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Stochastic Stability and Analytical Solution with Homotopy Perturbation Method of Multicompartment Non-Linear Epidemic Model with Saturated Rate

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Abstract

In this work, we consider a nonlinear epidemic model with a saturated incidence rate. we consider a population of size N(t) at time t, this population is divided into six subclasses, with $N(t)=S(t)+I(t)+I_1(t)+I_2(t)+I_3(t)+Q(t)$. Where S(t), I(t), $I_1(t)$, $I_2(t)$, $I_3(t)$, and Q(t) denote the sizes of the population susceptible to disease, infectious members, and quarantine members, respectively. We have made the following contributions: 1.The local stabilities of the infection-free equilibrium and endemic equilibrium are; analyzed, respectively. The stability of a disease-free equilibrium and the existence of other nontrivial equilibria can be determined by the ratio called the basic reproductive number. 2. We find the analytical solution of the nonlinear epidemic model by Homotopy perturbation method. 3. Finally the stochastic stabilities. The study of its sections are justified with theorems and demonstrations under certain conditions. In this work, we have used the different references cited in different studies in the three sections already mentioned.

Keywords: Homotopy perturbation method; Incidence rate; Local stability; Nonlinear epidemic model; Stochastic stability.

1. Introduction

This paper considers the following epidemic model with saturated incidence rate.

$$\begin{cases} \dot{S}(t) = \lambda + \nu - \rho - \mu + d \ S(t) - \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)}, \\ \dot{I}(t) = \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)} - \mu_0 + d \ I(t) - \sum_{i=1}^{3} \alpha_i I(t), \\ \dot{I}_1(t) = \alpha_1 I(t) - \mu_1 + d + \gamma_1 \ I_1(t), \\ \dot{I}_2(t) = \alpha_2 I(t) - \mu_2 + d + \gamma_2 \ I_2(t) \\ \dot{I}_3(t) = \alpha_3 I(t) - \mu_3 + d + \gamma_3 \ I_3(t) \\ \dot{Q}(t) = \sum_{i=1}^{3} \gamma_i I_i(t) - (\mu_4 + d) Q(t). \end{cases} \quad (1)$$

- Consider a population of size N(t) at time t, this population is divided into six subclasses, with $N(t)=S(t)+I(t)+I_1(t)+I_2(t)+I_3(t)+Q(t)$. Where S(t), I(t), $I_1(t)$, $I_2(t)$, $I_3(t)$ and Q(t) denote the sizes of the population susceptible to disease, infectious members and quarantine members, respectively.
- The positive constants μ represent rate of incidence. The positive constant β is the average numbers of contacts infective for S to I. The positive constant ν is the parameter of emigration.
- The positive constant r is the parameter of immigration.

- The positive constants γ_1, γ_2, and γ_3, are the numbers of transfer or conversion of infected people quarantined. The positive constant α_1, α_2 and α_3 are the average numbers of contacts for I to $I_{i,\ i=1,2,3}$.
- The positive constants μ, μ_0, μ_1, μ_2, μ_3 and μ_4 represent the death rates of susceptible, infectious and quarantine.
- Biologically, it is natural to assume that $\mu \leq \min\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$. The positive constant d is natural mortality rate.
- The formulation of the incidence rate $\dfrac{S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)}$ which $a_{i,\ i=0,\ 1,\ 2,\ 3}$ is saturated rate with the susceptible.

The initial condition of (1) is givens as:

$$S(\eta) = \Phi_1(\eta),\ I(\eta) = \Phi_2(\eta), I_1(\eta) = \Phi_3(\eta), I_2(\eta) = \Phi_4(\eta)$$
$$I_3(\eta) = \Phi_5(\eta), Q(\eta) = \Phi_6(\eta), -\tau \leq \eta \leq 0. \qquad (2)$$

Where $\Phi = \ \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6\ ^T \in \mathbb{C}$ such that:

$$S(\eta) = \Phi_1(\eta) = \Phi_1(0) \geq 0,\ I(\eta) = \Phi_2(\eta) = \Phi_2(0) \geq 0,$$
$$I_1(\eta) = \Phi_3(\eta) = \Phi_3(0) \geq 0, I_2(\eta) = \Phi_4(\eta) = \Phi_4(0) \geq 0,$$
$$I_3(\eta) = \Phi_5(\eta) = \Phi_5(0) \geq 0, Q(\eta) = \Phi_6(\eta) = \Phi_6(0) \geq 0.$$

Let $\mathbb{C}$ denote the Banach space $\mathbb{C}([-\tau, 0], \mathbb{R}^6)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^6$. With a biological meaning, we further assume that

$\Phi_i(\eta) = \Phi_i(0) \geq 0$, for i =1, 2, 3, 4, 5, 6.

Hence, system (1) rewritten as:

$$\begin{cases} \dot{S}(t) = \lambda + \nu - \rho - \ \mu + d\ S(t) - \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)}, \\ \dot{I}(t) = \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)} - \ \mu_0 + d\ I(t) - \sum_{i=1}^{3} \alpha_i I(t), \\ \dot{I}_i(t) = \alpha_i I(t) - \ \mu_i + d + \gamma_i\ I_i(t), \\ \dot{Q}(t) = \sum_{i=1}^{3} \gamma_i I_i(t) - \ \mu_4 + d\ Q(t). \end{cases} \qquad (3)$$

With the initial conditions (2), where

$\Phi_i(0) \geq 0, -\tau \leq \eta \leq 0$, for i =1, 2, 3, 4, 5, 6. (4)

The region $\Omega = \left\{ \begin{array}{l} S, I, I_1, I_2, I_3, Q\ \in \mathbb{R}^6_+, \\ S + I + I_1 + I_2 + I_3 + Q \leq N < \dfrac{\lambda + \nu - \rho}{\mu + d} \end{array} \right\}$ is positively invariant set of (1).

2. Equilibrium and Local Stability

An equilibrium point of system (3) satisfies.

$$\begin{cases} \lambda + \nu - \rho - \ \mu + d\ S(t) - \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)} = 0, \\ \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)} - \ \mu_0 + d\ I(t) - \sum_{i=1}^{3} \alpha_i I(t) = 0, \\ \alpha_i I(t) - \ \mu_i + d + \gamma_i\ I_i(t) = 0, i=1,2,3, \\ \sum_{i=1}^{3} \gamma_i I_i(t) - \ \mu_4 + d\ Q(t) = 0. \end{cases} \qquad (5)$$

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection, $I = 0; I_i = 0, i = 1, 2, 3.$

The system (3) has a disease-free equilibrium E_0.

$$E_0 = \left(S, \hat{I}, I_1, I_2, I_3, Q\right)^T = \left(\frac{\lambda + \nu - \rho}{\mu + d}, 0, 0, 0, 0, 0\right)^T. \quad (6)$$

Theorem 2.1

The disease-free equilibrium E_0 of the system (3) is locally asymptotically stable if $R_0<1$.

So E^* is the unique positive endemic equilibrium point which exists if $R_0>1$.

Proof

The eigenvalues can be determined by solving the characteristic equation of the linearization of (3) near E_0. Therefore, the eigenvalues are:

$$A_i = -\left(\mu_i + d + \gamma_i\right), i = 1,2,3; A_4 = -\left(\mu + d\right)$$

$$A_5 = -\left(\mu_4 + d\right), A_6 = \beta S - \left(\mu_0 + d + \sum_{i=1}^{3} \alpha_i\right)$$

In order to A_6 will be negative, and then we define the basic reproduction number of the infection R_0 as follows:

$$R_0 = \frac{\beta}{\mu + d} \times \frac{\lambda + \nu - \rho}{\mu_0 + d + \sum_{i=1}^{3} \alpha_i} \quad (7)$$

If $R_0<1$, $A_6<0$.

We have $A_i<0$, i=1, 2, 3, $A_4<0$, $A_5<0$ and $A_6<0$, if $R_0<1$.

Then E_0 of the system (3) is locally asymptotically stable.

In the presence of infection $I \neq 0; I_i \neq 0, i = 1,2,3$, substituting in the system, Ω also contains a unique positive, endemic equilibrium. $E^* = \left(S^*, I^*, I^*_i, Q^*\right)^T, \forall i = 1,2,3.$ Where

$$\begin{cases} S^* = \dfrac{1}{R_0} \times \dfrac{\lambda + \nu - \rho}{\mu + d} \times \left(1 + \left(a_0 + \displaystyle\sum_{i=1}^{3} \frac{a_i \alpha_i}{\mu_i + d + \gamma_i}\right) I^*\right), \\ I^* = \dfrac{-D + \sqrt{\Delta}}{2A}, \Delta = D^2 - 4AC, \\ I^*_i = \left(\dfrac{\alpha_i}{\mu_i + d + \gamma_i}\right) I^*, \forall i = 1,2,3, \\ Q^* = \dfrac{1}{\mu_4 + d} \times \left[\displaystyle\sum_{i=1}^{3} \frac{\gamma_i \alpha_i}{\mu_i + d + \gamma_i}\right] I^*, \\ A = \left(a_0 + \displaystyle\sum_{i=1}^{3} \frac{a_i \alpha_i}{\mu_i + d + \gamma_i}\right) \times \left(\dfrac{\lambda + \nu - \rho}{R_0}\right) \times \left[\dfrac{\beta}{\mu + d} + \displaystyle\sum_{i=1}^{3} \frac{a_i \alpha_i}{\mu_i + d + \gamma_i}\right], \\ D = (\lambda + \nu - \rho)\left[\dfrac{\beta}{R_0} + \left(a_0 + \displaystyle\sum_{i=1}^{3} \frac{a_i \alpha_i}{\mu_i + d + \gamma_i}\right)\left(\dfrac{2}{R_0} - 1\right)\right], \\ C = (\lambda + \nu - \rho) \times \left(\dfrac{2}{R_0} - 1\right). \end{cases} \quad (8)$$

So $E^* = \left(S^*, I^*, I^*_i, Q^*\right)^T, \forall i = 1,2,3.$ is the unique positive endemic equilibrium point which exists if $R_0>1$. W

Theorem 2.2

If $R_0>1$, the system (3) has a unique non-trivial equilibrium E^* which is locally asymptotically stable.

3. Solution of Model by HPM

We define the operator $L = \dfrac{d}{dt}$,

By applying the homotopy perturbation method to system, (3) we obtain the following form:

$$\begin{cases} LS(t) - LS^0(t) = q\left[\lambda + \nu - \rho - \ \mu + d \ S(t) - \dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)} - LS^0(t)\right], \\ LI(t) - LI^0(t) = q\left[\dfrac{\beta S(t)I(t)}{1 + a_0 I(t) + \sum_{i=1}^{3} a_i I_i(t)} - \ \mu_0 + d \ I(t) - \sum_{i=1}^{3} \alpha_i I(t) - LI^0(t)\right], \\ LI_i(t) - LI_i^0(t) = q\left[\alpha_i I(t) - (\mu_i + d + \gamma_i) I_i(t) - LI_i^0(t)\right], i=1,2,3, \\ LQ(t) - LQ^0(t) = q\left[\sum_{i=1}^{3} \gamma_i I_i(t) - (\mu_4 + d) Q(t) - LQ^0(t)\right]. \end{cases} \tag{9}$$

The initial condition is

$$S_0(t) = S(0), I_0(t) = I(0), \ I_{i \ 0}(t) = \ I_i \ (0), Q_0(t) = Q(0), \tag{10}$$

We assume the solution for system (8) in the form

$$\begin{cases} S^* \ t \ = S_0{}^*(t) + pS_1{}^*(t) + p^2 S_2{}^*(t) + \ldots \\ I^* \ t \ = I_0{}^*(t) + pI_1{}^*(t) + p^2 I_2{}^*(t) + \ldots \\ I_i^* \ t \ = \ I_i{}^*{}_0(t) + p \ I_i{}^*{}_1(t) + p^2 \ I_i{}^*{}_2(t) + \ldots \\ Q^* \ t \ = Q_0{}^*(t) + pQ_1{}^*(t) + p^2 Q_2{}^*(t) + \ldots \end{cases} \tag{11}$$

Using (10) in (8), and comparing the coefficients of the same power, then we obtain

$$\begin{cases} LS(t) - LS^0(t) = 0, \\ LI(t) - LI^0(t) = 0, \\ LI_i(t) - LI_i^0(t) = 0, i=1,2,3, \\ LQ(t) - LQ^0(t) = 0. \end{cases} \tag{12}$$

Moreover, we have

$$\begin{cases} LS^*{}_1(t) = \lambda + \nu - \rho - \ \mu + d \ S^*{}_0(t) - \dfrac{\beta S^*{}_0(t) I^*{}_0(t)}{1 + a_0 I^*{}_0(t) + \sum_{i=1}^{3} a_i \ I_i{}^*{}_0(t)} - LS^*{}_0(t), \\ LI^*{}_1(t) = \dfrac{\beta S^*{}_0(t) I^*{}_0(t)}{1 + a_0 I^*{}_0(t) + \sum_{i=1}^{3} a_i \ I_i{}^*{}_0(t)} - \ \mu_0 + d \ I^*{}_0(t) - \sum_{i=1}^{3} \alpha_i I^*{}_0(t) - LI^*{}_0(t), \\ L \ I_i{}^*{}_1(t) = \alpha_i I^*{}_0(t) - \ \mu_i + d + \gamma_i \ \ I_i{}^*{}_0(t) - L \ I_i{}^*{}_0(t), \ i=1,2,3, \\ LQ^*{}_1(t) = \sum_{i=1}^{3} \gamma_i (I_i)^*{}_0(t) - (\mu_4 + d) Q^*{}_0(t) - LQ^*{}_0(t). \end{cases} \tag{13}$$

With the conditions

$$S^*{}_1(t) = 0, I^*{}_1(t) = 0, \ I_i{}^*{}_1(t) = 0, Q^*{}_1(t) = 0, \tag{14}$$

We have

$$\begin{cases} LS^*_2(t) = -(\mu+d)S^*_1(t) - \dfrac{\beta\left(S^*_0(t)I^*_1(t)+S^*_1(t)I^*_0(t)\right)}{1+a_0I^*_1(t)+\sum_{i=1}^{3}a_i\left(I_i\right)^*_1(t)}, \\ LI^*_2(t) = \dfrac{\beta\left(S^*_0(t)I^*_1(t)+S^*_1(t)I^*_0(t)\right)}{1+a_0I^*_1(t)+\sum_{i=1}^{3}a_i\left(I_i\right)^*_1(t)} - (\mu_0+d)I^*_1(t) - \sum_{i=1}^{3}\alpha_i I^*_1(t), \\ L\left(I_i\right)^*_2(t) = \alpha_i I^*_1(t) - (\mu_i+d+\gamma_i)\left(I_i\right)^*_1(t),\ i=1,2,3, \\ LQ^*_2(t) = \sum_{i=1}^{3}\gamma_i(I_i)^*_1(t) - (\mu_4+d)Q^*_1(t). \end{cases} \tag{15}$$

With the conditions

$$S^*_2(t)=0, I^*_2(t)=0, \left(I_i\right)^*_2(t)=0, Q^*_2(t)=0, \tag{16}$$

Moreover, we have

$$\begin{cases} LS^*_3(t) = -(\mu+d)S^*_2(t) - \dfrac{\beta\left(S^*_0(t)I^*_2(t)+S^*_1(t)I^*_1(t)+S^*_2(t)I^*_0(t)\right)}{1+a_0I^*_2(t)+\sum_{i=1}^{3}a_i\left(I_i\right)^*_2(t)}, \\ LI^*_3(t) = \dfrac{\beta\left(S^*_0(t)I^*_2(t)+S^*_1(t)I^*_1(t)+S^*_2(t)I^*_0(t)\right)}{1+a_0I^*_2(t)+\sum_{i=1}^{3}a_i\left(I_i\right)^*_2(t)} - (\mu_0+d)I^*_2(t) - \sum_{i=1}^{3}\alpha_i I^*_2(t), \\ L\left(I_i\right)^*_3(t) = \alpha_i I^*_2(t) - (\mu_i+d+\gamma_i)(I_i)^*_2(t),\ i=1,2,3, \\ LQ^*_3(t) = \sum_{i=1}^{3}\gamma_i(I_i)^*_2(t) - (\mu_4+d)Q^*_2(t). \end{cases} \tag{17}$$

We suppose that p=1 in (10), we have the solution is in the form

$$\begin{cases} S^*(t) = S_0^*(t) + S_1^*(t) + S_2^*(t) + \dots \\ I^*(t) = I_0^*(t) + I_1^*(t) + I_2^*(t) + \dots \\ I_i^*(t) = \left(I_i\right)^*_0(t) + \left(I_i\right)^*_1(t) + \left(I_i\right)^*_2(t) + \dots \\ Q^*(t) = Q_0^*(t) + Q_1^*(t) + Q_2^*(t) + \dots \end{cases} \tag{18}$$

Finally the solution is

1. The zero order solution

$$S^*_0(t)=c_1, I^*_0(t)=c_2, \left(I_i\right)^*_0(t)=c_3, Q^*_0(t)=c_4, \tag{19}$$

2. The first order solution

$$\begin{cases} S^*_1(t) = \left[\lambda+\nu-\rho-(\mu+d)c_1 - \dfrac{\beta c_1c_2}{1+a_0c_2+\sum_{i=1}^{3}a_ic_3}\right]t, \\ I^*_1(t) = \left[\dfrac{\beta c_1c_2}{1+a_0c_2+\sum_{i=1}^{3}a_ic_3} - (\mu_0+d)c_2 - \sum_{i=1}^{3}\alpha_ic_2\right]t, \\ \left(I_i\right)^*_1(t) = \left[\alpha_ic_2 - (\mu_i+d+\gamma_i)c_3\right]t,\ i=1,2,3, \\ Q^*_1(t) = \left[\sum_{i=1}^{3}\gamma_ic_3 - (\mu_4+d)c_4\right]t. \end{cases} \tag{20}$$

3. The second order solution

$$
\begin{cases}
S^{*}_{2}(t) = -(\mu + d)\left[\left(\lambda + \nu - \rho - c_1\left(\mu + d + \dfrac{\beta c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3}\right)\right)t\right] \\
\quad - \dfrac{\beta\left\{ c_1\left[\left(\dfrac{\beta c_1 c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3} - \left(\mu_0 + d + \sum_{i=1}^{3}\alpha_i\right)c_2\right)t\right] + \left[\left(\lambda + \nu - \rho - c_1\left(\mu + d + \dfrac{\beta c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3}\right)\right)t\right]c_2 \right\}}{1 + a_0\left[\left(\dfrac{\beta c_1 c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3} - \left(\mu_0 + d + \sum_{i=1}^{3}\alpha_i\right)c_2\right)t\right] + \sum_{i=1}^{3} a_i\left[\left(\alpha_i c_2 - (\mu_i + d + \gamma_i)c_3\right)t\right]}, \\
I^{*}_{2}(t) = \dfrac{\beta\left\{ c_1\left[\left(\dfrac{\beta c_1 c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3} - \left(\mu_0 + d + \sum_{i=1}^{3}\alpha_i\right)c_2\right)t\right] + \left[\left((\lambda + \nu - \rho) - c_1\left(\mu + d + \dfrac{\beta c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3}\right)\right)t\right]c_2 \right\}}{1 + a_0\left[\left(\dfrac{\beta c_1 c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3} - \left(\mu_0 + d + \sum_{i=1}^{3}\alpha_i\right)c_2\right)t\right] + \sum_{i=1}^{3} a_i\left[\left(\alpha_i c_2 - (\mu_i + d + \gamma_i)c_3\right)t\right]} \\
\quad - \left(\mu_0 + d + \sum_{i=1}^{3}\alpha_i\right)\left[\left(\dfrac{\beta c_1 c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3} - (\mu_0 + d)c_2(t) - \sum_{i=1}^{3}\alpha_i c_2\right)t\right], \\
(I_i)^{*}_{2}(t) = \alpha_i\left[\left(\dfrac{\beta c_1 c_2}{1 + a_0 c_2 + \sum_{i=1}^{3} a_i c_3} - (\mu_0 + d)c_2(t) - \sum_{i=1}^{3}\alpha_i c_2\right)t\right] \\
\quad -(\mu_i + d + \gamma_i)\left[\left(\alpha_i c_2 - (\mu_i + d + \gamma_i)c_3\right)t\right], \ i=1,2,3, \\
Q^{*}_{2}(t) = \sum_{i=1}^{3}\gamma_i\left[\left(\alpha_i c_2 - (\mu_i + d + \gamma_i)c_3\right)t\right] - (\mu_4 + d)\left[\left(\sum_{i=1}^{3}\gamma_i c_3 - (\mu_4 + d)c_4\right)t\right].
\end{cases}
\tag{21}
$$

4. Stochastic Stability

The system (3), is transformed to the Itô Stochastic differential equations. We replace β by β+ab (t) where b (t) is white noise.

$$
\begin{cases}
dS = \left[\lambda + \nu - \rho - (\mu + d)S - \dfrac{\beta SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i}\right] dt - a\dfrac{\beta SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i} db, \\
dI = \left[\dfrac{\beta SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i} - (\mu_0 + d)I - \sum_{i=1}^{3} \alpha_i I\right] dt + a\dfrac{\beta SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i} db, \\
dI_i = \left[\alpha_i I - (\mu_i + d + \gamma_i) I_i\right] dt, \\
dQ = \left[\sum_{i=1}^{3} \gamma_i I_i - (\mu_4 + d) Q\right] dt.
\end{cases}
\tag{22}
$$

Theorem 4.1

If $\beta^2 - 2a^2(\mu + d) < 0$, S(t) converge exponentially almost surely to $\frac{\lambda + \nu - \rho}{\mu + d}$.

Proof

We use Itô formula to the first equation in system (21), we obtain

$$
d\log\left|S - \frac{\lambda+\nu-\rho}{\mu-d}\right| = \left[\frac{\lambda+\nu-\rho}{S - \frac{\lambda+\nu-\rho}{\mu+d}} - (\mu+d)\frac{S}{S - \frac{\lambda+\nu-\rho}{\mu+d}} - \frac{\frac{\beta SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i}}{S - \frac{\lambda+\nu-\rho}{\mu+d}} - \frac{1}{2}\times\left(\frac{a}{S - \frac{\lambda+\nu-\rho}{\mu-d}} \times \frac{SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i}\right)^2\right] dt
$$

$$
- \left[\frac{SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i} \times \frac{a}{S - \frac{\lambda+\nu-\rho}{\mu+d}}\right] db
$$

$$
d\log\left|S - \frac{\lambda+\nu-\rho}{\mu+d}\right| \le \left[-(\mu+d) - \frac{\beta SI}{\left(S - \frac{\lambda+\nu-\rho}{\mu+d}\right)\left(1 + a_0 I + \sum_{i=1}^{3} a_i I_i\right)} - \frac{1}{2}\times\left(\frac{a}{S - \frac{\lambda+\nu-\rho}{\mu+d}} \times \frac{SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i}\right)^2\right] dt
$$

$$
- \left[\frac{SI}{1 + a_0 I + \sum_{i=1}^{3} a_i I_i} \times \frac{a}{S - \frac{\lambda+\nu-\rho}{\mu+d}}\right] db
$$

We suppose that

$$F(x) = -\frac{1}{2}a^2x^2 - \beta x - \mu + d \;, x = \frac{SI}{\left(S - \frac{\lambda + \nu - \rho}{\mu + d}\right)\left(1 + a_0 I + \sum_{i=1}^{8} a_i I_i\right)} \tag{23}$$

If the determinant of the equation is negative, then for all x.

$$F(x) \leq \frac{\Delta}{a^2} dt, \text{With } \Delta = \beta^2 - 2a^2 \;\; \mu + d \tag{24}$$

We have

$$d\log\left(S - \frac{\lambda + \nu - \rho}{\mu + d}\right) \leq \frac{\Delta}{a^2} dt - \left| \frac{SI}{1 + a_0 I + \sum_{i=1}^{8} a_i I_i} \times \frac{a}{S - \frac{\lambda + \nu - \rho}{\mu + d}} \right| \text{db}. \tag{25}$$

With integration, we obtain

$$d\log\left(S - \frac{\lambda + \nu - \rho}{\mu + d}\right) \leq \frac{\Delta}{a^2} dt - a\int_0^t \frac{S(v)I(v)}{1 + a_0 I(v) + \sum_{i=1}^{8} a_i I_i(v)} \times \frac{1}{S(v) - \frac{\lambda + \nu - \rho}{\mu + d}} db(v). \tag{26}$$

Since

$$\lim_{t\to\infty} \int_0^t \frac{S(v)I(v)}{1 + a_0 I(v) + \sum_{i=1}^{8} a_i I_i(v)} \times \frac{1}{S(v) - \frac{\lambda + \nu - \rho}{\mu + d}} db(v) = 0,$$

, almost surely.

Therefore

$$\lim_{t\to\infty} \sup \frac{1}{t} \log\left(S - \frac{\lambda + \nu - \rho}{\mu + d}\right) \leq \frac{\Delta}{a^2}$$

, almost surely.

S(t) is exponentially almost stable. □

5. Conclusion

This paper addresses a the equilibrium and local stability of the epidemic model with saturated incidence rate, in the absence of infection, the system has a disease-free equilibrium, in the presence of infection the system, has a unique positive, endemic equilibrium. Then we applied the Homotopy perturbation method, we obtained The zero, first and second order solutions.

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