreys and extension of Jeffreys priors under the Square Error loss function and the Linear Exponential loss function. Comparisons are made between the methods using mean square error (MSE) to determine the best estimator under several conditions.

2. Methodology

2.1. Maximum Likelihood Estimation

In the maximum likelihood estimation, the Exponential distribution was used and π given as known. Suppose that X is a random variable with probability density function $f(x;\lambda)$, and $x_1, x_2, \dots, x_n$ is a random sample of size n.

The likelihood function is:

$$L(x;\lambda)=\prod_{i=1}^{n}\left(\left[\left(\lambda e^{-\lambda x_i}\right)(1-\pi)\right]^{c_i}\right)^{\delta_i}\left(\pi^{1-c_i}\left[(1-\pi)\left(e^{-\lambda x_i}\right)\right]^{c_i}\right)^{1-\delta_i} \tag{1}$$

where c_i and δ_i are indicators of cure and censoring respectively for the i^{th} patient, such that

$$\delta_i=\begin{cases}0 & : \text{if } x \text{ is censoring time}\\ 1 & : \text{otherwise}\end{cases}, \text{ and } \quad c_i=\begin{cases}0 & : \text{for cured}\\ 1 & : \text{otherwise}\end{cases}.$$

When $\delta_i = 1$, then $c_i = 1$, and if $\delta_i = 0$, c_i can be either one or zero, assuming that censoring is independent of failure times. However, in maximum likelihood approach we will use the Exponential distribution function to represent the distributional function for the data set, such that the survival function of uncured patients is $e^{-\lambda x}$ and the probability density function is $f(x) = \lambda e^{-\lambda x}$. Then the log likelihood function can be obtained by:

$$\ln L(x;\lambda)=\ln\lambda\sum_{i=1}^{n}\delta_i c_i-\ln\pi\sum_{i=1}^{n}(1-\delta_i)(1-c_i)-\lambda\sum_{i=1}^{n}c_i x_i+\ln(1-\pi)\sum_{i=1}^{n}c_i. \tag{2}$$

See Klein and Moeschberger [16] and Aljawadi, *et al.* [12].

Differentiating Equation (2) partially with respect to the parameter λ and equalizing to zero. The resulting equation is given respectively as,

$$\frac{\partial L}{\partial\lambda}=\frac{1}{\lambda}\sum_{i=1}^{n}\delta_i c_i-\sum_{i=1}^{n}c_i x_i \tag{3}$$

The Equation 3 can be simplified and written as follows:

$$\hat{\lambda}_M=\frac{\sum_{i=1}^{n}\delta_i c_i}{\sum_{i=1}^{n}c_i x_i} \tag{4}$$

2.2. Bayesian Estimation

In the Bayesian estimation we used three priors: Gamma prior, Jeffreys prior and extension of Jeffreys prior and each prior will be estimate the parameter under square error loss function and The Linear Exponential loss function as explain below,

2.2.1. Bayesian Estimation with Gamma Prior

The scale parameter is consideration as unknown, and we compute the Bayesian estimation of the scale parameter and it is assumed that λ has independent gamma prior as follows,

$$g(\lambda/b,d)=\lambda^{b-1}\exp(-d\lambda)$$

The posterior of Exponential distribution with cure fraction is given as

$$\prod\nolimits_1(\lambda\mid\text{data})=\frac{\lambda^{\sum_{i=1}^{n}c_i\delta_i+b-1}\exp\left(-\lambda\left(\sum_{i=1}^{n}c_i x_i+d\right)\right)(1-\pi)^{\sum_{i=1}^{n}c_i}\pi^{\sum_{i=1}^{n}(1-c_i)(1-\delta_i)}}{\int_0^\infty\lambda^{\sum_{i=1}^{n}c_i\delta_i+b-1}\exp\left(-\lambda\left(\sum_{i=1}^{n}c_i x_i+d\right)\right)(1-\pi)^{\sum_{i=1}^{n}c_i}\pi^{\sum_{i=1}^{n}(1-c_i)(1-\delta_i)}\,d\lambda}$$

A wide range of loss functions have been reported in literature review to describe various types of loss structures. In this study, we describe two loss functions: The symmetric loss function is square error loss function and the asymmetric loss function is Linear Exponential loss function (LINEX).

2.2.1.1. Square Error Loss Function

The square error loss function used to estimate the scale parameter of Exponential distribution as given respectively below,

$$\hat{\lambda}_{GS}=\int_0^\infty\lambda\prod\nolimits_1(\lambda\mid\text{data})\,d\lambda$$

$$= \frac{\sum_{i=1}^{n} \delta_i c_i + b}{\sum_{i=1}^{n} c_i x_i + d} \quad (5)$$

2.2.1.2. Linear Exponential Loss Function (LINEX)

The Linear Exponential loss function is under the assumption that the minimal loss occurs at $\hat{\lambda} = \lambda$ and is expressed as

$$L(\Delta) = \exp(r\Delta) - r\Delta - 1, \qquad r \neq 1$$

where $\Delta = (\hat{\lambda} - \lambda), \hat{\lambda}$ is an estimate of λ, and when $r > 1$ means overestimation and underestimation if $r < 1$. For r close to zero the Linear Exponential loss function approximated the square error loss function.

The posterior under LINEX loss function in equation above given as follows,

$$E_\lambda \left(L(\hat{\lambda} - \lambda) \right) \propto \exp(r\hat{\lambda}) \mathrm{E}_\lambda \left(\exp(r\hat{\lambda}) \right) - r \left(\hat{\lambda} - \mathrm{E}_\lambda(\lambda) \right) - 1$$

Therefore, the Bayesian estimation of scale parameter of Exponential distribution with type I censored data under LINEX loss function is:

$$\hat{\lambda}_{GL} = -\frac{1}{r} \ln \left(\int_0^\infty \exp(-r\lambda) \prod{}_1(\lambda \mid \text{data}) \; d\lambda \right)$$

$$= -\frac{\sum_{i=1}^{n} \delta_i c_i + b - 2}{r} \ln \left(\frac{d + \sum_{i=1}^{n} c_i x_i}{r + d + \sum_{i=1}^{n} c_i x_i} \right) \quad (6)$$

2.2.2. Bayesian Estimation under Jeffreys Prior

The Bayesian estimator with Jeffreys prior is describing below, where the Jeffreys prior is the square root of the determinant of the Fisher information matrix.

Then the Jeffreys prior is

$$g_1(\lambda) = k \frac{1}{\lambda}$$

where k is a constant, as explain in Al Omari [17]

The posterior of Exponential distribution with cure fraction is given as

$$\prod{}_2(\lambda \mid \text{data}) = \frac{\lambda^{\sum_{i=1}^{n} c_i \delta_i - 1} \exp\left(-\lambda \sum_{i=1}^{n} c_i x_i \right) (1-\pi)^{\sum_{i=1}^{n} c_i} \pi^{\sum_{i=1}^{n} (1-c_i)(1-\delta_i)}}{\int_0^\infty \lambda^{\sum_{i=1}^{n} c_i \delta_i - 1} \exp\left(-\lambda \sum_{i=1}^{n} c_i x_i \right) (1-\pi)^{\sum_{i=1}^{n} c_i} \pi^{\sum_{i=1}^{n} (1-c_i)(1-\delta_i)} \; d\lambda}$$

The square error loss function used to estimate the scale parameter of Exponential distribution as given respectively below,

$$\hat{\lambda}_{JS} = \int_0^\infty \lambda \prod{}_2(\lambda \mid \text{data}) \; d\lambda$$

$$= \frac{\sum_{i=1}^{n} \delta_i c_i}{\sum_{i=1}^{n} c_i x_i} \quad (7)$$

Therefore, the Bayesian estimation of scale parameter of Exponential distribution with cure fraction under LINEX loss function is:

$$\hat{\lambda}_{JL} = -\frac{1}{r} \ln \left(\int_0^\infty \exp(-r\lambda) \prod{}_2(\lambda \mid \text{data}) \; d\lambda \right)$$

$$= -\frac{\sum_{i=1}^{n} \delta_i c_i - 2}{r} \ln \left(\frac{\sum_{i=1}^{n} c_i x_i}{r + \sum_{i=1}^{n} c_i x_i} \right) \quad (8)$$

2.2.3. Bayesian Estimation under Extension of Jeffreys Prior

Extension of Jeffreys prior information is the Fisher information with the variable c where c is a positive real number as given below,

$$g_2(\lambda) = \frac{k}{\lambda^{2a}}$$

See Al Omari [15] for more detail.

The posterior of Exponential distribution with cure fraction is given as

$$\prod{}_3(\lambda \mid \text{data}) = \frac{\lambda^{\sum_{i=1}^{n} c_i\delta_i - 2a} \exp\left(-\lambda \sum_{i=1}^{n} c_i x_i\right)(1-\pi)^{\sum_{i=1}^{n} c_i} \pi^{\sum_{i=1}^{n}(1-c_i)(1-\delta_i)}}{\int_0^\infty \lambda^{\sum_{i=1}^{n} c_i\delta_i - 2a} \exp\left(-\lambda \sum_{i=1}^{n} c_i x_i\right)(1-\pi)^{\sum_{i=1}^{n} c_i} \pi^{\sum_{i=1}^{n}(1-c_i)(1-\delta_i)} \, d\lambda}$$

The square error loss function used to estimate the scale parameter of Exponential distribution as given respectively below,

$$\hat{\lambda}_{ES} = \int_0^\infty \lambda \prod{}_3(\lambda \mid \text{data}) \; d\lambda$$

$$= \frac{\sum_{i=1}^{n} \delta_i c_i - 2a + 1}{\sum_{i=1}^{n} c_i x_i} \qquad (9)$$

Therefore, the Bayesian estimation of scale parameter of Exponential distribution with cure fraction under LINEX loss function is:

$$\hat{\lambda}_{EL} = -\frac{1}{r} \ln\left(\int_0^\infty \exp(-r\lambda) \prod{}_3(\lambda \mid \text{data}) \; d\lambda \right)$$

$$= -\frac{\sum_{i=1}^{n} \delta_i c_i - 2a - 1}{r} \ln\left(\frac{\sum_{i=1}^{n} c_i x_i}{r + \sum_{i=1}^{n} c_i x_i} \right) \qquad (10)$$

3. Simulation Study

This section contains the simulation to assess the performance of the Maximum likelihood, Bayesian using gamma prior, Jeffreys prior and extension of Jeffreys prior to estimate the parameter of Exponential distribution with cure fraction based on right censored data. The comparison is based on mean squared error (MSE), and we have chosen n=20, 50 and 100 to represent small moderate and large sample size, and the following steps are employed.

1. Generate lifetime X with different sample sizes n=20, 50 and 100 from Exponential distribution and the value of scale parameter chosen was $\lambda = 2$ and 3.
2. Generate censored time C with different sample sizes n=20, 50 and 100 from Uniform distribution (0.4, 0.5).
3. The observed time T is the minimum of the failure and censored times, $T_i = \min(X_i, C_i)$, and we defined δ as follows,

$$\delta_i = 1 \quad if \;\; X \le C \quad \text{and}\; \delta_i = 0 \quad if \;\; X > C$$

4. Generate cure indicator Y with different sample sizes n=20, 50 and 100 from Uniform distribution (C, C+0.1) to obtain cure rates less than the censored rate. Then the cure indicator can be denoted as given below:

$$c_i = 1 \quad if \;\; X < Y \quad \text{and}\; c_i = 0 \quad if \;\; X \ge Y$$

5. The two values of extension of Jeffreys prior were a=0.4 and 1.4 and the hyper-parameters in gamma prior were b=d=1.
6. The value for the Linex loss function was r= 0.7 see Al Omari [15].
7. The maximum likelihood from equation 6 was used to estimate the parameter for the Exponential distribution.
8. The Bayesian with gamma prior under square error loss function was estimated the parameter for the Exponential distribution form equation (7) and under Linex loss function from equation (8).
9. The Bayesian with Jeffreys prior under square error loss function was estimated the parameter for the Exponential distribution form equation (9) and under Linex loss function from equation (10).
10. The Bayesian with extension of Jeffreys prior under square error loss function was estimated the parameter for the Exponential distribution form equation (11) and under Linex loss function from equation (12).
11. The number of replication used was R= 10,000 times. For each repetition the mean squared error (MSE) of scale parameter of Exponential distribution was calculated as follows,

$$\text{MSE}(\hat{\lambda}) = \frac{\sum_{i=1}^{R}\left(\hat{\lambda}_i - \hat{\lambda}\right)^2}{R},$$

The results are displayed in Tables for the different choices parameter, extension of Jeffreys prior.

4. Results and Discussion

Table-1. Estimate the parameter of Exponential distribution based right censored data with cure fraction.

Sizes	Estimators	$\lambda = 2$	$\lambda = 3$
20	MLE	1.90212	2.87336
	BGS	1.90245	2.87365
	BGL	1.90253	2.87377
	BJS	1.90212	2.87336
	BJL	1.90225	2.87349
	BES(*a*=0.4)	1.90206	2.87331
	BES(*a*=1.4)	1.90226	2.87352
	BEL(*a*=0.4)	1.90237	2.87361
	BEL(*a*=1.4)	1.90248	2.87372
50	MLE	1.92322	2.89446
	BGS	1.92351	2.89475
	BGL	1.92360	2.89487
	BJS	1.92322	2.89446
	BJL	1.92335	2.89459
	BES(*a*=0.4)	1.92316	2.89442
	BES(*a*=1.4)	1.92336	2.89460
	BEL(*a*=0.4)	1.92347	2.89471
	BEL(*a*=1.4)	1.92358	2.89482
100	MLE	1.94192	2.91316
	BGS	1.94219	2.91345
	BGL	1.94231	2.91357
	BJS	1.94192	2.91316
	BJL	1.94205	2.91329
	BES(*a*=0.4)	1.94186	2.91310
	BES(*a*=1.4)	1.94206	2.91332
	BEL(*a*=0.4)	1.94217	2.91341
	BEL(*a*=1.4)	1.94228	2.91352

Table-2. Mean square error (MSE) of the parameter of Exponential distribution based right censored data with cure fraction.

Sizes	Estimators	$\lambda = 2$	$\lambda = 3$
20	MLE	0.01921	0.01807
	BGS	0.01895	0.01783
	BGL	0.01881	0.01768
	BJS	0.01921	0.01807
	BJL	0.01920	0.01806
	BES(*a*=0.4)	0.01915	0.01802
	BES(*a*=1.4)	0.01909	0.01795
	BEL(*a*=0.4)	0.01898	0.01784
	BEL(*a*=1.4)	0.01883	0.01769
50	MLE	0.01408	0.01285
	BGS	0.01382	0.01259
	BGL	0.01368	0.01247
	BJS	0.01408	0.01285
	BJL	0.01407	0.01284
	BES(*a*=0.4)	0.01403	0.01281
	BES(*a*=1.4)	0.01391	0.01268
	BEL(*a*=0.4)	0.01385	0.01262
	BEL(*a*=1.4)	0.01375	0.01252
100	MLE	0.01108	0.01011
	BGS	0.01082	0.00984
	BGL	0.01068	0.00975
	BJS	0.01108	0.01011
	BJL	0.01107	0.01009
	BES(*a*=0.4)	0.01102	0.01004
	BES(*a*=1.4)	0.01093	0.00995
	BEL(*a*=0.4)	0.01084	0.00986
	BEL(*a*=1.4)	0.01079	0.00981

As appeared in Table 1, the gauge of the parameter λ of Exponential distribution with cure fraction based on right censored data is acquired utilizing Maximum likelihood (MLE), Bayesian with gamma prior under square error loss function (BGS), and Linear Exponential loss function (BSL). Bayesian with Jeffreys prior under square error loss function (BJS), and Linear Exponential loss function (BJL). Likewise Bayesian with extension of Jeffreys prior under square error loss function (BES), and Linear Exponential loss function (BEL).

In table 1, The maximum likelihood estimator (MLE) of the parameter λ of Exponential distribution with cure fraction based on right censored data for the mean square error is calculated and obtained for as 1.90212 for a sample size of 20 and λ=2. Having repeated the sample for 10,000 times, the mean square error of the parameter was determined in table 2 as 0.01921. The Bayesian with gamma prior under square error loss function (BGS) aimed at estimation of the parameter regarding the mean square error and was also obtained for lambda as 1.90245 and 0.01895 respectively for a sample of size 20 and λ=2.

Table 2 the gauge of the scale parameter λ of Exponential distribution with cure fraction based on right censored data was analyzed by mean squared error (MSE). The outcomes show that, the Bayesian with gamma prior under LINEX loss function is better compare to the others because it had the smallest value of mean square error in all cases. Bayesian with gamma prior

The Bayesian with extension of Jeffreys prior when the value of *a* was equal to 1.4 under LINEX loss function is better compared the MLE and Bayesian with Jeffreys prior and extension of Jeffreys prior (0.4). This implies that, as the value of extension of Jeffreys prior is kept below one, it exerts very minimal influence on the posterior distribution but as it increases to at least above one, the influence becomes significant on the posterior distribution from which Bayesian inference is drawn to give a very small mean squared error and as compared to maximum likelihood and that of Jeffreys prior.

From tables 1 and 2, when the sample size *n* increases the mean squared error (MSE) decreases for all cases of the parameter of Exponential distribution with cure fraction based on right censored data.

5. Conclusion

In this paper we have considered the Bayesian with three types of priors: gamma prior, Jeffreys prior and extension of Jeffreys prior with cure fraction based on right censored data. Comparisons are made between the Bayesian under two loss functions: square error loss function and LINEX loss function with maximum likelihood estimators based on simulation study and we observed that, the parameter of Exponential distribution with cure fraction based on right censored data are better estimated by Bayesian with gamma prior under LINEX loss function.

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