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Half Step Numerical Method for Solution of Second Order Initial Value Problems

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Abstract

This paper presents a half step numerical method for solving directly general second order initial value problems. The scheme is developed via collocation and interpolation technique invoked on power series polynomial. The proposed method is consistent, zero stable, order four and three. This method can estimate the approximate solution at both step and off step points simultaneously by using variable step size. Numerical results are given to show the efficiency of the proposed scheme over some existing schemes of same and higher order.

Keywords: Power series polynomial; Block method; Collocation; Interpolation; Zero stability; Consistency.

1. Introduction

Numerical Solution of second order Initial Value Problem (IVP) of the form

$$y'' = f\left(x, y(x), y'^{(x)}\right), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{1}$$

have been treated in a variety of ways including the use of polynomial and non- polynomial via collocation and interpolation approach, where$a \le x \le b$, $a = x_0 < x_1 < x_1 < x_2 < \cdots..x_{N-1}$, $N = \frac{b-a}{h}$,$N = ,0\,,1,2, \ldots.N-1$ and $h = x_{n-1} - x_n$is called the step length, where $y_0$is the solution at x_0 and $x_0$is the initial point, **f** is a continuous function within the interval of integration, the condition on the function f are such that existence and uniqueness of solution is guaranteed(Wend [1], and prime indicates differentiation with respect to x, while y(x) is the unknown function to be determined.

Adeniran and Ogundare [2], propose a power series polynomial induced one step hybrid numerical scheme with two off grid points for solving directly second order initial value problems, the scheme can estimate the approximate solution at both step and off-step points simultaneously by using variable step size.

Ojo and Okoro [3], use a Bernstein polynomial to develop one step hybrid scheme with one off-grid point via collocation and interpolation techniques for the direct solution of general second order ordinary differential equations. Adeniran and Olanegan [4], develop a continuous two-step method using trigonometric function as basis function with two discrete methods as end products, the performance of the method is demonstrated on some numerical examples to show accuracy and efficiency advantages.

Adeniran and Longe [5], in their paper construct a one-step hybrid numerical scheme with one off grid points for solving directly the general second order initial value problems. The scheme is developed using collocation and interpolation technique invoked on Lucas polynomial. Numerical results of the scheme show some efficiency over some existing schemes of same and higher order. This paper is concerned with the development of half step method using power series polynomial for numerical solution of second order IVP.

2. Development of the Method

Considering a power series approximate solution in the form:

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j, \tag{2}$$

where c and i are number of distinct collocation and interpolation points respectively. Substituting the second derivative of (2) into (1) gives:

$$f\left(x, y(x), y'^{(x)}\right) = \sum_{j=0}^{c+i-1} j(j-1) a_j x^{j-2}, \tag{3}$$

we consider grid point of step length of half, collocating (3) at points x=x_n , $x_{n+\frac{1}{4}}$, and$x_{n+\frac{1}{2}}$, interpolating (2) at x=x_n and $x_{n+\frac{1}{4}}$ give a system of five equations which are solved using Gaussian elimination method to obtained the parameters a_j's, j=0, 1,…,4. The parameters a_j's obtained are then substituted back into equation (2) to give a continuous half step method of the form

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{4}} y_{n+\frac{1}{4}} + h^2 \left(\beta_0 f_n + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}}\right), \tag{4}$$

where α and β are continuous coefficients. The continuous method (4) is used to generate the main method, that is, we evaluate at x = $x_{n+\frac{1}{2}}$ to obtain

$$y_{n+\frac{1}{2}} = -y_n + 2y_{n+\frac{1}{4}} + h^2\left(\frac{1}{192} f_n + \frac{1}{192} f_{n+\frac{1}{4}} + \frac{5}{96} f_{n+\frac{1}{2}}\right) \tag{5}$$

In order to incorporate the initial condition at (1) into the derived scheme (5), we differentiate (4) with respect to x and evaluate at point at x=x_n, x = $x_{n+\frac{1}{4}}$and x=$x_{n+\frac{1}{2}}$ to obtain:

$$\mathrm{h}y'_n = -4y_n + 4y_{n+\frac{1}{4}} - h^2\left(\frac{7}{96} f_n + \frac{1}{16} f_{n+\frac{1}{4}} - \frac{1}{96} f_{n+\frac{1}{2}}\right) \tag{6}$$

$$\mathrm{h}y'_{n+\frac{1}{4}} = -4y_n + 4y_{n+\frac{1}{4}} + \left(\frac{1}{32} f_n - \frac{5}{48} f_{n+\frac{1}{4}} - \frac{1}{96} f_{n+\frac{1}{2}}\right) \tag{7}$$

$$\mathrm{h}y'_{n+\frac{1}{2}} = -4y_n + 4y_{n+\frac{1}{4}} \mp \left(\frac{1}{96} f_n + \frac{13}{48} f_{n+\frac{1}{4}} + \frac{3}{48} f_{n+\frac{1}{2}}\right) \tag{8}$$

Combining the schemes derived in equation (5-8). The block method is employed

to simultaneously obtain value for $y_{n+\frac{1}{4}}$, $y_{n+\frac{1}{2}}$, $y'_{n+\frac{1}{4}}$ and $y'_{n+\frac{1}{2}}$, needed to implement equation (1), thus we have

$$\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4} h y'_n + h^2\left(\frac{7}{384} f_n + \frac{1}{64} f_{n+\frac{1}{4}} - \frac{1}{384} f_{n+\frac{1}{2}}\right)\\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2} h y'_n + h^2\left(\frac{1}{24} f_n + \frac{1}{12} f_{n+\frac{1}{4}} - 0 f_{n+\frac{1}{2}}\right)\\
y'_{n+\frac{1}{4}} &= y_n + h y'_n + h^2\left(\frac{5}{48h} f_n + \frac{1}{6h} f_{n+\frac{1}{4}} - \frac{1}{48h} f_{n+\frac{1}{2}}\right)\\
y'_{n+\frac{1}{2}} &= y_n + h y'_n + h^2\left(\frac{1}{12h} f_n + \frac{1}{3} f_{n+\frac{1}{4}} - \frac{1}{13h} f_{n+\frac{1}{2}}\right)
\end{aligned}$$

3. Analysis of the Method

We analyze the derived method which includes the order and error constant, consistency zero stability, and convergence of the method.

3.1. Order and Error Constant

We adopted the method proposed by Fatunla [6] and Lambert [7] to obtain the order of our method as $(3, 3, 3, 4)^T$ and error constant as $\left(\frac{1}{46080}, \frac{1}{23040}, \frac{1}{6144}, -\frac{1}{92160}\right)^T$

3.2. Consistency

According to Gurjinder, *et al.* [8], A linear multistep method is said to be

Consistent, if it has an order of convergence, say $p \geq 1$, thus, our derived methods are consistent, since all are of order three and four respectively.

3.3. Zero Stability

To obtain the zero stability of the method, we consider the following conditions:

i. A block method of linear multistep method is said to be stable if as $h \to 0$, the roots $r_j,\ j = 0(1)k$ of the first characteristics polynomial $\rho(R) = 0$, that is $\rho(R) = det\left[\sum A^{(i)} A^{k-1}\right] = 0$, and for those roots with $R \leq 1$, must have multiplicity equal to unity.(see Fatunla [6] for details.)

ii. If the block method be an $R \times R$ marix, then, it is zero stable as $h^{\mu} \to 0$, $|RA^0 - A^i| = 0$. For those roots with $|R_j| \leq 1$, the multiplicity must not exceed the order of the differential equation.

For the block method, as $h \to 0$

$$\begin{pmatrix} \lambda & -1 \\ 0 & \lambda - 1 \end{pmatrix} \tag{9}$$

Taking the determinant of equation (9) and setting it equal to zero, we have

$$\lambda(\lambda - 1) = 0 \tag{10}$$

Solving equation (10), gives $\lambda = 0,\ 1$.

3.4. Convergence

Theorem 3.1: *The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.*

Thus our block method is convergent since it is zero stable and consistent.

4. Numerical Implementation of the Scheme

The effectiveness and validity of our newly derived method was tested by applying it to some second order differential equations. All calculations and programs are carried out with the aid of Maple 2016 software.

Example1

We consider the non-linear initial value problem:

$$y'' - x(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = \frac{1}{2}$$

Whose exact solution is given by: $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$

Table-1. Showing the exact solutions and the computed results from the proposed methods for Example 1, h=0.1

X	Exact	Numerical	Error
0.1	1.0500417292784912682	1.0500417286203137097	$3.2627 * 10^{-10}$
0.2	1.1003353477310755806	1.1003353463682225667	$1.3629 * 10^{-9}$
0.3	1.1511404359364668053	1.1511404337684041921	$2.1681 * 10^{-9}$
0.4	1.2027325540540821910	1.2027325509092220570	$3.1449 * 10^{-9}$
0.5	1.2554128118829953416	1.2554128074875801923	$4.3954 * 10^{-9}$
0.6	1.3095196042031117155	1.3095195981268332349	$6.0763 * 10^{-9}$
0.7	1.3654437542713961691	1.3654437458319520626	$8.4394 * 10^{-9}$
0.8	1.4236489301936018068	1.4236489182848502431	$1.1909 * 10^{-9}$
0.9	1.4847002785940517416	1.4847002613646092135	$1.7229 * 10^{-8}$
0.10	1.5493061443340548457	1.5493061185565787340	$2.5777 * 10^{-9}$

Table-2. Comparison of error for Example 1 with existing literature

x	[5]	NPM
0.1	$1.051251 * 10^{-8}$	$3.2627 * 10^{-10}$
0.2	$2.176690 * 10^{-8}$	$1.3629 * 10^{-9}$
0.3	$3.462528 * 10^{-8}$	$2.1681 * 10^{-9}$
0.4	$5.022104 * 10^{-8}$	$3.1449 * 10^{-9}$
0.5	$7.018369 * 10^{-8}$	$4.3954 * 10^{-9}$
0.6	$9.700952 * 10^{-8}$	$6.0763 * 10^{-9}$
0.7	$1.3471588 * 10^{-7}$	$8.4394 * 10^{-9}$
0.8	$1.9005788 * 10^{-7}$	$1.1909 * 10^{-8}$
0.9	$2.749090 * 10^{-7}$	$1.7229 * 10^{-8}$
1.0	$4.1118559 * 10^{-7}$	$2.5777 * 10^{-8}$

Example 2

Considering a moderately stiff problem

$$y'' = y' \quad y(0) = 0, \quad y'(0) = -1$$

Whose exact solution is $y(x) = 1 - \exp(x)$

Table-3. Showing the exact solutions, computed results and error from the Example 2. h = 0:1.

x	Exact	Numerical	Error
0.1	-0.1051709180756476248	-0.1051709171209395283 2	$7.0148 * 10^{-10}$
0.2	-0.2214027784259734602 8	-0.2214027560671973128 4	$1.7915 * 10^{-09}$
0.3	-0.3498588075760031040$	-0.3498588041331602069 9	$3.0859 * 10^{-09}$
0.4	-0.4918246976412703178	-0.4918246926050317633 4	$4.6154 * 10^{-09}$
0.5	-0.6487212707001281468	-0.6487212637906759094 6	$6.4152 * 10^{-09}$
0.6	-0.8221188003905089749	-0.8221187912867920170 9	$8.5253 * 10^{-09}$
0.7	-1.0137527074704765216	-1.0137526958046328118 0	$1.0991 * 10^{-08}$
0.8	-1.2255409284924676046	-1.2255409138435289269 0	$1.3864 * 10^{-08}$
0.9	-1.4596031111569496638	-1.4596030930437400593 0	$1.7202 * 10^{-08}$
1.0	-1.7182818284590452354	-1.7182818063321730913 0	$2.1072 * 10^{-08}$

Table-4. Comparison of error for proposed scheme with existing literature for Example 2

x	Error in Anake, *et al.* [9]	Error in Yahaya and Badmus [10]	Error in Kayode and Adeyeye [11]	Error in Adeniran and Ogundare [2]	Error in NP
0.1	0.84 $*10^{-05}$	$0.87*10^{-04}$	$0.817*10^{-06}$	$2.22*10^{-08}$	$7.0148*10^{-10}$
0.2	$0.53*10^{-05}$	$0.32*10^{-03}$	$0.31*10^{-0}$	$1.25*10^{-07}$	$1.7915*10^{-09}$
0.3	$0.62*10^{-05}$	$0.22*10^{-02}$	0.6510^{-05}	3.25010^{-07}	3.085910^{-09}
0.4	$0.16*10^{-05}$	0.49 $*10^{-02}$	$0.6610^{-0\,5}$	6.42410^{-07}	4.615410^{-09}
0.5	$0.10*10^{-04}$	0.9110^{-02}	$0.11*10^{-0\,5}$	$1.099*10^{-06}$	$6.4152*10^{-09}$
0.6	$0.29*10^{-04}$	$0.14*10^{-01}$	$1.80*10^{-04}$	$1.7213*10^{-06}$	$8.5253*10^{-09}$
0.7	$0.59*10^{-04}$	$0.21*10^{-01}$	$0.26*10^{-04}$	$2.538*10^{-06}$	$1.0991*10^{-08}$
0.8	$0.10*10^{-03}$	$0.29*10^{-01}$	$0.37*10^{-04}$	$3.583*10^{-06}$	$1.3864*10^{-08}$
0.9	$0.15*10^{-03}$	$0.4*10^{-01}$	$0.51*10^{-04}$	$4.896*10^{-06}$	$1.7202*10^{-08}$
1.0	$0.23*10^{-03}$	$0.52*10^{-01}$	$0.67*10^{-04}$	$6.522*10^{-06}$	$2.1072*10^{-08}$

Example 3

We consider a highly stiff problem

$$y'' + 1001y' + 1000y = 0, \qquad y(0) = 1, \;\; y'(0) = 1$$

whose exact solution is $y(x) = exp(x)$

Table-5. Numerical result for Example 3 with h = 0:05

x	Exact	Numerical	Error
0.1	0.90483741803595957316	0.90483741802371542157	$3.5092*10^{-13}$
0.2	0.81873075307798185867	0.81873075305510954668	$1.0314*10^{-11}$
0.3	0.74081822068171786607	0.74081822065024362953	$1.9954*10^{-11}$
0.4	0.67032004603563930074	0.67032004599739126976	$2.7903*10^{-11}$
0.5	0.60653065971263342360	0.60653065966918341872	$3.4194*10^{-11}$
0.6	0.54881163609402643263	0.54881163604671052571	$3.9041*10^{-11}$
0.7	0.49658530379140951470	0.49658530374135690883	$4.2656*10^{-11}$
0.8	0.44932896411722159143	0.44932896406538162995	$4.5229*10^{-11}$
0.9	0.40656965974059911188	0.40656965968776524454	$4.6927*10^{-11}$
1.0	0.36787944117144232160	0.36787944111827317188	$4.7892*10^{-11}$

Table-6. Comparison of error for Example 3 with existing literature

x	Error in Abhulimen and Okunuga [12]	Error in Adeniran and Ogundare [2]	Error in NPM
0.1	-	$2.05*10^{-11}$	$3.5092*10^{-13}$
0.2	$0.26*10^{-05}$	$4.39*10^{-11}$	$1.0314*10^{-11}$
0.3	$0.40*10^{-05}$	$6.55*10^{-11}$	$1.9954*10^{-11}$
0.4	$0.53*10^{-05}$	$8.38*10^{-11}$	$2.7903*10^{-11}$
0.5	$0.66*10^{-05}$	$9.86*10^{-11}$	$3.4194*10^{-11}$
0.6	$0.79*10^{-05}$	$1.10*10^{-10}$	$3.9041*10^{-11}$
0.7	$0.93*10^{-05}$	$1.19*10^{-10}$	$4.2656*10^{-11}$
0.8	$0.11*10^{-04}$	$1.24*10^{-10}$	$4.5229*10^{-11}$
0.9	$0.12*10^{-04}$	$1.28*10^{-10}$	$4.6927*10^{-11}$
1.0	$0.13*10^{-04}$	$1.30*10^{-10}$	$4.7892*10^{-11}$

Example 4

We consider a highly oscillatory test problem:

$$y'' + \lambda^2 y = 0, \;\; y(0) = 1, \;\; y'(0) = 2 \qquad (12)$$

for $\lambda = 2$, the exact solution is known to be : $y(x) = cos(2x) + sin(2x)$.

Table-7. Numerical result for Example 4 with h=0.01

x	Exact	Numerical	Error
0.01	1.0197986733599108578	1.0197986733259586886	$4.3881*10^{-11}$
0.02	1.0391894408476120998	1.0391894407785766525	$7.9019*10^{-11}$
0.03	1.0581645464146487647	1.0581645463094435496	$1.1525*10^{-10}$
0.04	1.0767164002717920723	1.0767164001293760990	$1.5252*10^{-10}$
0.05	1.0948375819248539184	1.0948375817442325458	$1.9079*10^{-10}$
0.06	1.1125208431427856122	1.1125208429230113669	$2.3002*10^{-10}$
0.07	1.1297591108568736536	1.1297591105970470164	$2.7014*10^{-10}$
0.08	1.1465454899898729124	1.1465454896891430734	$3.1112*10^{-10}$
0.09	1.1628732662139455929	1.1628732658715111736	$3.5291*10^{-10}$
0.10	1.1787359086363028466	1.1787359082514125890	$3.9545*10^{-10}$

Table-8. Comparison of error for Example 4 with existing literature

X	[12]	[2]	NPM
0.01	-	0.00	4.3881 *10^{-11}
0.02	0:26*10^{-05}	0.00	7.9019*10^{-11}
0.03	0.40 *10^{-05}	0.00	1.1525*10^{-10}
0.04	0.53 *10^{-05}	0.00	1.5252*10^{-10}
0.05	0.66*10^{-05}	0.00	1.9079 *10^{-10}
0.06	0.79 *10^{-05}	0.00	2:3002 *10^{-10}
0.07	0.93*10^{-05}	0.00	2:7014 *10^{-10}
0.08	0:11*10^{-04}	0.00	3:1112 *10^{-10}
0.09	0:12 *10^{-04}	0:00	3:5291*10^{-10}
0:10	0:13 *10^{-04}	0:00	3:9545 *10^{-10}

5. Conclusion

The half method are derived via multistep collocation technique and implemented in block form. The method are of order 3 and 4, consistent, zero stable and convergent. The method is reliable, efficient display better accuracy than most of the existing ones in literature.

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