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A Simplified Novel Technique for Solving Linear Programming Problems with Triangular Fuzzy Variables via Interval Linear Programming Problems

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Abstract

This study proposes a novel technique for solving Linear Programming Problems with triangular fuzzy variables. A modified version of the well-known simplex method and Existing Method for Solving Interval Linear Programming problems are used for solving linear programming problems with triangular fuzzy variables. Furthermore, for illustration, some numerical examples and one real problem are used to demonstrate the correctness and usefulness of the proposed method. The proposed algorithm is flexible, easy and reasonable.

Keywords: Interval numbers; Triangular fuzzy variables; Linear programming.

1. Introduction

In the real-world environment, there are many problems which are concern to the linear programming models and sometimes it is necessary to formulate these models with parameters of uncertainty. Many numbers from these problems are linear programming problems with fuzzy variables. Interval analysis is an efficient and reliable tool that allows us to handle such problems effectively.

Linear programming problems with interval coefficients have been studied by several authors, such as Atanu and Tapan [1], Atanu, *et al.* [2], Bitran [3], Chanas and Kuchta [4], Nakahara, *et al.* [5], Steuer [6] and Tong Shaocheng [7]. Numerous methods for comparison of interval numbers can be found as in Atanu and Tapan [1], Atanu, *et al.* [2], Ganesan and Veeramani [8] Ganesan [9], etc. By taking maximum value range and minimum value range inequalities as constraint conditions, Tong Shaocheng [7] reduced the interval linear programming problem in to two classical linear programming problems and obtained an optimal interval solution to it. Ramesh and Ganesan [10], proposed a method for solving interval number linear programming problems without converting them to classical linear programming problems.

Bellman and Zadeh [11], introduced for the first time the concept of a fuzzy decision process as an intersection of the fuzzy objective function and resource constraints. From this idea, the authors of Mahdavi-Amiri and Nasseri [12], Seyed [13] introduced linear programming problems with fuzzy variables and semi-fully fuzzy linear programming problems. Some authors considered these problems and have developed various methods for solving these problems. Recently, some authors [12, 13] considered linear programming problems with trapezoidal fuzzy data and/or variables and semi-fully fuzzy linear programming problems and stated a fuzzy simplex algorithm to solve these problems. Moreover, they developed the duality results in fuzzy environment and presented a dual simplex algorithm for solving linear programming problems with trapezoidal fuzzy variables and semi-fully fuzzy linear programming problems. Furthermore, Seyed [13] showed that the presented dual simplex algorithm directly using the primal simplex tableau algorithm tenders the capability for sensitivity (or post optimality) analysis using primal simplex tableaus. In this paper, we attempt to develop the solving for Linear Programming Problems with triangular fuzzy variables via Interval Linear Programming problems without converting them to classical linear programming problems.

The rest of this paper is organized as follows: In section 2, we recall the definitions of interval numbers, some related results of interval arithmetic on them and the existing method for solving linear programming problem involving interval numbers. In section 3, we define introduced linear programming problems with fuzzy variables and semi-fully fuzzy linear programming problems. In section 4, a numerical example is provided to illustrate the theory developed in this paper.

2. Material and Method

In this section, some basic definitions, arithmetic operations for closed Intervals numbers and of linear programming problems involving interval numbers are presented [10, 14, 15].

2.1. Arithmetic Operations for Closed Intervals Numbers

In this section, some arithmetic operations for two intervals are presented [14, 16, 17].

Let us denoted by $\mathfrak{I}$ the class of all closed intervals in $\mathbb{R}$. If A is closed interval, we also adopt the notation $\bar{a} = [a^1, a^3]$, where a^1 and a^3 means the lower and upper bounds of $\bar{a}$ respectively and $a^1 \leq a^3$ with $a^1, a^3 \in \mathbb{R}$.

If $\bar{a} = a^1 = a^3 = a$, then $\bar{a} = [a, a] = a$. For any two intervals $\bar{a} = [a^1, a^3]$ and $\bar{b} = [b^1, b^3]$ and the arithmetic operations on $\bar{a}$ and $\bar{b}$ are defined as

Addition: $\bar{a} + \bar{b} = [\, a^1 + b^1, a^3 + b^3 \,]$

Subtraction: $\bar{a} - \bar{b} = [a^1 - b^3, a^3 - b^1]$

Multiplication :

$\bar{a} \times \bar{b} = [Min\{a^1 \times b^1, a^1 \times b^3, a^3 \times b^1, a^3 \times b^3\},\ Max\{a^1 \times b^1, a^1 \times b^3, a^3 \times b^1, a^3 \times b^3\}]$

Division: $\bar{a} \div \bar{b} = \left[Min\left\{\frac{a^1}{b^3}, \frac{a^1}{b^1}, \frac{a^3}{b^3}, \frac{a^3}{b^1}\right\},\ Max\left\{\frac{a^1}{b^3}, \frac{a^1}{b^1}, \frac{a^3}{b^3}, \frac{a^3}{b^1}\right\}\right]$ with $0 \notin \bar{b}$.

2.2. A New Interval Arithmetic

In this section, some arithmetic operations for two intervals are presented [10, 15].

Let $\mathfrak{R} = \{\bar{a} = [a^1, a^3]: a^1 \leq a^3\ with\ a^1, a^3 \in \mathbb{R}\}$ be the set of all proper intervals and $\bar{\mathfrak{R}} = \{\bar{a} = [a^1, a^3]: a^1 > a^3\ with\ a^1, a^3 \in \mathbb{R}\}$ be the set of all improper intervals on the real line $\mathbb{R}$. We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width (or half-width) of an interval number are defined as The mid-point and width (or half-width) of an interval number $\bar{a} = [a^1, a^3]$ are defined as $m(\bar{a}) = \left(\frac{a^3 + a^1}{2}\right)$ and $w(\bar{a}) = \left(\frac{a^3 - a^1}{2}\right)$. The interval number $\bar{a}$ can also be expressed in terms of its midpoint and width as

$\bar{a} = [a^1, a^3] = \langle m(\bar{a}), w(\bar{a})\rangle = \langle \frac{a^3 + a^1}{2}, \frac{a^3 - a^1}{2}\rangle$.

For any two intervals $\bar{a} = [a^1, a^3]$ and $\bar{b} = [b^1, b^3]$, the arithmetic operations on $\bar{a}$ and $\bar{b}$ are defined as:

Addition: $\bar{a} + \bar{b} = [a^1, a^3] + [b^1, b^3] = \langle m(\bar{a}) + m(\bar{b}), \max\{w(\bar{a}), w(\bar{b})\}\rangle$ Subtraction: $\bar{a} - \bar{b} = [a^1, a^3] - [b^1, b^3] = \langle m(\bar{a}) - m(\bar{b}), \max\{w(\bar{a}), w(\bar{b})\}\rangle$

Multiplication : $\bar{a} \times \bar{b} = [a^1, a^3] \times [b^1, b^3] = \langle m(\bar{a}) \times m(\bar{b}), \max\{w(\bar{a}), w(\bar{b})\}\rangle$

Division: $\bar{a} \div \bar{b} = [a^1, a^3] \div [b^1, b^3] = \langle m(\bar{a}) \div m(\bar{b}), \max\{w(\bar{a}), w(\bar{b})\}\rangle$ with $m(\bar{b}) \not\approx 0$.

2.3. Ramesh and Ganesan's Method for Solving Linear Programming Problem Involving Interval Numbers

In this section, to overcome all the limitations of the method, presented in Ladji, *et al.* [14], Ramesh and Ganesan [10] is presented to find the exact optimal solution of linear programming problems involving interval numbers in which all the parameters are represented by intervals numbers.

2.3.1. Formulation of a Linear Programming Problem Involving Interval Numbers (INLP)

Now we are in a position to prove interval analogue of some important relationships between the primal and dual linear programming problems. We consider the primal and dual linear programming problems involving interval numbers as follows [10, 14, 15]:

For all the rest of this paper, we will consider the following notations:

$$\bar{x} = [\bar{x}_j]_{n\times 1} = \left[[x_j^1, x_j^3]\right]_{n\times 1} = \left[\langle \frac{x_j^3 + x_j^1}{2}, \frac{x_j^3 - x_j^1}{2}\rangle\right]_{n\times 1} = [\langle m(\bar{x}_j), w(\bar{x}_j)\rangle]_{n\times 1},$$

$$\bar{c} = [\bar{c}_j]_{1\times n} = \left[[c_j^1, c_j^3]\right]_{1\times n} = \left[\langle \frac{c_j^3 + c_j^1}{2}, \frac{c_j^3 - c_j^1}{2}\rangle\right]_{1\times n} = [\langle m(\bar{c}_j), w(\bar{c}_j)\rangle]_{1\times n},$$

$$\bar{b} = [\bar{b}_i]_{m\times 1} = \left[[b_i^1, b_i^3]\right]_{m\times 1} = \left[\langle \frac{b_i^3 + b_i^1}{2}, \frac{b_i^3 - b_i^1}{2}\rangle\right]_{m\times 1} = [\langle m(\bar{b}_i), w(\bar{b}_i)\rangle]_{m\times 1} \text{ and}$$

$$A = [a_{ij}]_{m\times n} \text{ where } x_j^1, x_j^3,\ c_j^1, c_j^3,\ b_i^1, b_i^3 \text{ and } a_{ij} \text{ are real numbers } (\mathbb{R}).$$

We consider the primal (P) and dual (D) linear programming problems involving interval numbers **(INLP)** as follows:

$$(P)\ \begin{cases} \text{Max } \bar{Z}(\bar{x}) \approx \sum_{j=1}^{n} \bar{c}_j \times \bar{x}_j \\ \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} a_{ij} \times \bar{x}_j \preccurlyeq \bar{b}_i \end{cases} \quad \text{and} \quad (D)\ \begin{cases} \text{Min } \bar{W}(\bar{y}) \approx \sum_{i=1}^{m} \bar{b}_i \times \bar{y}_i \\ \text{Subject to the constraints} \\ \quad \sum_{i=1}^{m} a_{ij} \times \bar{y}_i \succcurlyeq \bar{c}_j \end{cases}$$

For all the rest of this paper, we will consider the following primal linear programming problem involving interval numbers **(INLP)**:

$$\begin{cases} \text{Min/Max } \bar{Z}(\bar{x}_1, \dots, \bar{x}_n) \approx \sum_{j=1}^{n} [c_j^1, c_j^3][x_j^1, x_j^3] \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij}\ [x_j^1, x_j^3] \begin{pmatrix} \lesssim \\ \approx \\ \gtrsim \end{pmatrix} [b_i^1, b_i^3] \\ 1 \le j \le n \text{ and } 1 \le i \le m \end{cases} \quad \textbf{(INLP1)} \text{ is equivalent to}$$

$$\text{and} \begin{cases} \text{Min/Max } \bar{Z}(\bar{x}_1, \dots, \bar{x}_n) \approx \sum_{j=1}^{n} \langle m(\bar{c}_j), w(\bar{c}_j)\rangle \langle m(\bar{x}_j), w(\bar{x}_j)\rangle \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij}\ \langle m(\bar{x}_j), w(\bar{x}_j)\rangle \begin{pmatrix} \lesssim \\ \approx \\ \gtrsim \end{pmatrix} \langle m(\bar{b}_i), w(\bar{b}_i)\rangle \\ 1 \le j \le n \text{ and } 1 \le i \le m \end{cases} \quad \textbf{(INLP2)}$$

2.3.2. Table $\bar{T}^*$ Optimal for Linear Programming Problem Involving Interval Numbers (INLP2)

In this section, we will describe how to determine the values of the primal (P) and dual (D) variables.

Calculation of the values of the primal (P) variables $\bar{x}^* = (\bar{x}_1^*\ \bar{x}_2^* \dots \bar{x}_{n+m}^*)^t$:

Assume that $\bar{T}^*$ is optimal, then the current basis is $\bar{x}_B^* = \{\bar{x}_{J_1}^*, \bar{x}_{J_2}^*, \dots, \bar{x}_{J_m}^*\}$ and the corresponding solution is $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ with $B_{N^*}^{-1} = (A_{n+1}^*, A_{n+2}^*, \dots, A_{n+m}^*)$.

Moreover, the current nonbasic variables is $\bar{x}_N^* = \{\bar{x}_d^*,\ \bar{x}_d^* \notin \bar{x}_B^*\}$ and the corresponding solution is $\bar{x}_N^* = \{\bar{x}_d^* \approx \bar{0},\ \bar{x}_d^* \in \bar{x}_N^*\}$. Hence the optimal solution to the problem can be written as $\bar{x}^* = (\bar{x}_1^*\ \bar{x}_2^* \dots \bar{x}_{n+m}^*)^t$ with the associated value of the objective function $\bar{Z}(\bar{x}^*) \approx \bar{c}\bar{x}^* = [Z^{*1}, Z^{*3}]$.

Calculation of the values of the dual (D) variables $\bar{y}^* = (\bar{y}_1^*\ \bar{y}_2^* \dots \bar{y}_{n+m}^*)$ in $\bar{T}^*$:

Assume that $\bar{T}^*$ is optimal, then the current basis is $\bar{x}_B^* = \{\bar{x}_{J_1}^*, \bar{x}_{J_2}^*, \dots, \bar{x}_{J_m}^*\}$ and the corresponding coefficient is matrix $\bar{C}_B^* = (\bar{c}_{J_1}^*\ \ \bar{c}_{J_2}^* \dots \bar{c}_{J_m}^*)$. So $\bar{y}_{m+j}^* = |\bar{C}_B^* A_j^* - \bar{c}_j|$ and $\bar{y}_i^* = |\bar{C}_B^* A_{n+i}^* - \bar{c}_{n+i}|$ with the associated value of the objective function Min $\bar{W}(\bar{y}^*) \approx \bar{y}^*\bar{b} = [W^{*1}, W^{*3}]$.

3. Main Results

In this section, we will describe our method of solving.

3.1. A New Interval Arithmetic for Triangular Fuzzy Numbers via Intervals Numbers

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

A number $\tilde{a} = (a^1, a^2, a^3)$ (where $a^1 \le a^2 \le a^3$) is said to be a triangular fuzzy number if its membership function is given by Iwona, *et al.* [16], Jagdeep and Amit [17], Seyed [13]:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a^1}{a^2 - a^1}, & a^1 \le x \le a^2, \\ \dfrac{x - a^3}{a^2 - a^3}, & a^2 \le x \le a^3, \end{cases}$$

Assume that $\tilde{a} = (a^1, a^2, a^3) = (a^2|\bar{a}) = \left(a^2 \middle| [a^1, a^3] = \langle \frac{a^3+a^1}{2}, \frac{a^3-a^1}{2} \rangle\right)$ and

$\tilde{b} = (b^1, b^2, b^3) = (b^2|\bar{b}) = \left(b^2 \middle| [b^1, b^3] = \langle \frac{b^3+b^1}{2}, \frac{b^3-b^1}{2} \rangle\right)$ are two triangular fuzzy numbers. For any two triangular fuzzy numbers $\tilde{a} = (a^2|\bar{a})$ and $\tilde{b} = (b^2|\bar{b})$, the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as:

Addition:

$$\tilde{a} + \tilde{b} = (a^2|[a^1, a^3]) + (b^2|[b^1, b^3]) = (a^2 + b^2|[a^1, a^3] + [b^1, b^3])$$

Subtraction:

$$\tilde{a} - \tilde{b} = (a^2|[a^1, a^3]) - (b^2|[b^1, b^3]) = (a^2 - b^2|[a^1, a^3] - [b^1, b^3])$$

Multiplication :

$$\tilde{a} \times \tilde{b} = (a^2|[a^1, a^3]) \times (b^2|[b^1, b^3]) = (a^2 \times b^2|[a^1, a^3] \times [b^1, b^3])$$

For all the rest of this paper, we will consider the following notations:

Assume that $\tilde{c}_j = (c_j^1, c_j^2, c_j^3)$, $\tilde{x}_j = (x_j^1, x_j^2, x_j^3)$ and $\tilde{b}_i = (b_i^1, b_i^2, b_i^3)$ are triangular fuzzy numbers

$$\tilde{x} = [\tilde{x}_j]_{n\times 1} = [(x_j^2|[x_j^1, x_j^3])]_{n\times 1} = [(x_j^2|\langle m(\bar{x}_j), w(\bar{x}_j)\rangle)]_{n\times 1},$$

$$\tilde{c} = [\tilde{c}_j]_{1\times n} = [(c_j^2|[c_j^1, c_j^3])]_{1\times n} = [(c_j^2|\langle m(\bar{c}_j), w(\bar{c}_j)\rangle)]_{1\times n},$$

$$\tilde{b} = [\tilde{b}_i]_{m\times 1} = [(b_i^2|[b_i^1, b_i^3])]_{m\times 1} = [(b_i^2|\langle m(\bar{b}_i), w(\bar{b}_i)\rangle)]_{m\times 1} \text{ and}$$

$A = [a_{ij}]_{m\times n}$ where x_j^1, x_j^3, c_j^1, c_j^3, b_i^1, b_i^3 and a_{ij} are real numbers ($\mathbb{R}$).

3.2. Formulation of a Linear Programming with Triangular Fuzzy Variables (FVLP) Problems

In this section, we introduce a kind of linear programming problems where the right-hand-side vector and the decision variables are a type of triangular fuzzy numbers, simultaneity. We name such problems as Linear Programming with Triangular Fuzzy Variables (FVLP) problems.

We consider the primal (P) and dual (D) Linear Programming with Triangular Fuzzy Variables (FVLP) problems as follows [13, 18, 19]:

$$(\text{P})\ \begin{cases} \text{Max } \tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n} c_j \times \tilde{x}_j \\ \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} a_{ij} \times \tilde{x}_j \preccurlyeq \tilde{b}_i \end{cases} \quad \text{and} \quad (\text{D})\ \begin{cases} \text{Min } \tilde{W}(y) \approx \sum_{i=1}^{m} \tilde{b}_i \times y_i \\ \text{Subject to the constraints, } y_i \in \mathbb{R} \\ \quad \sum_{i=1}^{m} a_{ij} \times y_i \succcurlyeq c_j \end{cases}$$

For all the rest of this paper, we will consider the following primal Linear Programming with Triangular Fuzzy Variables (FVLP) problem:

$$\begin{cases} \text{Min/Max } \tilde{Z}(\tilde{x}_1, \dots, \tilde{x}_n) \approx \sum_{j=1}^{n} c_j \left(x_j^2 \middle| \left[x_j^1, x_j^3\right]\right) \\ \qquad \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij}\ \left(x_j^2 \middle| \left[x_j^1, x_j^3\right]\right) \binom{\preccurlyeq}{\succcurlyeq}\ \left(b_i^2 \middle| \left[b_i^1, b_i^3\right]\right) \\ \qquad 1 \le j \le n \text{ and } 1 \le i \le m \end{cases} \quad \textbf{(FVLP1)} \quad \text{is equivalent}$$

$$\begin{cases} \text{Min/Max } \tilde{Z}(\tilde{x}_1, \dots, \tilde{x}_n) \approx \sum_{j=1}^{n} c_j \left(x_j^2 \middle| \langle m(\bar{x}_j), w(\bar{x}_j)\rangle\right) \\ \qquad \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij}\ \left(x_j^2 \middle| \langle m(\bar{x}_j), w(\bar{x}_j)\rangle\right) \binom{\preccurlyeq}{\succcurlyeq}\ \left(b_i^2 \middle| \langle m(\bar{b}_i), w(\bar{b}_i)\rangle\right) \\ \qquad 1 \le j \le n \text{ and } 1 \le i \le m \end{cases} \quad \textbf{(FVLP2)}$$

3.3. Formulation of a Semi-fully Fuzzy Linear Programming Problem (SFFLP)

In this section, we introduce a kind of linear programming problems where the coefficients in objective function, the right-hand-side vector and the decision variables are a type of triangular fuzzy numbers, simultaneity. We name such problems as Semi-Fully Fuzzy Linear Programming (SFFLP) problems.

We consider the primal (P) and dual (D) Semi-fully Fuzzy Linear Programming Problem (SFFLP) as follows [13, 18, 19]:

$$(\text{P})\ \begin{cases} \text{Max } \tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n} \tilde{c}_j \times \tilde{x}_j \\ \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} a_{ij} \times \tilde{x}_j \preccurlyeq \tilde{b}_i \end{cases} \quad \text{and} \quad (\text{D})\ \begin{cases} \text{Min } \tilde{W}(\tilde{y}) \approx \sum_{i=1}^{m} \tilde{b}_i \times \tilde{y}_i \\ \text{Subject to the constraints} \\ \quad \sum_{i=1}^{m} a_{ij} \times \tilde{y}_i \succcurlyeq \tilde{c}_j \end{cases}$$

For all the rest of this paper, we will consider the following primal Semi-Fully Fuzzy Linear Programming (SFFLP) problems:

$$\begin{cases} \text{Min/Max } \tilde{Z}(\tilde{x}_1, \dots, \tilde{x}_n) \approx \sum_{j=1}^{n} \left(c_j^2 \middle| \left[c_j^1, c_j^3\right]\right)\left(x_j^2 \middle| \left[x_j^1, x_j^3\right]\right) \\ \qquad \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij}\ \left(x_j^2 \middle| \left[x_j^1, x_j^3\right]\right) \binom{\preccurlyeq}{\succcurlyeq}\ \left(b_i^2 \middle| \left[b_i^1, b_i^3\right]\right) \\ \qquad 1 \le j \le n \text{ and } 1 \le i \le m \end{cases} \quad \textbf{(SFFLP1)}$$

$$\text{is equivalent to } \begin{cases} \text{Min/Max } \tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n} \left(c_j^2 \middle| \langle m(\bar{x}_j), w(\bar{x}_j)\rangle\right)\left(x_j^2 \middle| \langle m(\bar{x}_j), w(\bar{x}_j)\rangle\right) \\ \qquad \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij}\ \left(x_j^2 \middle| \langle m(\bar{x}_j), w(\bar{x}_j)\rangle\right) \binom{\preccurlyeq}{\succcurlyeq}\ \left(b_i^2 \middle| \langle m(\bar{b}_i), w(\bar{b}_i)\rangle\right) \\ \qquad 1 \le j \le n \text{ and } 1 \le i \le m \end{cases} \quad \textbf{(SFFLP2)}$$

3.4. Our Method for Solving Linear Programming Problems With Triangular Fuzzy Variables (FVLP) and Semi-Fully Fuzzy Linear Programming Problems (SFFLP)

In this section, a method to find a fuzzy optimal solution of Linear Programming problems with Triangular Fuzzy Variables (FVLP) and Semi-Fully Fuzzy Linear Programming problems (SFFLP) is presented.

3.4.1. Formulation of a Linear Programming (LP) Problem

For all the rest of this paper, we assume that $x^{*2} = (x_1^{*2},\ x_2^{*2}, \dots,\ x_{n+m}^{*2})$ is the unique optimal solution of **(LP)** with the associated value of the objective function $Z^{*2}\left(x^{*2}\right) = \sum_{j=1}^{n} c_j^2 \times x_j^{*2}$

$$\begin{cases} \qquad \text{Max } Z^2(x^2) = \sum_{j=1}^{n} c_j^2 \times x_j^2 \\ \qquad \text{Subject to the constraints} \\ \sum_{j=1}^{n} a_{ij} \times x_j^2 + x_{n+i}^2 = b_i^2\ , 1 \le i \le m \\ \qquad x_j^{*1} \le\ x_j^2 \le\ x_j^{*3} \\ \qquad 1 \le j \le n + m \\ \qquad x_j^2 \ \text{ are real numbers } (\mathbb{R}) \end{cases} \quad \textbf{(LP)}$$

Note 1. For any two triangular fuzzy numbers

$\tilde{a} = (a^1, a^2, a^3) = (a^2|\bar{a}) = (a^2|[a^1, a^3]) = (a^2|\langle m(\bar{a}), w(\bar{a})\rangle)$ and

$\tilde{b} = (b^1, b^2, b^3) = \left(b^2|\bar{b}\right) = (b^2|[b^1, b^3]) = \left(b^2|\langle m(\bar{b}), w(\bar{b})\rangle\right)$, then:

(i) $\tilde{a} = \tilde{b}$ if and only if $a^1 = b^1$, $a^2 = b^2$ and $a^3 = b^3$.

(ii) $\tilde{a} = \tilde{b}$ if and only if $a^2 = b^2$ and $\bar{a} = \bar{b}$.

(iii) $\tilde{a} = \tilde{b}$ if and only if $a^2 = b^2$ and $[a^1, a^3] = [b^1, b^3]$.
(iv) $\tilde{a} = \tilde{b}$ if and only if $a^2 = b^2$ and $m(\bar{a}) = m(\bar{b})$ and $w(\bar{a}) = w(\bar{b})$.
Now we can write the following propositions using Ramesh and Ganesan [10] and Ladji, *et al.* [14]:
Proposition 1. $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ is a optimal solution to the **(INLP2)** if and only if $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ is a optimal solution to the **(INLP1).**
Proof. Just assume that $\bar{a} = [a^1, a^3] = \langle m(\bar{a}), w(\bar{a})\rangle$.
We can write the following propositions:
Proposition 2. $\tilde{x}^* = (\tilde{x}_1^*\, \tilde{x}_2^* \dots \tilde{x}_{n+m}^*)^t$ is an optimal solution to the **(SFFLP2)** if and only if $\tilde{x}^* = (\tilde{x}_1^*\, \tilde{x}_2^* \dots \tilde{x}_{n+m}^*)^t$ is an optimal solution to the **(SFFLP1)**.
Proof. Assume that $\tilde{x}_j^* = (x_j^{*1}, x_j^{*2}, x_j^{*3})$ and
$\tilde{x}_j^* = (x_j^{*2}|\bar{x}_j^*) = (x_j^{*2}|\langle m(\bar{x}_j^*), w(\bar{x}_j^*)\rangle)$ for $1 \le j \le n$.
If $\tilde{x}_j^* = (x_j^{*1}, x_j^{*2}, x_j^{*3}) \in \{\tilde{x}_1^*,\ \tilde{x}_2^*,\ \dots,\ \tilde{x}_{n+m}^*\}$ is an optimal solution to the **(SFFLP2)**, then $\bar{x}_j^* = \langle m(\bar{x}_j^*), w(\bar{x}_j^*)\rangle$ is an optimal solution to the **(INLP2)** and $x^{*2} = (x_1^{*2},\ x_2^{*2}, \dots,\ x_{n+m}^{*2})$ is an optimal solution to the **(LP)** and vice versa and see **Note 1**.
Proposition 3. $\tilde{x}^* = (\tilde{x}_1^*\, \tilde{x}_2^* \dots \tilde{x}_{n+m}^*)^t$ is an optimal solution to the **(SFFLP2)** if and only if $\tilde{x}^* = (\tilde{x}_1^*\, \tilde{x}_2^* \dots \tilde{x}_{n+m}^*)^t$ is an optimal solution to the **(SFFLP1)**.
Proof. Assume that $\tilde{x}_j^* = (x_j^{*1}, x_j^{*2}, x_j^{*3})$ and $\tilde{x}_j^* = (x_j^{*2}|\bar{x}_j^*)$ for $1 \le j \le n$.
If $\tilde{x}_j^* = (x_j^{*1}, x_j^{*2}, x_j^{*3}) \in \{\tilde{x}_1^*,\ \tilde{x}_2^*,\ \dots,\ \tilde{x}_{n+m}^*\}$ is an optimal solution to the **(SFFLP2)**, then $\bar{x}_j^* = [x_j^{*1}, x_j^{*3}]$ is an optimal solution to the **(INLP1)** and
$x^{*2} = (x_1^{*2},\ x_2^{*2}, \dots,\ x_{n+m}^{*2})$ is an optimal solution to the **(LP)** and vice versa and see **Note 1**.
Corollary 1. If $\bar{x}_j^* = [x_j^{*1}, x_j^{*3}]$ is an optimal solution to the **(INLP1)** and
$x^{*2} = (x_1^{*2},\ x_2^{*2}, \dots,\ x_{n+m}^{*2})$ is an optimal solution to the **(LP)**, then
$\tilde{x}^* = (\tilde{x}_1^*\, \tilde{x}_2^* \dots \tilde{x}_{n+m}^*)^t$ is an optimal solution to the **(SFFLP2)** and **(SFFLP1).**

3.4.2. The Steps of Our Method

The steps of our method for solving Linear Programming problems with Triangular Fuzzy Variables (FVLP) and Semi-Fully Fuzzy Linear Programming problems (SFFLP) as follows:

Step 1 Consider a Linear Programming problem with Triangular Fuzzy Variables (FVLP) or Semi-Fully Fuzzy Linear Programming problem (SFFLP):

Identify the Linear Programming problem involving Interval Numbers **(INLP1)** and the Linear Programming problem **(LP)**

Step 2 Solving the primal Linear Programming problem involving Interval Numbers **(INLP1)** by Ramesh and Ganesan [10] and Ladji, *et al.* [14] to determine the optimal solution:

If $m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{x}_j^* = [x_j^{*1}, x_j^{*3}]$ with $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ and $\bar{b}_i = [b_i^1, b_i^3]$.

If $m(\bar{X}_B^*) < w(\bar{b}_i)$, then $\bar{x}_j^* = \langle m(\bar{x}_j^*), w(\bar{x}_j^*)\rangle$ with $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ and $\bar{b}_i = \langle m(\bar{b}_i), w(\bar{b}_i)\rangle$. In addition, the values of $\bar{Z}^* = \langle m(\bar{Z}^*), w(\bar{Z}^*)\rangle = [Z^{*1}, Z^{*3}]$.

Step 3 To determine the optimal solution of the dual (D) of Linear Programming problem involving Interval Numbers **(INLP1)**: $\bar{y}^* = (\bar{y}_1^*\, \bar{y}_2^* \dots \bar{y}_{n+m}^*)$,

If $m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{y}_i^* = [y_i^{*1}, y_i^{*3}]$ with $\bar{y}_{m+j}^* = |\bar{C}_B^* A_j^* - \bar{c}_j|$ and
$\bar{y}_i^* = |\bar{C}_B^* A_{n+i}^* - \bar{c}_{n+i}|$ and $\bar{c}_j = [c_j^1, c_j^3]$.

If $m(\bar{X}_B^*) < w(\bar{b}_i)$, then $\bar{y}_i^* = \langle m(\bar{y}_i^*), w(\bar{y}_i^*)\rangle$ with $\bar{y}_{m+j}^* = |\bar{C}_B^* A_j^* - \bar{c}_j|$ and
$\bar{y}_i^* = |\bar{C}_B^* A_{n+i}^* - \bar{c}_{n+i}|$ and $\bar{c}_j = \langle m(\bar{c}_j), w(\bar{c}_j)\rangle$.

In addition, the values of $\bar{W}^* = \langle m(\bar{W}^*), w(\bar{W}^*)\rangle = [W^{*1}, W^{*3}]$.

Step 4 Formulation of a classical primal (P) and dual (D) Linear Programming **(LP)** problem to determine the values of $x_j^{*2}, j = 1, \dots, n+m$ and $y_i^{*2} = |\Delta_{n+i}^*|$,
$i = 1, \dots, m$ and $y_{m+j}^{*2} = |\Delta_j^*|, j = 1, \dots, n$. In addition, the values of Z^{*2} and W^{*2}.

Step 5 Fuzzy optimal solution of Linear Programming problems with Triangular Fuzzy Variables **(FVLP)** or Semi-Fully Fuzzy Linear Programming problems **(SFFLP)**.

(i) Primal optimal solution:
$\tilde{x}_j^* = (x_j^{*2}|\bar{x}_j^*) = (x_j^{*2}|[x_j^{*1}, x_j^{*3}]) = (x_j^{*1}, x_j^{*2}, x_j^{*3})$, $j = 1, \dots, n+m$ with the associated value of the objective function Max $\tilde{Z}^* = (Z^{*1}, Z^{*2}, Z^{*3}) = (Z^{*2}|\bar{Z}^*)$.

(ii) then corresponding Dual optimal solution problem is given by
$\tilde{y}_i^* = (y_i^{*2}|\bar{y}_i^*) = (y_i^{*2}|[y_i^{*1}, y_i^{*3}]) = (y_i^{*1}, y_i^{*2}, y_i^{*3})$, $j = 1, \dots, n+m$ with the associated value of the objective function
Min$\tilde{W}^* = (W^{*1}, W^{*2}, W^{*3}) = (W^{*2}|\bar{W}^*)$.

(iii) Comparisons using Ranking function [17]:
$\mathcal{R}(\text{Max}\tilde{Z}^*) = \frac{1}{4}(Z^{*1} + 2Z^{*2} + Z^{*3})$ and $\mathcal{R}(\text{Min}\tilde{W}^*) = \frac{1}{4}(W^{*1} + 2W^{*2} + W^{*3})$.

NB: Let $\tilde{a} = (a^1, a^2, a^3)$ and $\tilde{b} = (b^1, b^2, b^3)$ then: $\tilde{a} \preccurlyeq \tilde{b}$ iff $\mathcal{R}(\tilde{a}) \le \mathcal{R}(\tilde{b})$,
$\tilde{a} \succcurlyeq \tilde{b}$ iff $\mathcal{R}(\tilde{a}) \ge \mathcal{R}(\tilde{b})$ and $\tilde{a} \approx \tilde{b}$ iff $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$.

4. Numerical Examples

Numerical examples are provided to illustrate the theory developed in this paper.

Example 4.1. Step 1

Consider the following Semi-Fully Fuzzy Primal Linear Programming (SFFLP) problem: $(\boldsymbol{P})$	**Then the corresponding dual problem is given by: $(\boldsymbol{D})$**
$\text{Max}\tilde{Z} \approx (29,30,31)\tilde{x}_1 + (22,23,24)\tilde{x}_2 + (28,29,30)\tilde{x}_3$ Subject to the constraints $6\tilde{x}_1 + 5\tilde{x}_2 + 3\tilde{x}_3 \preccurlyeq (25,26,27)$ $4\tilde{x}_1 + 2\tilde{x}_2 + 5\tilde{x}_3 \preccurlyeq (6,7,8)$ where $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ are triangular fuzzy numbers.	$\text{Min } \tilde{w} \approx (25,26,27)\tilde{y}_1 + (6,7,8)\tilde{y}_2$ Subject to the constraints $6\tilde{y}_1 + 4\tilde{y}_2 \succcurlyeq (29,30,31)$ $5\tilde{y}_1 + 2\tilde{y}_2 \succcurlyeq (22,23,24)$ $3\tilde{y}_1 + 5\tilde{y}_2 \succcurlyeq (28,29,30)$ where $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$ are triangular fuzzy numbers.

Step 2

$$\begin{cases} \text{Max } \bar{Z}(\bar{x}_1,\bar{x}_2,\bar{x}_3) \approx [29,31]\bar{x}_1 + [22,24]\bar{x}_2 + [28,30]\bar{x}_3 \\ 6\bar{x}_1 + 5\bar{x}_2 + 3\bar{x}_3 \preccurlyeq [25,27] \\ 4\bar{x}_1 + 2\bar{x}_2 + 5\bar{x}_3 \preccurlyeq [6,8] \\ \bar{x}_j \in \mathbb{R} \end{cases} \qquad \textbf{(INLP1)}$$

We convert the primal and dual linear programming problem involving interval numbers **(INLP2)** to its canonical form by adding slack variables $\overline{x_4}$ and $\overline{x_5}$ as follows: Max $\bar{Z}(\overline{x_1},\overline{x_2},\overline{x_3}) \approx \langle 30,1\rangle\overline{x_1} + \langle 23,1\rangle\overline{x_2} + \langle 29,1\rangle\,\overline{x_3}$ Subject to the constraint $6\,\overline{x_1} + 5\,\overline{x_2} + 3\,\overline{x_3} + \overline{x_4} \approx \langle 26,1\rangle$, $4\,\overline{x_1} + 2\,\overline{x_2} + 5\,\overline{x_3} + \overline{x_5} \approx \langle 7,1\rangle$.

Basic variable $\bar{x}_B^*$	**Coefficients $\bar{x}_B^*$ $\bar{C}_B^*$**	**$[29,31] = \langle 30,1\rangle$**	**$[22,24] = \langle 23,1\rangle$**	**$[28,30] = \langle 29,1\rangle$**	**0**	**0**	**Current values $\bar{X}_B^*$ $\langle m(\bar{X}_B^*), w(\bar{b}_i)\rangle$**
		A_1^*	A_2^*	A_3^*	A_4^*	A_5^*	
$\bar{x}_4$	0	-4	0	$\frac{-19}{2}$	1	$\frac{-5}{2}$	$\langle\frac{17}{2},1\rangle$
$\bar{x}_2$	$[22,24] = \langle 23,1\rangle$	2	1	$\frac{5}{2}$	0	$\frac{1}{2}$	$\langle\frac{7}{2},1\rangle$
$\bar{\Delta}_j^* = \bar{C}_B^*\bar{A}_j^* - \bar{c}_j$		$\langle 16,1\rangle$	0	$\langle\frac{57}{2},1\rangle$	0	$\langle\frac{23}{2},1\rangle$	$\bar{C}_B^*\bar{X}_B^* = \langle\frac{161}{2},1\rangle$

$$m(\bar{X}_B^*) > w(\bar{b}_i), \text{ then } \bar{x}_j^* = [x_j^{*1}, x_j^{*3}] \text{ with } \bar{x}_B^* \approx B_{N^*}^{-1}\bar{b} \text{ and } \bar{b}_\iota = [b_i^1, b_i^3]:$$

$$\text{We get the matrix } B_{N^*}^{-1} = \begin{pmatrix} 1 & \frac{-5}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \bar{b} = \begin{pmatrix} [25,27] \\ [6,8] \end{pmatrix} \text{ and } \bar{C}_B^* = (0,[22,24]):$$

$$\begin{pmatrix} \bar{x}_4^* \\ \bar{x}_2^* \end{pmatrix} \approx B_{N^*}^{-1}\bar{b} \approx \begin{pmatrix} 1 & \frac{-5}{2} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} [25,27] \\ [6,8] \end{pmatrix} \approx \begin{pmatrix} [5,12] \\ [3,4] \end{pmatrix} \text{ avec } \bar{x}_1^* \approx \bar{0}, \bar{x}_2^* \approx [3,4],$$

$\bar{x}_3^* \approx \bar{0}$, $\bar{x}_4^* \approx [5,12]$ and $\bar{x}_5^* \approx \bar{0}$ with Max $\bar{Z}(\bar{x}^*) \approx [66,96]$.

Step 3 Then the corresponding dual problem (D) is given by:

Min $\bar{W}(\overline{y_1},\overline{y_2}) \approx [25,27]\overline{y_1} + [6,8]\overline{y_2}$ Subject to the constraints

$6\,\overline{y_1} + 4\,\overline{y_2} \succcurlyeq [29,31]$, $5\,\overline{y_1} + 2\,\overline{y_2} \succcurlyeq [22,24]$, $3\,\overline{y_1} + 5\,\overline{y_2} \succcurlyeq [28,30]$ and

$\overline{y_\iota} \in \mathbb{R}$, $i = 1,2,3$. We convert the dual linear programming problem involving interval numbers to its canonical form by adding slack variables $\overline{y_3}$, $\overline{y_4}$ and $\overline{y_5}$ as follows: Min $\bar{W}(\overline{y_1},\overline{y_2}) \approx \langle 26,1\rangle,\overline{y_1} + \langle 7,1\rangle,\overline{y_2}$ Subject to the constraints

$6\,\overline{y_1} + 4\,\overline{y_2} - \overline{y_3} \approx \langle 30,1\rangle$, $5\,\overline{y_1} + 2\,\overline{y_2} - \overline{y_4} \approx \langle 23,1\rangle$,

$3\,\overline{y_1} + 5\,\overline{y_2} - \overline{y_5} \approx \langle 29,1\rangle$.

We have $m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{y}_i^* = [y_i^{*1}, y_i^{*3}]$ with $\bar{y}_{m+j}^* = |\bar{C}_B^*A_j^* - \bar{c}_j|$ and

$\bar{y}_i^* = |\bar{C}_B^*A_{n+i}^* - \bar{c}_{n+i}|$ and $\bar{c}_j = [c_j^1, c_j^3]$. We get $\bar{y}_1^* \approx \bar{0}$, $\bar{y}_2^* \approx [11,12]$,

$\bar{y}_3^* \approx [13,19]$, $\bar{y}_4^* \approx \bar{0}$ and $\bar{y}_5^* \approx [25,32]$ with Min $\bar{W}(\bar{y}^*) \approx [66,96]$.

Step 4 Formulation of a classical primal (P) Linear Programming **(LP)** problem

$$\begin{cases} \text{Max } Z^2 = 30x_1^2 + 23x_2^2 + 29x_3^2 \\ \quad \text{Subject to the constraints} \\ 6x_1^2 + 5x_2^2 + 3x_3^2 + x_4^2 = 26 \\ 4x_1^2 + 2x_2^2 + 5x_3^2 + x_5^2 = 7 \\ \quad 3 \le x_2^2 \le 4 \\ \quad 5 \le x_4^2 \le 12 \end{cases}$$ with $x_1^{*2} = 0, x_2^{*2} = \frac{7}{2}, x_3^{*2} = 0, x_4^{*2} = \frac{17}{2}, x_5^{*2} = 0, \quad y_1^{*2} = 0, y_2^{*2} = \frac{23}{2},$ $y_3^{*2} = 16, y_4^{*2} = 0$, and $y_5^{*2} = \frac{57}{2}$ with

Max Z^{*2} =Min$W^{*2} = \frac{161}{2}$.

Step 5 Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

$$\tilde{x}_1^* = (x_1^{*1}, x_1^{*2}, x_1^{*3}) = (x_1^{*2}|\bar{x}_1^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{x}_2^* = (x_2^{*1}, x_2^{*2}, x_2^{*3}) = (x_2^{*2}|\bar{x}_2^*) = \left(\tfrac{7}{2}\middle|[3,4]\right) = \left(3, \tfrac{7}{2}, 4\right),$$
$$\tilde{x}_3^* = (x_3^{*1}, x_3^{*2}, x_3^{*3}) = (x_3^{*2}|\bar{x}_3^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{x}_4^* = (x_4^{*1}, x_4^{*2}, x_4^{*3}) = (x_4^{*2}|\bar{x}_4^*) = \left(\tfrac{17}{2}\middle|[5,12]\right) = \left(5, \tfrac{17}{2}, 12\right),$$
$$\tilde{x}_5^* = (x_5^{*1}, x_5^{*2}, x_5^{*3}) = (x_5^{*2}|\bar{x}_5^*) = (0|[0,0]) = (0,0,0),$$
$$\text{Max}\tilde{Z} = (Z^{*1}, Z^{*2}, Z^{*3}) = (Z^{*2}|\bar{Z}^*) = \left(\tfrac{161}{2}\middle|[66,96]\right) = \left(66, \tfrac{161}{2}, 96\right).$$

(ii) Dual optimal solution:

$$\tilde{y}_1^* = (y_1^{*1}, y_1^{*2}, y_1^{*3}) = (y_1^{*2}|\bar{y}_1^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{y}_2^* = (y_2^{*1}, y_2^{*2}, y_2^{*3}) = (y_2^{*2}|\bar{y}_2^*) = \left(\tfrac{23}{2}\middle|[11,12]\right) = \left(11, \tfrac{23}{2}, 12\right),$$
$$\tilde{y}_3^* = (y_3^{*1}, y_3^{*2}, y_3^{*3}) = (y_3^{*2}|\bar{y}_3^*) = (16|[13,19]) = (13,16,19),$$
$$\tilde{y}_4^* = (y_4^{*1}, y_4^{*2}, y_4^{*3}) = (y_4^{*2}|\bar{y}_4^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{y}_5^* = (y_5^{*1}, y_5^{*2}, y_5^{*3}) = (y_5^{*2}|\bar{y}_5^*) = \left(\tfrac{57}{2}\middle|[25,32]\right) = \left(25, \tfrac{57}{2}, 32\right),$$
$$\text{Min}\widetilde{W} = (W^{*1}, W^{*2}, W^{*3}) = (W^{*2}|\bar{W}^*) = \left(\tfrac{161}{2}\middle|[66,96]\right) = \left(66, \tfrac{161}{2}, 96\right).$$

(iii) Comparisons using Ranking function [17]:

$$\text{Max}\tilde{Z} = \text{Min}\widetilde{W} = \left(66, \tfrac{161}{2}, 96\right) \text{ and } \mathcal{R}(\text{Max}\tilde{Z}) = \mathcal{R}(\text{Min}\widetilde{W}) = \tfrac{323}{4}.$$

Example 4.2. Step 1

Consider the following Semi-Fully Fuzzy Linear Programming (SFFLP) problem: (*P*)	**Then the corresponding dual problem is given by: (*D*)**
Max $\tilde{Z} = (5,6,8)\tilde{x}_1 + (4,4,4)\tilde{x}_2$ Subject to the constraints $3\tilde{x}_1 + 2\tilde{x}_2 \preccurlyeq (140,150,150)$ $4\tilde{x}_1 + 3\tilde{x}_2 \preccurlyeq (155,160,165)$ where $\tilde{x}_1$ and $\tilde{x}_2$ are triangular fuzzy numbers.	Min $\widetilde{W} = (140,150,150)\tilde{y}_1 + (155,160,165)\tilde{y}_2$ Subject to the constraints $3\tilde{y}_1 + 4\tilde{y}_2 \succcurlyeq (5,6,8)$ $2\tilde{y}_1 + 3\tilde{y}_2 \succcurlyeq (4,4,4)$ where $\tilde{y}_1$ and $\tilde{y}_2$ are triangular fuzzy numbers.

Step 2

$$\begin{cases} \text{Max } \bar{Z}(\bar{x}_1, \bar{x}_2) \approx [5,8]\bar{x}_1 + [4,4]\bar{x}_2 \\ \quad \text{Subject to the contraints} \\ \quad 3\,\bar{x}_1 + 2\,\bar{x}_2 \preccurlyeq [140,150] \\ \quad 4\,\bar{x}_1 + 3\,\bar{x}_2 \preccurlyeq [155,165] \\ \quad\quad \bar{x}_j \in \mathbb{R} \end{cases} \qquad \textbf{(INLP1)}$$

We convert the primal and dual linear programming problem involving interval numbers **(INLP2)** to its canonical form by adding slack variables $\overline{x_3}$ and $\overline{x_4}$ as follows: Max $\bar{Z}(\overline{x_1}, \overline{x_2}) \approx \langle\frac{13}{2}, \frac{3}{2}\rangle\overline{x_1} + \langle 4, 0\rangle\overline{x_2}$ Subject to the constraints

$$3\,\overline{x_1} + 2\,\overline{x_2} + \overline{x_3} \approx \langle 145, 5\rangle, 4\,\overline{x_1} + 3\,\overline{x_2} + \overline{x_4} \approx \langle 160, 5\rangle.$$
$$\text{Max } \bar{Z}(\overline{x_1}, \overline{x_2}) \approx \langle\tfrac{13}{2}, \tfrac{3}{2}\rangle\overline{x_1} + \langle 4, 0\rangle\overline{x_2} \text{ Subject to the constraints}$$
$$3\,\overline{x_1} + 2\,\overline{x_2} + \overline{x_3} \approx \langle 145, 5\rangle, 4\,\overline{x_1} + 3\,\overline{x_2} + \overline{x_4} \approx \langle 60, 5\rangle$$

Basic variables $\bar{x}_B^*$	**Coefficients** $\bar{x}_B^*$ $\bar{C}_B^*$	$[5,8] = \langle \frac{13}{2}, \frac{3}{2} \rangle$	$[4,4] = \langle 4, 0 \rangle$	**0**	**0**	**Current values** $\bar{X}_B^*$ $\langle m(\bar{X}_B^*), w(\bar{b}_i) \rangle$
		A_1^*	A_2^*	A_3^*	A_4^*	
$\bar{x}_3$	0	0	$\frac{-1}{4}$	1	$\frac{-3}{4}$	$\langle 25, \frac{3}{2} \rangle$
$\bar{x}_1$	$\frac{13}{2}$	1	$\frac{3}{4}$	0	$\frac{1}{4}$	$\langle 40, \frac{3}{2} \rangle$
$\bar{\Delta}_j^* = \bar{C}_B^* \bar{A}_j^* - \bar{c}_j$		0	$\langle \frac{7}{8}, \frac{3}{2} \rangle$	0	$\langle \frac{13}{8}, \frac{3}{2} \rangle$	$\bar{C}_B^* \bar{X}_B^* = \langle 260, \frac{3}{2} \rangle$

$m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{x}_j^* = [x_j^{*1}, x_j^{*3}]$ with $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ and $\bar{b}_i = [b_i^1, b_i^3]$:

We get the matrix $B_{N^*}^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -3 \\ 0 & 1 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} [140,150] \\ [155,165] \end{pmatrix}$ and $\bar{C}_B^* = (0,[5,8])$, $\begin{pmatrix} \bar{x}_3^* \\ \bar{x}_1^* \end{pmatrix} \approx B_{N^*}^{-1}\bar{b}$ avec $\bar{x}_1^* \approx [\frac{155}{4}, \frac{165}{4}]$, $\bar{x}_2^* \approx \bar{0}$, $\bar{x}_3^* \approx [\frac{65}{4}, \frac{135}{4}]$ and $\bar{x}_4^* \approx \bar{0}$ with

Max $\bar{Z}(\bar{x}^*) \approx \left[\frac{775}{4}, 330\right]$

Step 3 Then the corresponding dual problem (D) is given by:

Min $\bar{W}(\bar{y}_1, \bar{y}_2) \approx [140, 150]\bar{y}_1 + [155, 165]\bar{y}_2$ Subject to the constraints

$3\,\bar{y}_1 + 4\,\bar{y}_2 \geqslant [5, 8]$, $2\,\bar{y}_1 + 3\,\bar{y}_2 \geqslant [4, 4]$ and $\bar{y}_i \in \mathbb{R}$, $i = 1,2$.

We convert the dual linear programming problem involving interval numbers to its canonical form by adding slack variables $\bar{y}_3$ and $\bar{y}_4$ as follows:

Min $\bar{W}(\bar{y}_1, \bar{y}_2) \approx \langle 145, 5\rangle\bar{y}_1 + \langle 160, 5\rangle\bar{y}_2$ Subject to the constraints

$3\,\bar{y}_1 + 4\,\bar{y}_2 - \bar{y}_3 \approx \langle \frac{13}{2}, \frac{3}{2} \rangle$, $2\,\bar{y}_1 + 3\,\bar{y}_2 - \bar{y}_4 \approx \langle 4,0 \rangle$. We have $m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{y}_i^* = [y_i^{*1}, y_i^{*3}]$ with $\bar{y}_{m+j}^* = |\bar{C}_B^* A_j^* - \bar{c}_j|$ and $\bar{y}_i^* = |\bar{C}_B^* A_{n+i}^* - \bar{c}_{n+i}|$ and

$\bar{c}_j = [c_j^1, c_j^3]$. We get $\bar{y}_1^* \approx \bar{0}$, $\bar{y}_2^* \approx [\frac{5}{4}, 2]$, $\bar{y}_3^* \approx \bar{0}$ and $\bar{y}_4^* \approx \left[\frac{-1}{4}, 2\right]$ with

Min $\bar{W}(\bar{y}^*) \approx \left[\frac{775}{4}, 330\right]$.

Step 4 Formulation of a classical primal (P) Linear Programming **(LP)** problem

$$\begin{cases} \text{Max } Z^2 = 6x_1^2 + 4x_2^2 \\ \text{Subject to the constraints} \\ 3x_1^2 + 2x_2^2 + x_3^2 = 150 \\ 4x_1^2 + 3x_2^2 + x_4^2 = 160 \\ \frac{155}{4} \le x_1^2 \le \frac{165}{4} \\ \frac{65}{4} \le x_3^2 \le \frac{135}{4} \\ x_j^2 \ge 0 \end{cases}$$

we get $x_1^{*2} = 40$, $x_2^{*2} = 0$, $x_3^{*2} = 30$, $x_4^{*2} = 0$,

$y_1^{*2} = 0$, $y_2^{*2} = \frac{3}{2}$, $y_3^{*2} = 0$, and $y_4^{*2} = \frac{1}{2}$ with Max Z^{*2} =Min$W^{*2} = 240$.

Step 5 Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

$$\tilde{x}_1^* = (x_1^{*1}, x_1^{*2}, x_1^{*3}) = (x_1^{*2}|\bar{x}_1^*) = \left(40\middle|[\tfrac{155}{4}, \tfrac{165}{4}]\right) = \left(\tfrac{155}{4}, 40, \tfrac{165}{4}\right),$$
$$\tilde{x}_2^* = (x_2^{*1}, x_2^{*2}, x_2^{*3}) = (x_2^{*2}|\bar{x}_2^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{x}_3^* = (x_3^{*1}, x_3^{*2}, x_3^{*3}) = (x_3^{*2}|\bar{x}_3^*) = \left(30\middle|[\tfrac{65}{4}, \tfrac{135}{4}]\right) = \left(\tfrac{65}{4}, 30, \tfrac{135}{4}\right),$$
$$\tilde{x}_4^* = (x_4^{*1}, x_4^{*2}, x_4^{*3}) = (x_4^{*2}|\bar{x}_4^*) = (0|[0,0]) = (0,0,0),$$
$$\text{Max}\tilde{Z} = (Z^{*1}, Z^{*2}, Z^{*3}) = (Z^{*2}|\bar{Z}^*) = \left(240\middle|[\tfrac{775}{4}, 330]\right) = \left(\tfrac{775}{4}, 240, 330\right).$$

(ii) Dual optimal solution:

$$\tilde{y}_1^* = (y_1^{*1}, y_1^{*2}, y_1^{*3}) = (y_1^{*2}|\bar{y}_1^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{y}_2^* = (y_2^{*1}, y_2^{*2}, y_2^{*3}) = (y_2^{*2}|\bar{y}_2^*) = \left(\tfrac{3}{2}\middle|[\tfrac{5}{4}, 2]\right) = \left(\tfrac{5}{4}, \tfrac{3}{2}, 2\right),$$
$$\tilde{y}_3^* = (y_3^{*1}, y_3^{*2}, y_3^{*3}) = (y_3^{*2}|\bar{y}_3^*) = (0|[0,0]) = (0,0,0),$$
$$\tilde{y}_4^* = (y_4^{*1}, y_4^{*2}, y_4^{*3}) = (y_4^{*2}|\bar{y}_4^*) = \left(\tfrac{1}{2}\middle|[\tfrac{-1}{4}, 2]\right) = \left(\tfrac{-1}{4}, \tfrac{1}{2}, 2\right),$$
$$\text{Min}\tilde{W} = (W^{*1}, W^{*2}, W^{*3}) = (W^{*2}|\bar{W}^*) = \left(240\middle|[\tfrac{775}{4}, 330]\right) = \left(\tfrac{775}{4}, 240, 330\right).$$

(iii) Comparisons using Ranking function [17]:

$$\text{Max}\tilde{Z} = \text{Min}\tilde{W} = \left(\tfrac{775}{4}, 240, 330\right) \text{ and } \mathcal{R}(\text{Max}\tilde{Z}) = \mathcal{R}(\text{Min}\tilde{W}) = \tfrac{4015}{16}.$$

Example 4.3. Step 1

Consider the following Triangular Fuzzy Variables Linear Programming (TFVLP) problem: (P)	Then the corresponding dual problem is given by: (D)
Max $\tilde{Z} = 40\tilde{x}_1 + 30\tilde{x}_2$ Subject to the constraints $\tilde{x}_1 + \tilde{x}_2 \preccurlyeq (395, 400, 405)$ $2\tilde{x}_1 + \tilde{x}_2 \preccurlyeq (493, 500, 507)$ where $\tilde{x}_1$ and $\tilde{x}_2$are triangular fuzzy numbers.	Min $\widetilde{W} = (395, 400, 405)y_1 + (493, 500, 507)y_2$ Subject to the constraints $y_1 + 2y_2 \succcurlyeq 40$ $y_1 + y_2 \succcurlyeq 30$ where y_1 and y_2 are real numbers.

Step 2

$$\begin{cases} \text{Max } \tilde{Z} = 40\bar{x}_1 + 30\bar{x}_2 \\ \text{Subject to the contraints} \\ \bar{x}_1 + \bar{x}_2 \preccurlyeq [395, 405] \\ 2\bar{x}_1 + \bar{x}_2 \preccurlyeq [493, 507] \\ \bar{x}_j \in \mathbb{R} \end{cases} \qquad \textbf{(INLP1)}$$

We convert the primal and dual linear programming problem involving interval numbers (INLP2) to its canonical form by adding slack variables $\overline{x_3}$ and $\overline{x_4}$ as follows: Max $\tilde{Z} = 40\bar{x}_1 + 30\bar{x}_2$ Subject to the constraints $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \approx \langle 400, 5\rangle$

$$2\bar{x}_1 + \bar{x}_2 + \bar{x}_4 \approx \langle 500, 7\rangle.$$

Basic variables $\bar{x}_B^*$	Coefficients $\bar{x}_B^*$ $\bar{C}_B^*$	40 A_1^*	30 A_2^*	0 A_3^*	0 A_4^*	Current values $\bar{X}_B^*$ $\langle m(\bar{X}_B^*), w(\bar{b}_i)\rangle$
$\bar{x}_2$	30	0	1	2	-1	$\langle 300,7\rangle$
$\bar{x}_1$	40	1	0	-1	1	$\langle 100,7\rangle$
$\bar{\Delta}_j^* = \bar{C}_B^*\bar{A}_j^* - \bar{c}_j$		0	0	20	10	$\bar{C}_B^*\bar{X}_B^* = \langle 13000,7\rangle$

$m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{x}_j^* = [x_j^{*1}, x_j^{*3}]$ with $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ and $\bar{b}_i = [b_i^1, b_i^3]$:

We get the matrix $B_{N^*}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} [395, 405] \\ [493, 507] \end{pmatrix}$ and $\bar{C}_B^* = (30, 40)$. We have $\begin{pmatrix} \bar{x}_2^* \\ \bar{x}_1^* \end{pmatrix} \approx B_{N^*}^{-1}\bar{b}$ avec $\bar{x}_1^* \approx [88,112]$, $\bar{x}_2^* \approx [283, 317]$, $\bar{x}_3^* \approx \bar{0}$, and $\bar{x}_4^* \approx \bar{0}$ with Max $\bar{Z}(\bar{x}^*) \approx [12010,\ 13990]$.

Step 3 Then the corresponding dual problem (D) is given by:

Min $\bar{w} = [395, 405]y_1 + [493, 507]y_2$

Subject to the constraints $y_1 + 2y_2 \succcurlyeq 40$, $y_1 + y_2 \succcurlyeq 30$ and $y_i \in \mathbb{R}$, $i = 1,2$.

We convert the dual linear programming problem involving interval numbers to its canonical form by adding slack variables y_3 and y_4 as follows:

Min $\bar{w} = \langle 400, 5\rangle y_1 + \langle 500, 7\rangle y_2$ Subject to the constraints $y_1 + 2y_2 - y_3 \approx 40$, $y_1 + y_2 - y_4 \approx 30$, and $y_i \in \mathbb{R}$, $i = 1, \dots, 4$. We have $m(\bar{X}_B^*) > w(\bar{b}_i)$, then $\bar{y}_i^* = [y_i^{*1}, y_i^{*3}]$ with $\bar{y}_{m+j}^* = |\bar{C}_B^*A_j^* - \bar{c}_j|$ and $\bar{y}_i^* = |\bar{C}_B^*A_{n+i}^* - \bar{c}_{n+i}|$ and $\bar{c}_j = [c_j^1, c_j^3]$. We get $y_1^* \approx 20$, $y_2^* \approx 10$, $y_3^* \approx 0$ and $y_4^* \approx 0$ with Min $\overline{W}(y^*) \approx [12830,\ 13170]$.

Step 4 Formulation of a classical primal (P) Linear Programming **(LP)** problem

$$\begin{cases} \text{Max } Z^2 = 40x_1^2 + 30x_2^2 \\ \text{Subject to the constraints} \\ x_1^2 + x_2^2 + x_3^2 = 400 \\ 2x_1^2 + x_2^2 + x_4^2 = 500 \\ 88 \le x_1^2 \le 112 \\ 283 \le x_2^2 \le 317 \\ x_j^2 \ge 0 \end{cases} \qquad \text{we get } x_1^{*2} = 100,\ x_2^{*2} = 300,$$

$x_3^{*2} = 0$, $x_4^{*2} = 0$, $y_1^{*2} = 20$, $y_2^{*2} = 10$, $y_3^{*2} = 0$, and $y_4^{*2} = 0$ with Max Z^{*2} =Min$W^{*2} = 13000$.

Step 5 Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

$$\tilde{x}_1^* = (x_1^{*1}, x_1^{*2}, x_1^{*3}) = (x_1^{*2}|\bar{x}_1^*) = (100|[88,112]) = (88, 100,112),$$
$$\tilde{x}_2^* = (x_2^{*1}, x_2^{*2}, x_2^{*3}) = (x_2^{*2}|\bar{x}_2^*) = (300|[283,317]) = (283, 300,317),$$
$$\tilde{x}_3^* = (x_3^{*1}, x_3^{*2}, x_3^{*3}) = (x_3^{*2}|\bar{x}_3^*) = (0|[0,0]) = (0, 0,0),$$
$$\tilde{x}_4^* = (x_4^{*1}, x_4^{*2}, x_4^{*3}) = (x_4^{*2}|\bar{x}_4^*) = (0|[0,0]) = (0, 0,0),$$
$$\text{Max}\tilde{Z} = (Z^{*1}, Z^{*2}, Z^{*3}) = (Z^{*2}|\bar{Z}^*) = (13000|[12010, 13990]) = (12010,13000,13990).$$

(ii) Dual optimal solution:

$y_1^* = 20, y_2^* = 10, y_3^* = 0, \; y_4^* = 0,$
$\text{Min}\widetilde{W} = (W^{*1}, W^{*2}, W^{*3}) = (W^{*2} | \overline{W}^*) = (13000|[12830, 13170]) = (12830,13000,13170).$
(iii) Comparisons using Ranking function [17]:

$$\mathcal{R}(\text{Max}\tilde{Z}) = \frac{1}{4}(12010 + 26000 + 13990) = 13000 \text{ and}$$

$$\mathcal{R}(\text{Min}\widetilde{W}) = \frac{1}{4}(12830 + 26000 + 13170) = 13000$$

Example 4.4. Step 1

Consider the following Triangular Fuzzy Variables Linear Programming (FVLP) problem: (P)	Then the corresponding dual problem is given by: (D)
Max $\tilde{Z} = 4\tilde{x}_1 + 3\tilde{x}_2$ Subject to the constraints $\tilde{x}_1 + 2\tilde{x}_2 \preccurlyeq (4, 8, 12)$ $2\tilde{x}_1 + \tilde{x}_2 \preccurlyeq (6, 9, 12)$ where $\tilde{x}_1$ and $\tilde{x}_2$ are triangular fuzzy numbers.	Min $\widetilde{W} = (4, 8, 12)y_1 + (6, 9, 12)y_2$ Subject to the constraints $y_1 + 2y_2 \succcurlyeq 4$ $2y_1 + y_2 \succcurlyeq 3$ where y_1 and y_2 are real numbers.

Step 2

$$\begin{cases} \text{Max } \bar{Z}(\bar{x}) = 4\bar{x}_1 + 3\bar{x}_2 \\ \text{Subject to the contraints} \\ \quad \bar{x}_1 + 2\bar{x}_2 \preccurlyeq [4, 12] \\ \quad 2\bar{x}_1 + \bar{x}_2 \preccurlyeq [6, 12] \\ \quad \bar{x}_j \in \mathbb{R} \end{cases} \qquad \textbf{(INLP1)}$$

We convert the primal and dual linear programming problem involving interval numbers (INLP2) to its canonical form by adding slack variables $\overline{x_3}$ and $\overline{x_4}$ as follows: Max $\tilde{Z} = 4\bar{x}_1 + 3\bar{x}_2$ Subject to the constraints $\bar{x}_1 + 2\bar{x}_2 + \bar{x}_3 \approx \langle 8, 4 \rangle,$

$$2\bar{x}_1 + \bar{x}_2 + \bar{x}_4 \approx \langle 9, 3 \rangle.$$

Basic variables $\bar{x}_B^*$	Coefficients $\bar{x}_B^*$ $\bar{C}_B^*$	4 A_1^*	3 A_2^*	0 A_3^*	0 A_4^*	Current values $\bar{X}_B^*$ $\langle m(\bar{X}_B^*), w(\bar{b}_i)\rangle$
$\bar{x}_2$	3	0	1	$\frac{2}{3}$	$\frac{-1}{3}$	$\langle \frac{7}{3}, 4 \rangle$
$\bar{x}_1$	4	1	0	$\frac{-1}{3}$	$\frac{2}{3}$	$\langle \frac{10}{3}, 4 \rangle$
$\bar{\Delta}_j^* = \bar{C}_B^* \bar{A}_j^* - \bar{c}_j$		0	0	$\frac{2}{3}$	$\frac{5}{3}$	$\bar{C}_B^* \bar{X}_B^* = \langle \frac{61}{3}, 4 \rangle$

$m(\bar{X}_B^*) < w(\bar{b}_i)$, then $\bar{x}_j^* = \langle m(\bar{x}_j^*), w(\bar{x}_j^*)\rangle$ with $\bar{x}_B^* \approx B_{N^*}^{-1}\bar{b}$ and $\bar{b}_i = \langle m(\bar{b}_i), w(\bar{b}_i)\rangle$: We get the matrix $B_{N^*}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$
$\bar{b} = \begin{pmatrix} \langle 8,4 \rangle \\ \langle 9,3 \rangle \end{pmatrix}$ and $\bar{C}_B^* = (3, 4)$. We have $\begin{pmatrix} \bar{x}_2^* \\ \bar{x}_1^* \end{pmatrix} \approx B_{N^*}^{-1}\bar{b}$ avec $\bar{x}_1^* \approx \langle \frac{10}{3}, 4 \rangle = [\frac{-2}{3}, \frac{22}{3}]$, $\bar{x}_2^* \approx \langle \frac{7}{3}, 4 \rangle = \left[\frac{-5}{3}, \frac{19}{3}\right]$, $\bar{x}_3^* \approx \bar{0}$, and $\bar{x}_4^* \approx \bar{0}$ with Max $\bar{Z}(\bar{x}^*) \approx \langle \frac{61}{3}, 4 \rangle = [\frac{49}{3}, \frac{73}{3}]$

Step 3 Then the corresponding dual problem (D) is given by:
Min $\widetilde{W} = \langle 8, 4 \rangle y_1 + \langle 9, 3 \rangle y_2$ Subject to the constraints $y_1 + 2y_2 - y_3 \approx 4$, $2y_1 + y_2 - y_4 \approx 3$ and $y_i \in \mathbb{R}$, $i = 1, \ldots, 4$. If $m(\bar{X}_B^*) < w(\bar{b}_i)$, then $\bar{y}_i^* = \langle m(\bar{y}_i^*), w(\bar{y}_i^*)\rangle$ with $\bar{y}_{m+j}^* = |\bar{C}_B^* A_j^* - \bar{c}_j|$ and $\bar{y}_i^* = |\bar{C}_B^* A_{n+i}^* - \bar{c}_{n+i}|$ and $\bar{c}_j = \langle m(\bar{c}_j), w(\bar{c}_j)\rangle$. We get $y_1^* \approx \frac{2}{3}$, $y_2^* \approx \frac{5}{3}$, $y_3^* \approx 0$ and $y_4^* \approx 0$ with Min $\overline{W}(y^*) \approx \langle \frac{61}{3}, 4 \rangle = \left[\frac{49}{3}, \frac{73}{3}\right]$.

Step 4 Formulation of a classical primal (P) Linear Programming **(LP)** problem

$$\begin{cases} \text{Max } Z^2 = 4x_1^2 + 3x_2^2 \\ \text{Subject to the constraints} \\ \quad x_1^2 + 2x_2^2 + x_3^2 = 8 \\ \quad 2x_1^2 + x_2^2 + x_4^2 = 9 \\ \quad \frac{-2}{3} \le x_1^2 \le \frac{22}{3} \\ \quad \frac{-5}{3} \le x_2^2 \le \frac{19}{3} \\ \quad x_j^2 \in \mathbb{R} \end{cases}$$

we have $x_1^{*2} = \frac{10}{3}, x_2^{*2} = \frac{7}{3}, x_3^{*2} = 0, x_4^{*2} = 0,$

$y_1^{*2} = \frac{2}{3}, y_2^{*2} = \frac{5}{3}, y_3^{*2} = 0$, and $y_4^{*2} = 0$ with Max Z^{*2} =Min$W^{*2} = \frac{61}{3}$.

Step 5 Fuzzy optimal solutions and comparisons using Ranking function

(i) Primal optimal solution:

$$\tilde{x}_1^* = (x_1^{*1}, x_1^{*2}, x_1^{*3}) = (x_1^{*2}|\bar{x}_1^*) = \left(\frac{10}{3}\middle|[\frac{-2}{3}, \frac{22}{3}]\right) = \left(\frac{-2}{3}, \frac{10}{3}, \frac{22}{3}\right),$$
$$\tilde{x}_2^* = (x_2^{*1}, x_2^{*2}, x_2^{*3}) = (x_2^{*2}|\bar{x}_2^*) = \left(\frac{7}{3}\middle|[\frac{-5}{3}, \frac{19}{3}]\right) = \left(\frac{-5}{3}, \frac{7}{3}, \frac{19}{3}\right),$$
$$\tilde{x}_3^* = (x_3^{*1}, x_3^{*2}, x_3^{*3}) = (x_3^{*2}|\bar{x}_3^*) = (0|[0,0]) = (0, 0,0),$$
$$\tilde{x}_4^* = (x_4^{*1}, x_4^{*2}, x_4^{*3}) = (x_4^{*2}|\bar{x}_4^*) = (0|[0,0]) = (0, 0,0),$$
$$\text{Max}\tilde{Z} = (Z^{*1}, Z^{*2}, Z^{*3}) = (Z^{*2}|\bar{Z}^*) = \left(\frac{61}{3}\middle|[\frac{49}{3}, \frac{73}{3}]\right) = \left(\frac{49}{3}, \frac{61}{3}, \frac{73}{3}\right).$$

(ii) Dual optimal solution:

$$y_1^* = \frac{2}{3}, y_2^* = \frac{5}{3}, y_3^* = 0, y_4^* = 0.$$
$$\text{Min}\widetilde{W} = (W^{*1}, W^{*2}, W^{*3}) = (W^{*2}|\bar{W}^*) = \left(\frac{61}{3}\middle|[\frac{49}{3}, \frac{73}{3}]\right) = \left(\frac{49}{3}, \frac{61}{3}, \frac{73}{3}\right).$$

(iii) Comparisons using Ranking function [17]:

$$\mathcal{R}(\text{Max}\tilde{Z}) = \mathcal{R}(\text{Min}\widetilde{W}) = \frac{1}{4}\left(\frac{49}{3} + \frac{122}{3} + \frac{73}{3}\right) = \frac{61}{3}.$$

5. Conclusion

We introduced the notation of primal Linear Programming problems with triangular Fuzzy Variables and primal Semi-Fully Fuzzy Linear Programming problems as the way of primal linear programming problems involving interval numbers. We discuss the solution concepts of primal Linear Programming problems with triangular Fuzzy Variables and primal Semi-Fully Fuzzy Linear Programming problems without converting them to classical linear programming problems. Under new arithmetic operations between interval numbers and fuzzy numbers, we have found the optimal solution of dual linear programming problems with triangular fuzzy variables and semi-fully fuzzy linear programming problems.

These results will be useful for fully fuzzy linear programming problems. A numerical example is provided to show that both primal and dual problems have optimal solutions and the two Ranking values are equal.

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