Academic Journal of Applied Mathematical Sciences
ISSN(e): 2415-2188, ISSN(p): 2415-5225
Vol. 7, Issue. 2, pp: 113-128, 2021
URL: https://arpgweb.com/journal/journal/17
DOI: https://doi.org/10.32861/ajams.72.113.128

Academic Research Publishing Group

Original Research Open Access

Approaches for Solving Fully Fuzzy Rough Multi-Objective Nonlinear Programming Problems

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Article History
Received: February 2, 2021
Revised: February 21, 2021
Accepted: March 4, 2021
Published: March 6, 2021

Abstract

Practical nonlinear programming problem often encounters uncertainty and indecision due to various factors that cannot be controlled. To overcome these limitations, fully fuzzy rough approaches are applied to such a problem. In this paper, an effective two approaches are proposed to solve fully fuzzy rough multi-objective nonlinear programming problem (FFRMONLP) where all the variables and parameters are fuzzy rough triangular numbers. The first, based on a slice sum technique, a fully fuzzy rough multi-objective nonlinear problem has turned into five an equivalent multi-objective nonlinear programming (FFMONLP) problems. The second proposed method for solving FFRMONLP problems is α-cut approach, where the triangular fuzzy rough variables and parameters of FFRMONLP problem are converted into rough interval variables and parameters by α-level cut, moreover the rough MONLP problem turns into four MONLP problems. Furthermore, the weighted sum method is used in both proposed approaches to convert multi-objective nonlinear problems into an equivalent nonlinear programming problem. Finally, the effectiveness of the proposed procedure is demonstrated by numerical examples.

Keywords: Fuzzy programming; Triangular fuzzy number; Multi-objective nonlinear programming; Fully Fuzzy programming; Fuzzy rough programming.

1. Introduction

Mathematical model data cannot be unambiguously collected in many real-world problems. This uncertainty may occur in a vague sense, rough, or both. Moreover, when the parameters are imprecise scalar quantities, it is very appropriate to implement ambiguous quantities to model these situations. The concept of fuzzy quantities in decision making was introduced by Bellman and Zadeh [1], Zimmermann [2], Fortemps [3] and Sakawa have proposed fuzzy programming approach to solve crisp multi-objective linear and nonlinear programming problems. Dong and Wan [4], proposed A new method for solving fuzzy multi-objective linear programming Problems. Ammar, *et al.* [5], suggested a solution for fuzzy multi-objective nonlinear programming problem, Pérez-Cañedo, *et al.* [6] proposed a fuzzy epsilon-constraint method that yields Pareto optimal fuzzy solutions of fuzzy variable and fully fuzzy MOLP problems, in which all parameters and decision variables take on LR fuzzy numbers. A new algorithm was proposed for solving fully fuzzy multi-objective linear programming problem which first converted it into the multi-objective interval linear programming problem by Sharma and Aggarwal [7].

Rough Set Theory is a new mathematical theory introduced by Pawlak to deal with ambiguity or uncertainty in the early (1980). Youness [8], Feng, *et al.* [9] Mathematical programming proposed in the rough environment in several aspects. Lu, *et al.* [10] The concept of a rough interval is introduced to represent uncertain double information for many parameters. Garai, *et al.* [11], developed a multi-objective multi-item inventory model with fuzzy rough coefficients. Elsisy and Elsayed [12], develop bilevel multi-objective nonlinear programming problem (BMNPP), in which the objective functions have fuzzy nature and the constraints represented as a rough set. Midya and Roy [13], introduced an analysis of the interval programming utilizing rough interval Pandian, *et al.* [14] proposed level-bound method for solving fully fuzzy interval integer transportation problem Ammar and Eljarbi [15] proposed an algorithm for solving fuzzy rough multi-objective integer linear fractional programming problem.

In this paper, we introduce fully fuzzy rough MONLP problem such that all coefficients and variables in both the objective functions and constraints are fuzzy rough intervals. Basic notions of fuzzy number, fuzzy rough intervals, a triangular fuzzy rough number and the α-cut of a fuzzy rough interval, are given in section 2. We give solution proceedings to characterize the rough solution set of FFRMONLP problem with triangular fuzzy rough intervals in both the objective and constraints functions in section 3. The slice-sum algorithm to deduce the fully fuzzy rough solution set of the fully fuzzy rough MONLP problem is given in section 4. In section5 introduced an α-cut approach for solving FFRMONLP problem. For the above two approaches numerical examples are given.

2. Preliminary

Several necessary basic concepts are recalled in this section which can be established in Bellman and Zadeh [1], Buckley and Feuring [16], Das, *et al.* [17], and Zimmermann [2], Ammar and Muamer [18].

Defination1. A fuzzy number $\tilde{A}=(a^{\ell},a^{m},a^{u})$ is said to be a triangular fuzzy number (T.F.N) if it has the following membership function where a^{ℓ},a^{m},a^{u}

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a^{\ell}}{a^{m}-a^{\ell}} & for \quad a^{\ell}\le x\le a^{m}\\ \dfrac{a^{u}-x}{a^{u}-a^{m}} & for \quad a^{m}\le x\le a^{u}\\ 0, & otherwise\end{cases}$$

Defination2. Let (a^{ℓ},a^{m},a^{u}) and (b^{ℓ},b^{m},b^{u}) be two positive triangular fuzzy numbers, then

- $(a^{\ell},a^{m},a^{u})+(b^{\ell},b^{m},b^{u})=(a^{\ell}+b^{\ell},a^{m}+b^{m},a^{u}+b^{u})$.
- $(a^{\ell},a^{m},a^{u})-(b^{\ell},b^{m},b^{u})=(a^{\ell}-b^{u},a^{m}-b^{m},a^{u}-b^{\ell})$.
- $k\,(a^{\ell},a^{m},a^{u})=(ka^{\ell},ka^{m},ka^{u})$ for $k>0$.
- $k\,(a^{\ell},a^{m},a^{u})=(ka^{u},ka^{m},ka^{\ell})$ for $k<0$.
- $(a^{\ell},a^{m},a^{u})\times(b^{\ell},b^{m},b^{u})=(a^{\ell}\times b^{\ell},a^{m}\times b^{m},a^{u}\times b^{u})$.
- $(a^{\ell},a^{m},a^{u})\div(b^{\ell},b^{m},b^{u})=(a^{\ell}\div b^{u},a^{m}\div b^{m},a^{u}\div b^{\ell})$.

Let $F(R)$ be the set of all real triangular fuzzy numbers.

Definition 3. Let X be denote a compact set of real numbers. A fuzzy rough interval $\tilde{X}^{R}$ is defined as $\tilde{X}^{R}=[\tilde{X}^{LAI}:\tilde{X}^{UAI}]$ where $\tilde{X}^{LAI}$ and $\tilde{X}^{UAI}$ are fuzzy set called lower and upper approximation fuzzy numbers of $\tilde{X}^{R}$ with $\tilde{X}^{LAI}\subseteq\tilde{X}^{UAI}$.

Definition 4. A fuzzy rough interval $\tilde{A}^{R}=[\tilde{A}^{L}:\tilde{A}^{U}]$ is said to be normalized if $\tilde{A}^{L}$ and $\tilde{A}^{U}$ are normal.

Definition 5. Let $\tilde{A}^{R}=[\tilde{A}^{L}:\tilde{A}^{U}]$ and $\tilde{B}^{R}=[\tilde{B}^{L}:\tilde{B}^{U}]$ are two fuzzy rough intervals in $\in\mathbb{R}^{n}$. We say $\tilde{A}^{R}\cong\tilde{B}^{R}$ iff $\tilde{A}^{L}\cong\tilde{B}^{L}$ and $\tilde{A}^{U}\cong\tilde{B}^{U}$.

Definition 6. The α-cut of a fuzzy rough interval $\tilde{A}^{R}$ is defined as :
$(\tilde{A}^{R})_{\alpha}=[\tilde{A}^{L}_{\alpha}:\tilde{A}^{U}_{\alpha}]$ where $\tilde{A}^{L}_{\alpha}$ and $\tilde{A}^{U}_{\alpha}$ are intervals with $\tilde{A}^{L}_{\alpha}\subseteq_{LR}\tilde{A}^{U}_{\alpha}$.

Definition 7. A fuzzy rough number $\tilde{A}^{R}$ is a triangular fuzzy rough number denoted by $\tilde{A}^{R}=[(a^{\ell\ell},a^{m},a^{u\ell}):(a^{\ell u},a^{m},a^{uu})]$ where $a^{\ell\ell},a^{m},a^{u\ell},a^{\ell u},a^{m}$ and a^{uu} $\in\mathbb{R}^{n}$ such that $a^{\ell u}\le a^{\ell\ell}\le a^{m}\le a^{u\ell}\le a^{uu}$ and its membership function can be defined as:

Note that $\tilde{A}^{L}=(a^{\ell\ell},a^{m},a^{u\ell})$, $\tilde{A}^{U}=(a^{\ell u},a^{m},a^{uu})$ and $\tilde{A}^{L}\subseteq\tilde{A}^{U}$. where $\mu_{\tilde{A}^{L}}(x)$ and $\mu_{\tilde{A}^{U}}(x)$ are membership functions of lower and upper approximation triangular fuzzy number respectively.

We can define $(\tilde{A}^{R})_{\alpha}$ for any fuzzy rough with triangular fuzzy numbers $\tilde{A}^{R}=[(a^{\ell\ell},a^{m},a^{u\ell}):(a^{\ell u},a^{m},a^{uu})]$ as: $(\tilde{A}^{R})_{\alpha}=[a^{\ell\ell}(\alpha),a^{u\ell}(\alpha):a^{\ell u}(\alpha),a^{uu}(\alpha)]$ where

$$\tilde{A}^{L}_{\alpha}=[a^{\ell\ell}+(a^{m}-a^{u\ell})\alpha,a^{u\ell}+(a^{m}-a^{u\ell})\alpha]$$

and $$\tilde{A}^{U}_{\alpha}=[a^{\ell u}+(a^{m}-a^{uu})\alpha,a^{uu}+(a^{m}-a^{uu})\alpha],\alpha\in[0,1]$$

Definition 8. Let $\tilde{A}^{R}\ge 0$ and $\tilde{B}^{R}\ge 0$ be two fuzzy rough intervals , then

- $\tilde{A}^R \oplus \tilde{B}^R = [(\tilde{A}^L \oplus \tilde{B}^L):(\tilde{A}^U \oplus \tilde{B}^U)].$
- $\tilde{A}^R \ominus \tilde{B}^R = [(\tilde{A}^L \ominus \tilde{B}^L):(\tilde{A}^U \ominus \tilde{B}^U)].$
- $\tilde{A}^R \otimes \tilde{B}^R = [(\tilde{A}^L \otimes \tilde{B}^L):(\tilde{A}^U \otimes \tilde{B}^U)].$
- $\tilde{A}^R \oslash \tilde{B}^R = [(\tilde{A}^L \oslash \tilde{B}^L):(\tilde{A}^U \oslash \tilde{B}^U)].$

3. Formulation of Fully Fuzzy Rough MONLP Problem

The fully fuzzy rough multi-objective nonlinear programming problem is defined as follows:

$$\left.\begin{aligned} &(FFRMONLP_1):\quad \min \tilde{f}_r^R(\tilde{x}^R) = \sum_{j=1}^{m} \tilde{c}_{rj}^R (\tilde{x}_j^R)^{\delta}, r = 1, 2, ..., k\ ;\ \delta \in Z^+ \\ &\text{s.t.} \\ &\sum_{j=1}^{m} \tilde{a}_{ij}^R (\tilde{x}_j^R)^{\delta} \le \tilde{b}_i^R, \\ &\tilde{x}_j^R \ge 0, i \in I = \{1, 2, ..., n\};\ j \in J = \{1, 2, ..., m\} \end{aligned}\right\} \quad (1)$$

Where $\tilde{c}_{rj}^R, \tilde{a}_{ij}^R, \tilde{b}_i^R\ and\ (\tilde{x}_j^R)\quad i \in I, j \in J, \delta_j \in \mathbf{Z}^+, r = 1,2,...k,$ are fuzzy rough coefficient, parameters and variables, respectively.

$$\left.\begin{aligned} \tilde{c}_{rj}^R &= [(c_{rj}^{\ell\ell}, c_{rj}^m, c_{rj}^{u\ell}):(c_{rj}^{\ell u}, c_{rj}^m, c_{rj}^{uu})], \\ (\tilde{x}_j^R)^{\delta} &= [((x_j^{\ell\ell})^{\delta}, (x_j^m)^{\delta}, (x_j^{u\ell})^{\delta}):((x_j^{\ell u})^{\delta}, (x_j^m)^{\delta}, (x_j^{uu})^{\delta})], \\ \tilde{a}_{ij}^R &= [(a_{ij}^{\ell\ell}, a_{ij}^m, a_{ij}^{u\ell}):(a_{ij}^{\ell u}, a_{ij}^m, a_{ij}^{uu})], \\ \tilde{b}_i^R &= [(b_i^{\ell\ell}, b_i^m, b_i^{u\ell}):(b_i^{\ell u}, b_i^m, b_i^{uu})]. \end{aligned}\right\} \quad (2)$$

The problem (1) can be written as:

$$\left.\begin{aligned} &(FFRMONLP_2):\min [\tilde{f}_r^L(x):\tilde{f}_r^U(x)] = \sum_{j=1}^{m} [\tilde{c}_{rj}^L:\tilde{c}_{rj}^U]\otimes[(\tilde{x}_{rj}^L)^{\delta}:(\tilde{x}_{rj}^U)^{\delta}], r = 1,..., k \\ &\text{s. t.} \\ &\sum_{j=1}^{m}[\tilde{a}_{ij}^L:\tilde{a}_{ij}^U]\otimes[(\tilde{x}_{rj}^L)^{\delta}:(\tilde{x}_{rj}^U)^{\delta}] \le [\tilde{b}_i^L:\tilde{b}_i^U] \\ &\tilde{x}_j^L, \tilde{x}_j^U \ge 0, i \in I;\ j \in J, \delta \in \mathbf{Z}^+ \end{aligned}\right\} \quad (3)$$

Suppose that $\tilde{c}_{rj}^L,\ \tilde{c}_{rj}^U,\ \tilde{a}_{ij}^L,\ \tilde{a}_{ij}^U,\ \tilde{b}_i^L,\ \tilde{b}_i^U$ are triangular fuzzy numbers, $\tilde{x}_{rj}^L, \tilde{x}_{rj}^U$ are triangular fuzzy variables. Then problem (2) could be written as:

$$\left.\begin{aligned} &(FRMONLP_3):\min [(f_r^{\ell\ell}, f_r^m, f_r^{u\ell}):(f_r^{\ell u}, f_r^m, f_r^{uu})], r = 1, 2, ..., k \\ &\text{s.t.} \\ &\sum_{j=1}^{m}[(a_{ij}^{\ell\ell}, a_{ij}^m, a_{ij}^{u\ell}):(a_{ij}^{\ell u}, a_{ij}^m, a_{ij}^{uu})]\otimes[((x_j^{\ell\ell})^{\delta}, (x_j^m)^{\delta}, (x_j^{u\ell})^{\delta}):((x_j^{\ell u})^{\delta}, (x_j^m)^{\delta}, (x_j^{uu})^{\delta})] \\ &\qquad\qquad \le [(b_i^{\ell\ell}, b_i^m, b_i^{u\ell}):(b_i^{\ell u}, b_i^m, b_i^{uu})] \\ &x_j^{\delta} \ge 0, i \in I;\ j \in J, \delta \in \mathbf{Z}^+. \end{aligned}\right\} \quad (4)$$

where $(f_r^{\ell\ell}, f_r^m, f_r^{u\ell}) = (c_{rj}^{\ell\ell}, c_{rj}^m, c_{rj}^{u\ell})\otimes((x_j^{\ell\ell})^{\delta}, (x_j^m)^{\delta}, (x_j^{u\ell})^{\delta})$

$$(f_r^{\ell u}, f_r^m, f_r^{uu}) = (c_{rj}^{\ell u}, c_{rj}^m, c_{rj}^{uu})\otimes((x_j^{\ell u})^{\delta}, (x_j^m)^{\delta}, (x_j^{uu})^{\delta}).$$

The problem (3) is equivalent to the following crisp MONLP problems

$$\left.\begin{array}{l}(FFRMONLP_4):\ \min\ (f_r^{\ell\ell},\ f_r^{u\ell},\ f_r^{m},\ f_r^{\ell u},\ f_r^{uu}),r=1,\ 2,\ ...,\ k\\ \quad \text{s.t.}\\ \quad \sum_{j=1}^{m} a_{ij}^{\ell\ell}(x_j^{\ell\ell})^{\delta} \le b_i^{\ell\ell},\quad \sum_{j=i}^{m} a_{ij}^{u\ell}(x_j^{u\ell})^{\delta} \le b_i^{u\ell},\quad \sum_{j=1}^{m} a_{ij}^{m}(x_j^{m})^{\delta} \le b_i^{m},\\ \quad \sum_{j=1}^{m} a_{ij}^{\ell u}(x_j^{\ell u})^{\delta} \le b_i^{\ell u},\quad \sum_{j=1}^{m} a_{ij}^{uu}(x_j^{uu})^{\delta} \le b_i^{uu}.\\ \quad x_j^{\delta} \ge 0, i \in I\ ;\ j \in J, \delta \in \mathbf{Z}^+\end{array}\right\} \quad (5)$$

where

$$f_r^{\ell\ell} = \sum_{j=1}^{m} c_{rj}^{\ell\ell}(x_j^{\ell\ell})^{\delta},\qquad f_r^{u\ell} = \sum_{j=1}^{m} c_{rj}^{u\ell}(x_j^{u\ell})^{\delta},\qquad f_r^{m} = \sum_{j=1}^{m} c_{rj}^{m}(x_j^{m})^{\delta},$$

$$f_r^{\ell u} = \sum_{j=1}^{m} c_{rj}^{\ell u}(x_j^{\ell u})^{\delta},\qquad f_r^{uu} = \sum_{j=1}^{m} c_{rj}^{uu}(x_j^{uu})^{\delta}.$$

Definition9: The $\tilde{x}_j^{*R} = [(x_j^{*\ell\ell}, x_j^{*m}, x_j^{*u\ell}) : (x_j^{*\ell u}, x_j^{*m}, x_j^{*uu})]$ triangular fuzzy rough vector which satisfies the conditions in problem (1) is called a fuzzy rough efficient solution of problem (5), if and only if there does not exist another $\tilde{x}^{R} = [(x^{\ell\ell}, x^{m}, x^{u\ell}) : (x^{\ell u}, x^{m}, x^{uu})] \in \tilde{M}^{R}$ such that

$$f_r(x^{*uu}) \le f_r(x^{uu}), f_r(x^{*u\ell}) \le f_r(x^{u\ell}),$$
$$f_r(x^{*m}) \le f_r(x^{m}), f_r(x^{*\ell\ell}) \le f_r(x^{\ell\ell})$$
$$\text{and } f_r(x^{*\ell u}) \le f_r(x^{\ell u})\ \forall r = 1,2,...k$$

And for at least one r=1,2,…,k follows:

$$f_r(x^{*uu}) < f_r(x^{uu}), f_r(x^{*u\ell}) < f_r(x^{u\ell}),$$
$$f_r(x^{*m}) < f_r(x^{m}), f_r(x^{*\ell\ell}) < f_r(x^{\ell\ell})$$
$$\text{and } f_r(x^{*\ell u}) < f_r(x^{\ell u}).$$

From the $(FFRMONLP_4)$ problem will be constructed five FMONLP problems as:

$$\left.\begin{array}{l}\text{FFMNPLP}^{uu}:\ \min\ f_r^{uu} = \sum_{j=1}^{m} c_{rj}^{uu}(x_j^{uu})^{\delta}, r=1,\ 2,\ ...,\ k\\ \quad \text{s. t.}\\ \quad \sum_{j=1}^{m} a_{ij}^{uu}(x_j^{uu})^{\delta} \le b_i^{uu}\\ \quad x_j^{uu} \ge 0, i \in I\ ;\ \delta \in \mathbf{Z}^+\end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l}\text{FFMONLP}^{u\ell}:\quad \min\ f_r^{u\ell} = \sum_{j=1}^{m} c_{rj}^{u\ell}(x_j^{u\ell})^{\delta}, r=1,\ 2,\ ...,\ k\\ \quad \text{s. t.}\\ \quad \sum_{j=1}^{m} a_{ij}^{u\ell}(x_j^{u\ell})^{\delta} \le b_i^{u\ell},\\ \quad x_j^{u\ell} \le x_j^{*uu}; x_j^{u\ell} \ge 0, i \in I, \delta \in \mathbf{Z}^+\end{array}\right\} \quad (7)$$

$$\left.\begin{aligned}&\text{FFMONLP}^{m}: \quad \min f_r^{m} = \sum_{j=1}^{m} c_{rj}^{m} (x_j^{m})^{\delta}, r = 1, 2, ..., k\\ &\text{s. t.}\\ &\sum_{j=1}^{m} a_{ij}^{m} (x_j^{m})^{\delta} \le \text{b}_i^{m},\\ &\text{x}_j^{*m} \le \text{x}_j^{*u\ell}, \text{x}_j^{*m} \ge 0, i \in I;\ j \in J, \delta \in \mathbf{Z}^{+}\end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned}&\text{FFMONLP}^{\ell\ell}: \quad \min f_r^{\ell\ell} = \sum_{j=1}^{m} c_{rj}^{\ell\ell} (x_j^{\ell\ell})^{\delta}, r = 1, 2, ..., k\\ &\text{s. t.}\\ &\sum_{j=1}^{m} a_{ij}^{\ell\ell} (x_j^{\ell\ell})^{\delta} \le \text{b}_i^{\ell\ell},\\ &\text{x}_j^{\ell\ell} \le \text{x}_j^{*m};\ \text{x}_j^{\ell\ell} \ge 0, i \in I, \delta \in \mathbf{Z}^{+}\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}&\text{FFMOLNP}^{\ell u}: \quad \min f_r^{\ell u} = \sum_{j=1}^{m} c_{rj}^{\ell u} (x_j^{\ell u})^{\delta}, r = 1, 2, ..., k\\ &\text{s. t.}\\ &\sum_{j=1}^{m} a_{ij}^{\ell u} (x_j^{\ell u})^{\delta} \le \text{b}_i^{\ell u},\\ &\text{x}_j^{\ell u} \le \text{x}_j^{*\ell\ell}; \text{x}_j^{\ell u} \ge 0, i \in I,\ \delta \in \mathbf{Z}^{+}\end{aligned}\right\} \quad (10)$$

By weighting method the last five problems FFMOLNP^{uu}, $\text{FFMOLNP}^{u\ell}$, FFMOLNP^{m} $\text{FFMOLNP}^{\ell\ell}$ *and* $\text{FFMOLNP}^{\ell u}$ may be converted into the equivalent the following N.L.P: For $\text{w}_r \in W = \{\text{w}_r : \sum_{r=1}^{k} w_r = 1, \text{w}_r \ge 0\}$

$$\left.\begin{aligned}&FFP^{uu}(\text{w}): \quad \min \left(\sum_{r=1}^{k} w_r \sum_{j=1}^{m} c_{rj}^{uu} (x_j^{uu})^{\delta}\right)\\ &\text{s. t.}\\ &\sum_{j=1}^{m} a_{ij}^{uu} (x_j^{uu})^{\delta} \le \text{b}_i^{uu}\\ &\text{x}_j^{uu} \ge 0, i \in I;\ w_r \in W, \delta \in \mathbf{Z}^{+}\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}&FFP^{u\ell}(\text{w}): \quad \min \left(\sum_{r=1}^{k} w_r \sum_{j=1}^{m} c_{rj}^{u\ell} (x_j^{u\ell})^{\delta}\right)\\ &\text{s.t.}\\ &\sum_{j=1}^{m} a_{ij}^{u\ell} (x_j^{u\ell})^{\delta} \le \text{b}_i^{u\ell},\\ &\text{x}_j^{u\ell} \le \text{x}_j^{*uu},\\ &\text{x}_j^{u\ell} \ge 0, i \in I;\ w_r \in \text{W}, \delta \in \mathbf{Z}^{+}\end{aligned}\right\} \quad (12)$$

where x_j^{*uu} is the solution of $FFP^{uu}(\text{w})$ problem.

$$\left.\begin{array}{l} FFP^{m}(\mathrm{w}): \quad \min \left(\sum_{r=1}^{k} w_r \sum_{j=1}^{m} c_{rj}^{m} (x_j^{m})^{\delta}\right) \\ \quad \text{s. t.} \\ \quad \sum_{j=1}^{m} a_{ij}^{m} (x_j^{m})^{\delta} \le \mathrm{b}_i^{m}, \\ \quad \mathrm{x}_j^{m} \le \mathrm{x}_j^{*u\ell}; \\ \quad \mathrm{x}_j^{m} \ge 0, i \in I;\ w_r \in \mathrm{W}, \delta \in \mathbf{Z}^{+} \end{array}\right\} \qquad (13)$$

where $\mathrm{x}_j^{*u\ell}$ is the solution of $FFP^{u\ell}(\mathrm{w})$ problem.

$$\left.\begin{array}{l} FFP^{\ell\ell}(\mathrm{w}): \quad \min \left(\sum_{r=1}^{k} w_r \sum_{j=1}^{m} c_{rj}^{\ell\ell} (x_j^{\ell\ell})^{\delta}\right) \\ \quad \text{s. t.} \\ \quad \sum_{j=1}^{m} a_{ij}^{\ell\ell} (x_j^{\ell\ell})^{\delta} \le \mathrm{b}_i^{\ell\ell}, \\ \quad \mathrm{x}_j^{\ell\ell} \le \mathrm{x}_j^{*m}, \\ \quad \mathrm{x}_j^{\ell\ell} \ge 0, i \in I;\ w_r \in W, \delta \in \mathbf{Z}^{+} \end{array}\right\} \qquad (14)$$

where x_j^{*m} is the solution of $FFP^{m}(\mathrm{w})$ problem.

$$\left.\begin{array}{l} FFP^{\ell u}(\mathrm{w}): \quad \min \sum_{r=1}^{k} w_r \sum_{j=1}^{m} c_{rj}^{\ell u} (x_j^{\ell u})^{\delta} \\ \quad \text{s. t.} \\ \quad \sum_{j=1}^{m} a_{ij}^{\ell u} (x_j^{\ell u})^{\delta} \le \mathrm{b}_i^{\ell u}, \\ \quad \mathrm{x}_j^{\ell u} \le \mathrm{x}_j^{*\ell\ell}, \\ \quad \mathrm{x}_j^{\ell u} \ge 0, i \in I, w_r \in W, \delta \in \mathbf{Z}^{+} \end{array}\right\} \qquad (15)$$

where $\mathrm{x}_j^{*\ell\ell}$ is the solution of $FFP^{\ell\ell}(\mathrm{w})$ problem.

Theorem 1. Bellman and Zadeh [1] For w*∈W, if $\mathbf{x}^{*}(w^{*}) \in \mathbb{R}^n$ is an optimal solution of N.L.P problems $FFP^{uu}(\mathrm{w}^*), FFP^{u\ell}(\mathrm{w}^*), FFP^{m}(\mathrm{w}^*), FFP^{\ell\ell}(\mathrm{w}^*)$ *and* $FFP^{\ell u}(\mathrm{w}^*)$. Then $\mathbf{x}^{*} \in \mathbb{R}^n$ an efficient solution of the corresponding MONLP problems $FMONLP^{uu}, FMONLP^{u\ell}, FMONLP^{m}, FMONLP^{\ell\ell}$ *and* $FMONLP^{\ell u}$.

Theorem 2. Let $x^{*uu}, x^{*u\ell}, \mathrm{x}^{*m}, x^{*\ell\ell}$ *and* $x^{*\ell u} \in \mathbb{R}^n$ be efficient solutions of $FFP^{uu}(\mathrm{w}^*), FFP^{u\ell}(\mathrm{w}^*), FFP^{m}(\mathrm{w}^*), FFP^{\ell\ell}(\mathrm{w}^*)$ *and* $FFP^{\ell u}(\mathrm{w}^*)$ respectively, then $\tilde{x}^{R} = [(\mathrm{x}^{*\ell\ell}, x^{*m}, x^{*u\ell}) : (x^{*\ell u}, x^{*m}, x^{*uu})]$ is an efficient fuzzy rough solution of the problem $(FFRMONLP_1)$.

Proof: Let $\tilde{x}^{R} = [(\mathrm{x}^{*\ell\ell}, \mathrm{x}^{*m}, \mathrm{x}^{*u\ell}) : (\mathrm{x}^{*\ell u}, \mathrm{x}^{*m}, \mathrm{x}^{*uu})]$ be a feasible solution of problem (1), clearly, $x^{*uu}, x^{*u\ell} x^{*m}, x^{*\ell\ell}$ and $x^{*\ell u}$ are feasible solutions of $FFP^{uu}(w^*), FFP^{u\ell}(w^*), FFP^{m}(w^*), FFP^{\ell\ell}(w^*)$ *and* $FFP^{\ell u}(w^*)$ respectively.

Assume that $x^{*uu}, x^{*u\ell} x^{*m}, x^{*\ell\ell}$ and $x^{*\ell u}$ are efficient solutions of problems $FFP^{uu}(w^*), FFP^{u\ell}(w^*), FFP^{m}(w^*), FFP^{\ell\ell}(w^*)$ *and* $FFP^{\ell u}(w^*)$, then

$$f_r(x^{*uu}) \le f_r(y^{uu}), f_r(x^{*u\ell}) \le f_r(y^{u\ell}),$$
$$f_r(x^{*m}) \le f_r(y^{m}), f_r(x^{*\ell\ell}) \le f_r(y^{\ell\ell})$$
$$\text{and} f_r(x^{*\ell u}) \le f_r(y^{\ell u})$$

for all. $r = 1, 2, \ldots k$ For all $\tilde{y}^R = (y^{\ell\ell}, y^m, y^{u\ell}) : (y^{\ell u}, y^m, y^{uu})$ feasible solutions of $FFP^{uu}(w^*), FFP^{u\ell}(w^*), FFP^{m}(w^*), FFP^{\ell\ell}(w^*)$ and $FFP^{\ell u}(w^*)$ and for at least one r follows:

$$f_r(x^{*uu}) < f_r(y^{uu}), f_r(x^{*u\ell}) < f_r(y^{u\ell}),$$
$$f_r(x^{*m}) < f_r(y^{m}), f_r(x^{*\ell\ell}) < f_r(y^{\ell\ell})$$
$$\text{and} f_r(x^{*\ell u}) < f_r(y^{\ell u})$$

This implies that $f_r^*(\tilde{x}^R) \le f_r^*(\tilde{y}^R)$ for all $r = 1, 2, \ldots k$ and $f_r^*(\tilde{x}^R) < f_r^*(\tilde{y}^R)$ for at least one r. Therefore, $\tilde{x}^R = [(x^{*\ell\ell}, x^{*m}, x^{*u\ell}) : (x^{*\ell u}, x^{*m}, x^{*uu})]$ is an efficient fuzzy solution to the given problem $(FFRMONLP_1)$.

3.1. Algorithm Solution for FRMONLP Problem

1. Consider the problem in the form $(FFRMONLP_4)$.
2. Transfer the $(FFRMONLP\,4)$ to five forms as $FFMONLP^{uu}$, $FFMONLP^{u\ell}$, $FFMONLP^{m}$, $FFMONLP^{\ell\ell}$ and $FMONLP^{\ell u}$ problems.
3. Use one of the secularization methods, say the weights method to convert each problem $FFMONLP^{uu}$, $FFMONLP^{u\ell}$, $FFMONLP^{m}$, $FFMONLP^{\ell\ell}$ and $FFMONLP^{\ell u}$ with a single objective in the form $FFP^{uu}(w), FFP^{u\ell}(w), FFP^{m}(w), FFP^{\ell\ell}(w)$ and $FFP^{\ell u}(w)$.
4. For $w = w^* \in W$ Find the optimal solution of each nonlinear programming problems $FFP^{uu}(w^*), FFP^{u\ell}(w^*), FFP^{m}(w^*), FFP^{\ell\ell}(w^*)$ and $FFP^{\ell u}(w^*)$.
5. Obtain the set of efficient fuzzy rough solutions to the given problem, using the results of last step by theorem(3)and(4) is

$$\tilde{x}^R = [(x^{*\ell\ell}, x^{*m}, x^{*u\ell}) : (x^{*\ell u}, x^{*m}, x^{*uu})]$$

and fuzzy rough value is: $\tilde{f}_r^R(x^*) = [(f(x^{*\ell\ell}), f(x^{*m}), f(x^{*u\ell}) : (f(x^{*\ell u}), f(x^{*m}), f(x^{*uu})].$

Example 3.1: Consider the following FRMONLP:

$$\left.\begin{aligned} &\text{Min } \tilde{f}_r^R(x) = (\tilde{c}_{11}^R(\tilde{x}_1^R)^2 + \tilde{c}_{12}^R(\tilde{x}_2^R)^2, \tilde{c}_{21}^R\tilde{x}_1^R + \tilde{c}_{22}^R\tilde{x}_2^R) \\ &s.t. \\ &\tilde{a}_{11}^R\tilde{x}_1^R + \tilde{a}_{12}^R\tilde{x}_2^R \le \tilde{b}_1^R, \\ &\tilde{a}_{21}^R\tilde{x}_1^R + \tilde{a}_{22}^R\tilde{x}_2^R \le \tilde{b}_2^R, \\ &\tilde{x}_2^R,\ \tilde{x}_2^R \ge 0. \end{aligned}\right\} \quad (16)$$

where

$$\tilde{x}_1^R = (x_1^{\ell\ell}, x_1^m, x_1^{u\ell}) : (x_1^{\ell u}, x_1^m, x_1^{uu}),\ \tilde{x}_2^R = (x_2^{\ell\ell}, x_2^m, x_2^{u\ell}) : (x_2^{\ell u}, x_2^m, x_2^{uu}),$$

$$\tilde{c}_{11}^R = (2,3,4):(1,3,5),\ \tilde{c}_{12}^R = (3,4,5):(2,4,7),\ \tilde{c}_{21}^R = (4,5,6):(2,5,8),\ \tilde{c}_{22}^R = (5,6,7):(1,6,9),$$

$$\tilde{a}_{11}^R = (3,4,5):(1,4,6),\ \tilde{a}_{12}^R = (1,2,3):(0.5,2,5),$$

$$\tilde{a}_{21}^R = (0.5,1,3):(0.25,1,4),\ \tilde{a}_{22}^R = (1,3.5,5):(0.5,3.5,7),$$

$$\tilde{b}_1^R = (20,30,45):(15,30,55),\ \tilde{b}_2^R = (14,15,30):(5,15,40).$$

$$\left.\begin{array}{l}
\min \begin{pmatrix} [(2,3,4):(1,3,5)]\otimes[((x_1^{\ell\ell})^2,(x_1^m)^2,(x_1^{u\ell})^2):((x_1^{\ell u})^2,(x_1^m)^2,(x_1^{uu})^2)]+ \\ [(3,4,5):(2,4,7)]\otimes((x_2^{\ell\ell})^2,(x_2^m)^2,(x_2^{u\ell})^2):((x_2^{\ell u})^2,(x_2^m)^2,(x_2^{uu})^2), \\ [(4,5,6):(2,5,8)]\otimes(x_1^{\ell\ell},x_1^m,x_1^{u\ell}):(x_1^{\ell u},x_1^m,x_1^{uu})+ \\ [(5,6,7):(1,6,9)]\otimes(x_2^{\ell\ell},x_2^m,x_2^{u\ell}):(x_2^{\ell u},x_2^m,x_2^{uu}) \end{pmatrix} \\
\text{s.t.} \\
[(3,4,5):(1,4,6)]\otimes(x_1^{\ell\ell},x_1^m,x_1^{u\ell}):(x_1^{\ell u},x_1^m,x_1^{uu})+ \\
[(1,2,3):(0.5,2,5)]\otimes(x_2^{\ell\ell},x_2^m,x_2^{u\ell}):(x_2^{\ell u},x_2^m,x_2^{uu}) \le [(20,30,45):(15,30,55)], \\
[(0.5,1,3):(0.25,1,4)]\otimes(x_1^{\ell\ell},x_1^m,x_1^{u\ell}):(x_1^{\ell u},x_1^m,x_1^{uu})+ \\
[(1,3.5,5):(0.5,3.5,7)]\otimes(x_2^{\ell\ell},x_2^m,x_2^{u\ell}):(x_2^{\ell u},x_2^m,x_2^{uu}) \le [(14,15,30):(5,15,40)], \\
\quad x_1, x_2 \ge 0.
\end{array}\right\} \quad (17)$$

Can be written as:

$$\left.\begin{array}{l}
(FFRMONLP): \\
\min \left([(f_1^{\ell\ell},f_1^m,f_1^{u\ell}):(f_1^{\ell u},f_1^m,f_1^{uu})],\ [(f_2^{\ell\ell},f_2^m,f_2^{u\ell}):(f_2^{\ell u},f_2^m,f_2^{uu})] \right) \\
\text{s.t} \\
[(3,4,5):(1,4,6)]\otimes(x_1^{\ell\ell},x_1^m,x_1^{u\ell}):(x_1^{\ell u},x_1^m,x_1^{uu})+ \\
[(1,2,3):(0.5,2,5)]\otimes(x_2^{\ell\ell},x_2^m,x_2^{u\ell}):(x_2^{\ell u},x_2^m,x_2^{uu}) \le [(20,30,45):(15,30,55)], \\
[(0.5,1,3):(0.25,1,4)]\otimes(x_1^{\ell\ell},x_1^m,x_1^{u\ell}):(x_1^{\ell u},x_1^m,x_1^{uu})+ \\
[(1,3.5,5):(0.5,3.5,7)]\otimes(x_2^{\ell\ell},x_2^m,x_2^{u\ell}):(x_2^{\ell u},x_2^m,x_2^{uu}) \le [(14,15,30):(5,15,40)], \\
\quad x_1, x_2 \ge 0.
\end{array}\right\} \quad (18)$$

where

$$f_1^m = 3(x_1^m)^2 + 4(x_2^m)^2,\ f_2^m = 5x_1^m + 6x_2^m,$$

$$f_1^{uu} = 5(x_1^{uu})^2 + 7(x_2^{uu})^2, f_2^{uu} = 8x_1^{uu} + 9x_2^{uu},$$

$$f_1^{u\ell} = 4(x_1^{u\ell})^2 + 5(x_2^{u\ell})^2, f_2^{u\ell} = 6x_1^{u\ell} + 7x_2^{u\ell},$$

$$f_1^{\ell\ell} = 2(x_1^{\ell\ell})^2 + (3x_2^{\ell\ell})^2, f_2^{\ell\ell} = 4x_1^{\ell\ell} + 5x_2^{\ell\ell},$$

$$f_1^{\ell u} = (x_1^{\ell u})^2 + 2(x_2^{\ell u})^2,\ f_2^{\ell u} = 2x_1^{\ell u} + 1x_2^{\ell u}.$$

$$\left.\begin{array}{ll}
\text{FFMONLP}^{uu}: & \min f^{uu} = (5(x_1^{uu})^2 + 7(x_2^{uu})^2, 8x_1^{uu} + 9x_2^{uu}) \\
& \text{s. t.} \\
& 6x_1^{uu} + 5x_2^{uu} \le 55, \\
& 4x_1^{uu} + 7x_2^{uu} \le 40, \\
& x_j^{uu} \ge 0,\ j = 1, 2
\end{array}\right\} \quad (19)$$

$$\left.\begin{aligned} &\text{FFMONLP}^{u\ell}: \quad \min f_r^{u\ell} = (4(x_1^{u\ell})^2 + 5(x_2^{u\ell})^2, 6x_1^{u\ell} + 7x_2^{u\ell}) \\ &\text{s. t.} \\ &5x_1^{u\ell} + 3x_2^{u\ell} \le 45, \\ &3x_1^{u\ell} + 5x_2^{u\ell} \le 30, \\ &x_j^{u\ell} \le x_j^{*uu},\ x_j^{u\ell} \ge 0,\ j = 1, 2. \end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned} &\text{FFMONLP}^{m}: \quad \min f_r^{m} = (3(x_1^{m})^2 + 4(x_2^{m})^2, 5x_1^{m} + 6x_2^{m}) \\ &\text{s. t.} \\ &4x_1^{m} + 2x_2^{m} \le 30, \\ &x_1^{m} + 3.5x_2^{m} \le 15 \\ &x_j^{m} \le x_j^{*u\ell},\ x_j^{m} \ge 0,\ j = 1, 2 \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} &\text{FFMONLP}^{\ell\ell}: \quad \min f_r^{\ell\ell} = (2(x_1^{\ell\ell})^2 + 3(x_2^{\ell\ell})^2, 4x_1^{\ell\ell} + 5x_2^{\ell\ell}) \\ &\text{s. t.} \\ &3x_1^{\ell\ell} + x_2^{\ell\ell} \le 20, \\ &0.5x_1^{\ell\ell} + x_2^{\ell\ell} \le 14, \\ &x_j^{\ell\ell} \le x_j^{*m},\ x_j^{\ell\ell} \ge 0,\ j = 1, 2. \end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned} &\text{FFMONLP}^{\ell u}: \quad \min f_r^{\ell u} = ((x_1^{\ell u})^2 + 2(x_2^{\ell u})^2, 2x_1^{\ell u} + x_2) \\ &\text{s. t.} \\ &x_1^{\ell u} + 0.5x_2^{\ell u} \le 15, \\ &0.25x_1^{\ell u} + 0.5x_2^{\ell u} \le 5, \\ &x_j^{\ell u} \le x_j^{*\ell\ell},\ x_j^{\ell u} \ge 0,\ j = 1, 2. \end{aligned}\right\} \quad (23)$$

For $w^* = w_1 = w_2 = 0.5$ The last five problem FFMONLP^{uu}, $\text{FFMONLP}^{u\ell}$, FFMONLP^{m}, $\text{FMONLP}^{\ell\ell}$ and $\text{FMONLP}^{\ell u}$ will converted into the equivalent N.L.P. problems

$$\left.\begin{aligned} &FFP^{uu}(w^*): \quad \min\ 2.5(x_1^{uu})^2 + 3.5(x_2^{uu})^2 + 4x_1^{uu} + 4.5x_2^{uu} \\ &\text{s. t.} \\ &6x_1^{uu} + 5x_2^{uu} \le 55, \\ &4x_1^{uu} + 7x_2^{uu} \le 40, \\ &x_j^{uu} \ge 0,\ j = 1, 2 \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} &FFP^{u\ell}(w^*): \quad \min\ 2(x_1^{u\ell})^2 + 2.5(x_2^{u\ell})^2, +3x_1^{u\ell} + 3.5x_2^{u\ell} \\ &\text{s. t.} \\ &5x_1^{u\ell} + 3x_2^{u\ell} \le 45, \\ &3x_1^{u\ell} + 5x_2^{u\ell} \le 30 \\ &x_j^{u\ell} \le x_j^{*uu},\ x_j^{u\ell} \ge 0,\ j = 1, 2 \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} &FFP^{m}(w^*): \quad \min\ 1.5(x_1^{m})^2 + 2(x_2^{m})^2, +2.5x_1^{m} + 6x_2^{m} \\ &\text{s. t.} \\ &2x_1^{m} + x_2^{m} \le 15, \\ &x_1^{m} + 3.5x_2^{m} \le 15, \\ &x_j^{m} \le x_j^{*u\ell},\ x_j^{m} \ge 0,\ j = 1, 2 \end{aligned}\right\} \quad (26)$$

$$\left.\begin{array}{l} FFP^{\ell\ell}(w^*):\quad \min\ (x_1^{\ell\ell})^2+1.5(x_2^{\ell\ell})^2+2x_1^{\ell\ell}+2.5x_2^{\ell\ell} \\ \text{s. t.} \\ 3x_1^{\ell\ell}+x_2^{\ell\ell}\le 20, \\ 0.5x_1^{\ell\ell}+x_2^{\ell\ell}\le 14, \\ x_j^{\ell\ell}\le x_j^{*m},\ x_j^{\ell\ell}\ge 0,\ j=1,2 \end{array}\right\} \quad (27)$$

$$\left.\begin{array}{l} FFP^{\ell u}(w^*):\quad \min 0.5(x_1^{\ell u})^2+(x_2^{\ell u})^2+x_1^{\ell u}+0.5x_2^{\ell u} \\ \text{s. t.} \\ x_1^{\ell u}+0.5x_2^{\ell u}\le 15, \\ 0.25x_1^{\ell u}+0.5x_2^{\ell u}\le 5, \\ x_j^{\ell u}\le x_j^{*\ell\ell},x_j^{\ell u}\ge 0\,j=1,2. \end{array}\right\} \quad (28)$$

The fuzzy rough optimal solutions are:

$$\tilde{x}_1^{*R}=[(6.364,7.046,8.409):(6.364,7.046,8.409)],$$

$$\tilde{x}_2^{*R}=[(0.909,0.909,0.909):(0.909,0.909,0.909)].$$

where the fuzzy rough efficient values range solutions for

$$\tilde{f}^{*R}=[(56.74,99.19,171.89):(27.89,99.19,217.37)]$$

And the fuzzy possibly optimal values range solution is

$$(FFP^{*\ell u}(w^*),FFP^{*m}(w^*),FFP^{*uu}(w^*))=(27.89,99.19,217.37)$$

The fuzzy surely optimal values range solutions are:

$$(FFP^{*\ell\ell}(w^*),FFP^{*m}(w^*),FFP^{*u\ell}(w^*))=(56.74,99.19,171.89).$$

In addition, the completely satisfactory solutions are:

$$(x_1^{*\ell\ell},x_1^{m},x_1^{*u\ell})=(6.364,7.046,8.409),\ (x_2^{*\ell\ell},x_2^{m},x_2^{*u\ell})=(0.909,0.909,0.909).$$

And the rather satisfactory solutions are:

$$(x_1^{\ell u},x_1^{m},x_1^{uu})=(6.364,7.046,8.409),\ (x_2^{\ell u},x_2^{m},x_2^{uu})=(0.909,0.909,0.909).$$

4. The α-Cut Approach for (FRMONLP) Problem

The multi-objective nonlinear programming problems with fuzzy rough coefficient, parameters and decision variables are defined as follows:

$$\left.\begin{array}{l} (FFRMONLP5):\min\ \tilde{F}_r^R(\tilde{x}^R)=\sum_{j=1}^{m}\tilde{C}_{rj}^R(\tilde{x}_j^R)^{\delta},r=1,\ 2,\ ...,\ k\ ;\ \delta\in Z^+ \\ \text{s.t.} \\ \sum_{j=1}^{m}\tilde{A}_{ij}^R(\tilde{x}_j^R)^{\delta}\le\tilde{B}_i^R\ , \\ \tilde{x}_j^R\ge 0,i\in I=\{1,\ 2,\ ...,\ n\};\ \ j\in J=\{1,\ 2,\ ...,\ m\} \end{array}\right\} \quad (29)$$

where $\tilde{C}_{rj}^R,\tilde{A}_{ij}^R,\tilde{B}_i^R,i\in I,j\in J,\delta_j\in\mathbf{Z}^+,r=1,2,...k,$ are fuzzy rough coefficient, and parameters, respectively? The problem 29 can be converting as:

$$\left.\begin{array}{l} (FFRMONLP6):\min\,[\tilde{f}_r^L(x):\tilde{f}_r^U(x)]=\sum_{j=1}^{m}[\tilde{C}_{rj}^L:\tilde{C}_{rj}^U]\otimes[(\tilde{x}_j^L)^{\delta}:(\tilde{x}_j^U)^{\delta}],r=1,\ 2,\ ...,\ k \\ \text{s.t.} \\ \sum_{j=1}^{m}[\tilde{A}_{ij}^L:\tilde{A}_{ij}^U]\otimes[(\tilde{x}_j^L)^{\delta}:(\tilde{x}_j^U)^{\delta}]\le[\tilde{B}_i^L:\tilde{B}_i^U] \\ \tilde{x}_j^L,\tilde{x}_j^U\ge 0,i\in I;\ \ j\in J,\delta\in\mathbf{Z}^+ \end{array}\right\} \quad (30)$$

where

$$\tilde{C}_{rj}^{L}=(c_{rj}^{\ell\ell},c_{rj}^{m},c_{rj}^{u\ell}),\tilde{C}_{rj}^{U}=(c_{rj}^{\ell u},c_{rj}^{m},c_{rj}^{uu}),$$
$$\tilde{A}^{L}=(a^{\ell\ell},a^{m},a^{u\ell}),\tilde{A}^{U}=(a^{\ell u},a^{m},a^{uu}),$$
$$\tilde{B}_{i}^{L}=(b_{i}^{\ell\ell},b_{i}^{m},b_{i}^{u\ell}),\tilde{B}^{U}=(b_{i}^{\ell u},b_{i}^{m},b_{i}^{uu}),$$

are triangular fuzzy numbers, and $\tilde{x}_{j}^{L}=((x_{j}^{\ell\ell})^{\delta},(x_{j}^{m})^{\delta},(x_{j}^{u\ell})^{\delta}),$ $\tilde{x}_{j}^{U}=((x_{j}^{\ell u})^{\delta},(x_{j}^{m})^{\delta},(x_{j}^{uu})^{\delta})$ are triangular fuzzy variables

$$\left.\begin{aligned}&(FFRMONLP_{\alpha}):\min\,[\tilde{f}_{r\alpha}^{L}(x):\tilde{f}_{r\alpha}^{U}(x)]=\sum_{j=1}^{m}[(C_{rj}^{L})_{\alpha}:(C_{rj}^{U})_{\alpha}]\otimes[(x_{j}^{L})_{\alpha}^{\delta}:(x_{j}^{U})_{\alpha}^{\delta}]\\&\text{s.t.}\\&\sum_{j=1}^{m}[[(A_{ij}^{L})_{\alpha}:(A_{ij}^{U})_{\alpha}]\otimes[(x_{j}^{L})_{\alpha}^{\delta}:(x_{j}^{U})_{\alpha}^{\delta}]\le[(B_{i}^{L})_{\alpha}:(B_{ij}^{U})_{\alpha}],\\&(x_{j}^{L})_{\alpha}^{\delta},(x_{j}^{U})_{\alpha}^{\delta}\ge 0,i\in I;\ j\in J,\delta\in\mathbf{Z}^{+},\alpha\in(0,1],r=1,2,\ldots,k.\end{aligned}\right\}\quad(31)$$

The α-cuts of coefficient, parameters and variables [see.Definition7] as:

$$(\tilde{C}_{rj}^{L})_{\alpha}=((c_{rj}^{\ell\ell})_{\alpha},(c_{rj}^{u\ell})_{\alpha}),(\tilde{C}_{rj}^{U})_{\alpha}=((c_{rj}^{\ell u})_{\alpha},(c_{rj}^{uu})_{\alpha}),$$
$$(\tilde{A}_{rj}^{L})_{\alpha}=((a_{rj}^{\ell\ell})_{\alpha},(a_{rj}^{u\ell})_{\alpha}),(\tilde{A}_{rj}^{U})_{\alpha}=((a_{rj}^{\ell u})_{\alpha},(a_{rj}^{uu})_{\alpha}),$$
$$(\tilde{B}_{i}^{L})_{\alpha}=((b_{i}^{\ell\ell})_{\alpha},(b_{i}^{u\ell})_{\alpha}),(\tilde{B}_{i}^{U})_{\alpha}=((b_{i}^{\ell u})_{\alpha},(b_{i}^{uu})_{\alpha}),$$
$$(\tilde{x}_{j}^{L})_{\alpha}=((x_{j}^{\ell\ell})_{\alpha}^{\delta},(x_{j}^{u\ell})_{\alpha}^{\delta}),(\tilde{x}_{j}^{U})_{\alpha}=((x_{j}^{\ell u})_{\alpha}^{\delta},(x_{j}^{uu})_{\alpha}^{\delta}).$$

Therefore

$$\left.\begin{aligned}&(FRMONLP_{\alpha}):\min f_{r\alpha}^{R}(x)=\sum_{j=1}^{m}[((c_{rj}^{\ell\ell})_{\alpha},(c_{rj}^{u\ell})_{\alpha}):((c_{rj}^{\ell u})_{\alpha},(c_{rj}^{uu})_{\alpha})]\otimes\\&\qquad[((x_{j}^{\ell\ell})_{\alpha}^{\delta},(x_{j}^{u\ell})_{\alpha}^{\delta}):((x_{j}^{\ell u})_{\alpha}^{\delta},(x_{j}^{uu})_{\alpha}^{\delta})],r=1,2,\ldots,k\\&\text{s.t.}\\&\sum_{j=1}^{m}[((a_{ij}^{\ell\ell})_{\alpha},(a_{ij}^{u\ell})_{\alpha}):((a_{ij}^{\ell u})_{\alpha},(a_{ij}^{uu})_{\alpha})]\otimes[((x_{j}^{\ell\ell})_{\alpha}^{\delta},(x_{j}^{u\ell})_{\alpha}^{\delta}):((x_{j}^{\ell u})_{\alpha}^{\delta},(x_{j}^{uu})_{\alpha}^{\delta})]]\le\\&[((b_{i}^{\ell\ell})_{\alpha},(b_{i}^{u\ell})_{\alpha}):((b_{i}^{\ell u})_{\alpha},(b_{i}^{uu})_{\alpha})],\\&(x_{j}^{\ell\ell})_{\alpha}^{\delta},(x_{j}^{u\ell})_{\alpha}^{\delta},(x_{j}^{\ell u})_{\alpha}^{\delta},(x_{j}^{uu})_{\alpha}^{\delta}\ge 0,i\in I;\ \delta\in\mathbf{Z}^{+},\alpha\in(0,1]\end{aligned}\right\}\quad(32)$$

where $f_{r\alpha}^{R}(x)=[(f_{r\alpha}^{\ell\ell}(x),f_{r\alpha}^{u\ell}(x):(f_{r\alpha}^{\ell u}(x),f_{r\alpha}^{uu}(x)]$.

Therefore, the $(FRMONLP_{\alpha})$ problem (32) decomposes to multi-objective nonlinear programming problem (MONLP) defines as follows:

$$\left.\begin{aligned}&MONLP_{\alpha}^{uu}:\min f_{r\alpha}^{uu}(x)=\sum_{j=1}^{m}(c_{rj}^{uu})_{\alpha}(x_{j}^{uu})_{\alpha}^{\delta},r=1,2,\ldots,k,\\&\text{s.t.}\\&\sum_{j=1}^{m}(a_{ij}^{uu})_{\alpha}(x_{j}^{uu})_{\alpha}^{\delta}\le(b_{i}^{uu})_{\alpha},\\&(x_{j}^{uu})_{\alpha}\ge 0,i\in I,\delta\in\mathbf{Z}^{+},\alpha\in(0,1]\end{aligned}\right\}\quad(33)$$

$$\left.\begin{aligned}&MONLP_{\alpha}^{u\ell}:\min f_{r\alpha}^{UL}(\mathrm{x})=\sum_{j=1}^{m}(c_{rj}^{u\ell}(\alpha))((\mathrm{x}_{j}^{u\ell}))^{\delta},r=1,\,2,\,\ldots,\,k\,,\\&\text{s.t.}\\&\sum_{j=1}^{m}(a_{ij}^{u\ell}(\alpha))\,(\mathrm{x}_{j}^{u\ell}(\alpha))^{\delta}\le b_{i}^{u\ell}(\alpha),\\&\mathrm{x}_{j}^{u\ell}(\alpha)\le \mathrm{x}^{*uu}_{j}\,(\alpha),\\&\mathrm{x}_{j}^{u\ell}(\alpha)\ge 0,i\in I,\delta\in\mathbf{Z}^{+},\alpha\in(0,1]\end{aligned}\right\}\qquad(34)$$

$$\left.\begin{aligned}&MONLP_{\alpha}^{\ell\ell}:\min f_{r\alpha}^{\ell\ell}(x)=\sum_{j=1}^{m}((c_{rj}^{\ell\ell})_{\alpha})((\mathrm{x}_{j}^{\ell\ell})_{\alpha})^{\delta},r=1,\,2,\,\ldots,\,k\\&\text{s.t.}\\&\sum_{j=1}^{m}((a_{ij}^{\ell\ell})_{\alpha})((\mathrm{x}_{j}^{\ell\ell})_{\alpha})^{\delta}\le(b_{i}^{\ell\ell})_{\alpha},\\&(\mathrm{x}_{j}^{\ell\ell})_{\alpha}\le(\mathrm{x}_{j}^{u\ell})_{\alpha}\;,\\&(\mathrm{x}_{j}^{\ell\ell})_{\alpha}\ge 0,i\in I\,;\;\;j\in J,\delta\in\mathbf{Z}^{+},\alpha\in(0,1]\end{aligned}\right\}\qquad(35)$$

$$\left.\begin{aligned}&MONLP_{\alpha}^{\ell u}:\min f_{r\alpha}^{\ell u}(\mathrm{x})=\sum_{j=1}^{m}((c_{rj}^{\ell u})_{\alpha})((\mathrm{x}_{j}^{\ell u})_{\alpha})^{\delta},r=1,\,2,\,\ldots,\,k\,,\\&\text{s.t.}\\&\sum_{j=1}^{m}((a_{ij}^{\ell u})_{\alpha})((\mathrm{x}_{j}^{\ell u})_{\alpha})^{\delta}\le(b_{i}^{\ell u})_{\alpha},\\&(\mathrm{x}_{j}^{\ell u})_{\alpha}\le(\mathrm{x}_{j}^{*\ell\ell})_{\alpha},(\mathrm{x}_{j}^{\ell u})_{\alpha}\ge 0,i\in I,\delta\in\mathbf{Z}^{+},\alpha\in(0,1]\end{aligned}\right\}\qquad(36)$$

Now, using weighting sum method to convert the MONLP problem to the nonlinear programming N.L.P. problem:

$$\left.\begin{aligned}&P_{\alpha}^{uu}(\mathrm{w}^{*}):\min\sum_{r=1}^{k}w_{r}\sum_{j=1}^{m}((c_{rj}^{uu})_{\alpha})((\mathrm{x}_{j}^{uu})_{\alpha})^{\delta}\\&\text{s.t.}\\&\sum_{j=1}^{m}((a_{ij}^{uu})_{\alpha})((\mathrm{x}_{j}^{uu})_{\alpha})^{\delta}\le(b_{i}^{uu})_{\alpha}\\&(\mathrm{x}_{j}^{uu})_{\alpha}\ge 0,i\in I\,;\;\;\delta\in\mathbf{Z}^{+},\alpha\in(0,1],w_{r}\in W\end{aligned}\right\}\qquad(37)$$

$$\left.\begin{aligned}&P_{\alpha}^{u\ell}(\mathrm{w}^{*}):\min\sum_{r=1}^{k}w_{r}\sum_{j=1}^{m}((c_{rj}^{u\ell})_{\alpha})((\mathrm{x}_{j}^{u\ell})_{\alpha})^{\delta}\\&\text{s.t.}\\&\sum_{j=1}^{m}((a_{ij}^{u\ell})_{\alpha})\,((\mathrm{x}_{j}^{u\ell})_{\alpha})^{\delta}\le(b_{i}^{u\ell})_{\alpha},\\&(\mathrm{x}_{j}^{u\ell})_{\alpha}\le(\mathrm{x}_{j}^{*uu})_{\alpha},\\&(\mathrm{x}_{j}^{u\ell})_{\alpha}\ge 0,i\in I\;,\delta\in\mathbf{Z}^{+},w_{r}\in W\;,\alpha\in(0,1]\end{aligned}\right\}\qquad(38)$$

$$\left.\begin{aligned}
&P_\alpha^{\ell\ell}(\mathrm{w}^*):\min\sum_{r=1}^{k} w_r\Big(\sum_{j=1}^{m}((c_{rj}^{\ell\ell})_\alpha)((\mathrm{x}_j^{\ell\ell})_\alpha)^\delta\\
&\quad \text{s.t.}\\
&\quad \sum_{j=1}^{m}(\,(a_{ij}^{\ell\ell})_\alpha)((\mathrm{x}_j^{\ell\ell})_\alpha)^\delta \le (b_i^{\ell\ell})_\alpha,\\
&\quad (\mathrm{x}_j^{\ell\ell})_\alpha \le (\mathrm{x}_j^{*\mathrm{u}\ell})_\alpha,\\
&\quad (\mathrm{x}_j^{\ell\ell})_\alpha \ge 0, i\in I\,; w_r\in W\ ,\delta\in\mathbf{Z}^+,\alpha\in(0,1]
\end{aligned}\right\}\quad (39)$$

$$\left.\begin{aligned}
&P_\alpha^{\ell u}(\mathrm{w}^*):\min\sum_{r=1}^{k} w_r\sum_{j=1}^{m}((c_{rj}^{\ell u})_\alpha)((\mathrm{x}_j^{\ell u})_\alpha)^\delta\\
&\quad \text{s.t.}\\
&\quad \sum_{j=1}^{m}((a_{ij}^{\ell u})_\alpha)((\mathrm{x}_j^{\ell u})_\alpha)^\delta \le (b_i^{\ell u})_\alpha\\
&\quad (\mathrm{x}_j^{\ell u})_\alpha \le (\mathrm{x}_j^{*\ell\ell})_\alpha,\\
&\quad (\mathrm{x}_j^{\ell u})_\alpha \ge 0, i\in I\,;w_r\in W\ ,\delta\in\mathbf{Z}^+,\alpha\in(0,1]
\end{aligned}\right\}\quad (40)$$

By theorem (3), the optimal solutions of $P_\alpha^{uu}(\mathrm{w}^*)$, $P_\alpha^{u\ell}(\mathrm{w}^*)$, $P_\alpha^{\ell\ell}(\mathrm{w}^*)$, and $P_\alpha^{\ell u}(\mathrm{w}^*)$ problems are the efficient solutions of $MONLP_\alpha^{uu}$, $MONLP_\alpha^{u\ell}$, $MONLP_\alpha^{\ell\ell}$ *and* $MONLP_\alpha^{\ell u}$ problems, respectively.

4.1. Algorithm Solution for FRMONLP Problem

1. Convert the problem to the form the $(FRMONLP_\alpha)$.
2. Use α-level cuts to deal with a fuzziness of fuzzy rough parameters decision variables as form $(FRMONLP_\alpha)$,
3. Use decompose technique for (FRMONLP$_\alpha$) problem (32) to get the MONLP problem $MONLP_\alpha^{uu}, MONLP_\alpha^{u\ell}, MONLP_\alpha^{\ell\ell}$ *and* $MONLP_\alpha^{\ell u}$
4. Use one of the secularization methods say the weights method to convert each problem $MONLP_\alpha^{uu}, MONLP_\alpha^{u\ell}, MONLP_\alpha^{\ell\ell}$ *and* $MONLP_\alpha^{\ell u}$ with a single objective in the form $\mathrm{P}_\alpha^{uu}(\mathrm{w}), \mathrm{P}_\alpha^{u\ell}(\mathrm{w}), \mathrm{P}_\alpha^{\ell\ell}(\mathrm{w})$ and $\mathrm{P}_\alpha^{u\ell}(\mathrm{w})$.

For $w=\mathrm{w}^*\in W$ Find the optimal solution of each nonlinear programming problems $\mathrm{P}_\alpha^{uu}(\mathrm{w}^*), \mathrm{P}_\alpha^{u\ell}(\mathrm{w}^*), \mathrm{P}_\alpha^{\ell\ell}(\mathrm{w}^*)$ and $\mathrm{P}_\alpha^{u\ell}(\mathrm{w}^*)$.

Example 2: Consider the following FRMONLP

$$\text{Min }(\tilde{c}_{11}^R(\tilde{x}_1^R)^2+\tilde{c}_{12}^R(\tilde{x}_2^R)^2,\ \tilde{c}_{21}^R\tilde{x}_1^R+\tilde{c}_{22}^R\tilde{x}_2^R)$$

$$\begin{aligned}
&\text{s.t.}\\
&\tilde{a}_{11}^R\tilde{x}_1^R+\tilde{a}_{12}^R\tilde{x}_2^R\le\tilde{b}_1^R,\\
&\tilde{a}_{21}^R\tilde{x}_1^R+\tilde{a}_{22}^R\tilde{x}_2^R\le\tilde{b}_2^R,\\
&\tilde{x}_1^R,\ \tilde{x}_2^R\ge 0
\end{aligned}$$

$$\tilde{c}_{11} = [(2,3,4):(1,3,5)],\ \tilde{c}_{12} = (3,4,5):(2,4,7),\ \tilde{c}_{21} = (4,5,6):(2,5,8),$$
$$\tilde{c}_{22} = (5,6,7):(1,6,9),\ \tilde{a}_{11} = (3,4,5):(1,4,6),\ \tilde{a}_{12} = (1,2,3):(0.5,2,5),$$
$$\tilde{a}_{21} = (0.5,1,3):(0.25,1,4),\ \tilde{a}_{22} = (1,3.5,5):(0.5,3.5,7),$$
$$\tilde{b}_1 = (20,30,45):(15,30,55),\ \tilde{b}_2 = (14,15,30):(5,15,40).$$

The problem is writing as:

$$\min \begin{cases} [(2,3,4):(1,3,5)] \otimes [((x_1^{\ell\ell})^2, (x_1^{m})^2, (x_1^{u\ell})^2) : ((x_1^{\ell u})^2, (x_1^{m})^2, (x_1^{uu})^2)] + \\ [(3,4,5):(2,4,7)] \otimes [((x_2^{\ell\ell})^2, (x_2^{m})^2, (x_2^{u\ell})^2) : ((x_2^{\ell u})^2, (x_2^{m})^2, (x_2^{uu})^2)], \\ [(4,5,6):(2,5,8)] \otimes [(x_1^{\ell\ell}, x_1^{m}, x_1^{u\ell}) : (x_1^{\ell u}, x_1^{m}, x_1^{uu})] + \\ [(5,6,7):(1,6,9)] \otimes [(x_2^{\ell\ell}, x_2^{m}, x_2^{u\ell}) : (x_2^{\ell u}, x_2^{m}, x_2^{uu})] \end{cases}$$

$s.t.$

$$\begin{cases} [(3,4,5):(1,4,6)] \otimes [(x_1^{\ell\ell}, x_1^{m}, x_1^{u\ell}) : (x_1^{\ell u}, x_1^{m}, x_1^{uu})] + \\ [(1,2,3):(0.5,2,5)] \otimes [(x_2^{\ell\ell}, x_2^{m}, x_2^{u\ell}) : (x_2^{\ell u}, x_2^{m}, x_2^{uu})] \le [(20,30,45):(15,30,55)], \end{cases}$$

$$\begin{cases} [(0.5,1,3):(0.25,1,4)] \otimes [(x_1^{\ell\ell}, x_1^{m}, x_1^{u\ell}) : (x_1^{\ell u}, x_1^{m}, x_1^{uu})] + \\ [(1,3.5,5):(0.5,3.5,7)] \otimes [(x_2^{\ell\ell}, x_2^{m}, x_2^{u\ell}) : (x_2^{\ell u}, x_2^{m}, x_2^{uu})] + \le [(14,15,30):(5,15,40)]. \end{cases}$$

For $\alpha = 0.5$ we get:

$$\min \begin{cases} [(1.5,3.5):(0,4)] \otimes [((x_1^{\ell\ell})_\alpha^2, (x_1^{u\ell})_\alpha^2) : ((x_1^{\ell u})_\alpha^2, (x_1^{uu})_\alpha^2)] + \\ [(2.5,4.5):(0.5,5.5)] \otimes [((x_2^{\ell\ell})_\alpha^2, (x_2^{u\ell})_\alpha^2) : ((x_2^{\ell u})_\alpha^2, (x_2^{uu})_\alpha^2)], \\ [(3.5,5.5):(0.5,6.5)] \otimes [(x_1^{\ell\ell})_\alpha, (x_1^{u\ell})_\alpha : ((x_1^{\ell u})_\alpha, (x_1^{uu})_\alpha)] + \\ [(4.5,6.5):(-0.5,7.5)] \otimes [(x_2^{\ell\ell})_\alpha, (x_2^{u\ell})_\alpha : ((x_2^{\ell u})_\alpha, (x_2^{uu})_\alpha)] \end{cases}$$

s.t.

$$\begin{cases} [(2.5,4.5):(0,5)] \otimes [(x_1^{\ell\ell})_\alpha, (x_1^{u\ell})_\alpha : ((x_1^{\ell u})_\alpha, (x_1^{uu})_\alpha)] + \\ [(0.5,2.5):(-1,3.5)] \otimes [(x_2^{\ell\ell})_\alpha, (x_2^{u\ell})_\alpha : ((x_2^{\ell u})_\alpha, (x_2^{uu})_\alpha)] \le [(13,37.5):(2.5,42.5)], \\ [(-0.5,2):(-1.25,2.5)] \otimes [(x_1^{\ell\ell})_\alpha, (x_1^{u\ell})_\alpha : ((x_1^{\ell u})_\alpha, (x_1^{uu})_\alpha)] + \\ [(0.25,4.25):(-1.25,5.25)] \otimes [(x_2^{\ell\ell})_\alpha, (x_2^{u\ell})_\alpha : ((x_2^{\ell u})_\alpha, (x_2^{uu})_\alpha)] \le [(6.5,22.5):(-7.5,27.5)] \end{cases}$$

The above problem can be written as the following problems:

$$MONLP_\alpha^{uu} : \min\ (4(x_1^{uu})_\alpha^2 + 5.5(x_2^{uu})_\alpha^2, 6.5(x_1^{uu})_\alpha + 7.5(x_2^{uu})_\alpha)$$

s.t.

$$5(x_1^{uu})_\alpha + 3.5(x_2^{uu})_\alpha \le 42.5$$
$$2.5(x_1^{uu})_\alpha + 5.25(x_2^{uu})_\alpha \le 27.5,$$
$$(x_1^{uu})_\alpha, (x_2^{uu})_\alpha \ge 0.$$

$$MONLP_\alpha^{u\ell} : \min\ (3.5(x_1^{u\ell})_\alpha^2 + 4.5(x_2^{u\ell})_\alpha^2, 5.5(x_1^{u\ell})_\alpha + 6.5(x_2^{u\ell})_\alpha)$$

s.t.

$$4.5(x_1^{u\ell})_\alpha + 2.5(x_2^{u\ell})_\alpha \le 37.5,$$
$$2(x_1^{u\ell})_\alpha + 4.25(x_2^{u\ell})_\alpha \le 22.5,$$
$$(x_1^{u\ell})_\alpha, (x_2^{u\ell})_\alpha \ge 0$$

$$MONLP_\alpha^{\ell\ell} : \min\ (1.5(x_1^{\ell\ell})_\alpha^2 + 2.5(x_2^{\ell\ell})_\alpha^2, 3.5(x_1^{\ell\ell})_\alpha + 4.5(x_2^{\ell\ell})_\alpha)$$

s.t.

$$2.5(x_1^{\ell\ell})_\alpha + 0.5(x_2^{\ell\ell})_\alpha \le 13,$$
$$-0.5(x_1^{\ell\ell})_\alpha + 0.25(x_2^{\ell\ell})_\alpha \le 6.5,$$
$$(x_1^{\ell\ell})_\alpha, (x_2^{\ell\ell})_\alpha \ge 0$$

$$MONLP_{\alpha}^{\ell u}: \min\ (0.5(x_1^{\ell u})_{\alpha}^2, 0.5(x_1^{\ell u})_{\alpha} - 0.5(x_2^{\ell u})_{\alpha}$$
$$\text{s.t.}$$
$$-(x_1^{\ell u})_{\alpha} \le 2.5$$
$$-1.25(x_1^{\ell u})_{\alpha} - 1.25(x_2^{\ell u})_{\alpha} \le -7.5.$$
$$(x_1^{\ell u})_{\alpha}, (x_2^{\ell u})_{\alpha} \ge 0$$

For $w_1 = w_2 = 0.5$ we getting the corresponded weighting nonlinear programm-ing problems $P_{\alpha}^{uu}(w^*)$, $P_{\alpha}^{u\ell}(w^*)$, $P_{\alpha}^{\ell\ell}(w^*)$ and $P_{\alpha}^{\ell u}(w^*)$ for $MONLP_{\alpha}^{uu}$, $MONLP_{\alpha}^{u\ell}$ $MONLP_{\alpha}^{\ell\ell}$ and $MONLP_{\alpha}^{\ell u}$, respectively can be described as follows:

$$P_{\alpha}^{uu}(w^*): \min\ 2(x_1^{uu})_{\alpha}^2 + 2.75(x_2^{uu})_{\alpha}^2 + 3.25(x_1^{uu})_{\alpha} + 3.75(x_2^{uu})_{\alpha})$$
$$\text{s.t.}$$
$$5(x_1^{uu})_{\alpha} + 3.5(x_2^{uu})_{\alpha} \le 42.5$$
$$2.5(x_1^{uu})_{\alpha} + 5.25(x_2^{uu})_{\alpha} \le 27.5,$$
$$(x_1^{uu})_{\alpha}, (x_2^{uu})_{\alpha} \ge 0.$$

Then, the efficient solution of $MONLP_{\alpha}^{uu}$ is $(x_1^{*uu})_{\alpha} = 7.04, (x_2^{*uu})_{\alpha} = 2.08$ and the minimum value of $f_{\alpha}^{uu}(x^{*uu}) = 141.81$.

$$P_{\alpha}^{u\ell}(w^*): \min\ 1.75(x_1^{u\ell})_{\alpha}^2 + 2.25(x_2^{u\ell})_{\alpha}^2 + 2.75(x_1^{u\ell})_{\alpha} + 3.25\ (x_2^{u\ell})_{\alpha}$$
$$\text{s.t.}$$
$$4.5(x_1^{u\ell})_{\alpha} + 2.5(x_2^{u\ell})_{\alpha} \le 37.5,$$
$$2(x_1^{u\ell})_{\alpha} + 4.25(x_2^{u\ell})_{\alpha} \le 22.5,$$
$$(x_1^{*u\ell})_{\alpha} \le 7.042,$$
$$(x_2^{*u\ell})_{\alpha} \le 2.083\ ,$$
$$(x_1^{u\ell})_{\alpha}, (x_2^{u\ell})_{\alpha} \ge 0$$

Then, the efficient solution of $MONLP_{\alpha}^{u\ell}$ is $(x_1^{*u\ell})_{\alpha} = 6.82, (x_2^{*u\ell})_{\alpha} = 2.08$ and the minimum value of $f_{\alpha}^{*u\ell}(x) = 116.76$

$$P_{\alpha}^{\ell\ell}(w^*): \min\ 0.75(x_1^{\ell\ell})_{\alpha}^2 + 1.25(x_2^{\ell\ell})_{\alpha}^2 + 1.75(x_1^{\ell\ell})_{\alpha} + 2.25(x_2^{\ell\ell})_{\alpha}$$
$$\text{s.t.}$$
$$2.5(x_1^{\ell\ell})_{\alpha} + 0.5(x_2^{\ell\ell})_{\alpha} \le 13,$$
$$-0.5(x_1^{\ell\ell})_{\alpha} + 0.25(x_2^{\ell\ell})_{\alpha} \le 6.5,$$
$$(x_1^{*\ell\ell})_{\alpha} \le 6.82,$$
$$(x_2^{*\ell\ell})_{\alpha} \le 2.08,$$
$$(x_1^{\ell\ell})_{\alpha}, (x_2^{\ell\ell})_{\alpha} \ge 0$$

So, the efficient solution of $MONLP_{\alpha}^{\ell\ell}$ is $(x_1^{*\ell\ell})_{\alpha} = 4.78, (x_2^{*\ell\ell})_{\alpha} = 2.08$ and the minimum value of $f_{\alpha}^{\ell\ell}(x^{*\ell\ell})_{\alpha} = 35.59$ and

$$P_{\alpha}^{\ell u}(w^*): \min\ 0.25(x_1^{\ell u})_{\alpha}^2 + 0.25(x_1^{\ell u})_{\alpha} - 0.25(x_2^{\ell u})_{\alpha}$$
$$\text{s.t.}$$
$$-1.25(x_1^{\ell u})_{\alpha} - 1.25(x_2^{\ell u})_{\alpha} \le -7.5.$$
$$(x_1^{*\ell u})_{\alpha} \le 4.78,$$
$$(x_2^{*\ell u})_{\alpha} \le 2.08,$$
$$(x_1^{\ell u})_{\alpha}, (x_1^{\ell u})_{\alpha} \ge 0.$$

Then, the α-cut fuzzy rough optimal solutions are:

$$(x_1^{*R})_\alpha = [(4.78, 6.82) : (4.78, 7.04)]$$

$$(x_2^{*R})_\alpha = [(2.08, 2.08) : (1.22, 2.08)]$$

Where the α-cut fuzzy rough efficient values range solutions for

$$\tilde{f}_\alpha^{*R} = [(35.59, 116.76) : (1.26, 141.81)]$$

And the α-cut fuzzy possibly optimal values range solution is

$$(P_\alpha^{\ell u}(w^*), P_\alpha^{uu}(w^*)) = (1.26, 141.81)$$

The α-cut fuzzy surely optimal values range solutions are:

$$(P_\alpha^{\ell\ell}(w^*), P_\alpha^{u\ell}(w^*)) = (35.59, 116.76).$$

In addition, the α-cut completely satisfactory solutions are:

$$((x_1^{*\ell\ell})_\alpha, (x_1^{*u\ell})_\alpha) = (4.78, 6.82),\ ((x_2^{*\ell\ell})_\alpha,, (x_2^{*u\ell})_\alpha) = (2.08, 2.08).$$

And the α-cut rather satisfactory solutions are:

$$((x_1^{*\ell u})_\alpha, (x_1^{*uu})_\alpha) = (4.78, 7.04),\ ((x_2^{*\ell u})_\alpha, (x_2^{*uu})_\alpha) = (1.22, 2.08).$$

Acknowledgments

The authors would like to express their gratitude to the anonymous reviewers for the comments and suggestions which greatly helped them to improve the paper.

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