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Exponentiated Cubic Transmuted Weibull Distribution: Properties and Application

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Abstract

This work is focused on the four parameters Exponentiated Cubic Transmuted Weibull distribution which mostly found its application in reliability analysis most especially for data that are non-monotone and Bi-modal. Structural properties such as moment, moment generating function, Quantile function, Renyi entropy, and order statistics were investigated. Maximum likelihood estimation technique was used to estimate the parameters of the distribution. Application to two real life data sets shows the applicability of the distribution in modeling real data.

Keywords: Exponentiated cubic transmuted distribution; Reliability analysis; Moment; Likelihood estimation; Moment generating function.

1. Introduction

Statistical distributions are very useful tools in analyzing and predicting real-life phenomena. In recent times, several distributions have been suggested and studied. There is always new ground for development in statistical distributions to blend with the current situations which allows for wider applications that can be achieved by inducing flexibility into the standard probability distribution to allow for fitting specific real-world scenarios. This has served as a motivating factor for many researchers to work towards developing new and more flexible distributions. There are several ways to extend standard probability distributions, and one of the most popular methods is the use of distribution generators such as the exponentiated method by Lehmann [1]; the Marshall-Olkin method developed by Marshall and Olkin [2]; the beta distribution method proposed by Alexander, *et al.* [3] and Eugene, *et al.* [4]; the gamma distribution method by Cordeiro, *et al.* [5], Ristic and Balakrishnan [6], and Zografos and Balakrishnan [7]; the McDonald method proposed and studied by MacDonald [8]; and the exponentiated generalized method developed by De Andrade, *et al.* [9]. The cubic rank transmutation map was proposed and studied by Granzotto, *et al.* [10]. The Weibull distribution mostly used for modeling lifetime data and phenomenon with monotone failure rates. However, in real life problems one may encounter real life data which is bimodal and also exhibits non-monotone failure rate which the Weibull distribution does not provide a reasonable parametric fit. Then the need to extend the Weibull distribution is of interest to allow for a wider class of applications. Several variant of Weibull distribution have been proposed and discuss in literature and these includes: Cubic Transmuted Weibull (CTW) distribution was studied by Abed Al-Kadim [11] the Additive Weibull (AW) distribution was developed and studied by Xie and Lai [12], Afify, *et al.* [13] studied the Transmuted Complementary Weibull Geometric (TWG) distribution. Transmuted Modified Weibull (TMW) distribution was studied by Khan and King [14], developed and studied the Modified Weibull (MW) distribution, the Transmuted Generalised Inverse Weibull (TGIW) distribution was studied by Merovci, *et al.* [15], the Beta Transmuted Weibull (BTW) distributionwas proposed and studied by Pal and Tiensuwan [16], the beta modified Weibull (BMW) distribution by Silva, *et al.* [17], Elbatal and Aryal [18] proposed and studied the transmuted additive Weibull distribution etc. The main focus of this study is to redefine the Weibull distribution to obtain Exponentiated Cubic Transmuted Weibull distribution which a more flexible distribution and that it can be used to model real life data even those that possesses bi-modal property.

A random variable X is said to follow a Weibull distribution if its cumulative distribution function (cdf) is given by

$$G(x) = 1 - e^{-\alpha x^{\beta}}, \;\; x \in [0, \infty) \qquad (1)$$

And the probability density function (pdf) given as

$$g(x) = \alpha x^{\beta-1} e^{-\alpha x^{\beta}}, \;\; x \in [0, \infty) \qquad (2)$$

With shape parameter α and the scale parameter β.

1.1. Cubic Transmuted Family of Distributions

Rahman, *et al.* [19], proposed an extension of the quadratic transmuted distributions called the cubic transmuted family of distribution meant to address the problem of bi-modality of the data which the quadratic family of distribution is not capable of handling. This was done by adding one or more parameter in (refqrtm) to the transmuted family of distribution. The cdf of cubic transmuted family of distribution is given by

$$F(x) = G(x) + \lambda_1 G(x)[1 - G(x) + \lambda_2 G^2(x)(1 - G(x))],$$

Equivalently, it can be written as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x) \quad (3)$$

Where, $\lambda_1 \in [-1,1], \lambda_2 \in [-1,1]$, and $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

Al-kadim and Mohammed (2017), proposed and studied a case of (3) by letting $\lambda_2 = -\lambda_1 = \lambda$ in (3). The cdf is given by

$$F(x) = (1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x), \quad |\lambda| \leq 1 \quad (4)$$

The density function corresponding to (4), is given by

$$f(x) = g(x)[(1 + \lambda) - 4\lambda G(x) + 3\lambda G^2(x)], \quad |\lambda| \leq 1 \quad (5)$$

In this article, we redefine the Cubic transmuted family of distributions proposed and studied by Abed Al-Kadim [11] using the cdf of Exponentiated generalised-G (EXG-G) family proposed by Cordeiro, *et al.* [5]. The cdf of EXG-G class is given by

$$W(x; a, b) = 1 - \left[1 - F(x)^{\theta}\right]^{b} \quad (6)$$

Where a and θ are positive shape parameters. Taking $b = 1$ in (6), we have another class of EXP-G family called the Lehmann type 1 which cdf is given by

$$W(x) = [F(x)]^{\theta} \quad (7)$$

And the corresponding pdf is given by

$$w(x) = \theta f(x)[F(x)]^{\theta - 1} \quad (8)$$

1.2. Generalized Exponentiated Cubic Transmuted Family of Distributions

Another family of distribution can be derived called the Exponentiated Cubic transmuted family of distribution by Putting (4) in (7). The cdf Generalized Exponentiated Cubic Transmuted (GECT) family of distribution given by

$$W(x) = [(1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta} \quad (9)$$

And the associated pdf corresponding to (9) is given by

$$w(x) = \theta g(x)[(1 + \lambda) - 4\lambda G(x) + 3\lambda G^2(x)][(1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta - 1} \quad (10)$$

The new added shape parameter θ is to further furnish/induce flexibility into the Cubic transmuted family of distribution. The hazard function $(hrf)$for the GECT family can be obtained by

$$h(x) = \frac{\theta g(x)[(1 + \lambda) - 4\lambda G(x) + 3\lambda G^2(x)][(1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta - 1}}{1 - [(1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta}} \quad (11)$$

1.3. Expansion for the PDF and the CDF

To simplify the expression given in (10), we use the Binomial series expansion given by

$$(1 + x)^p = \sum_{i=0}^{\infty} \binom{p}{i} x^p \quad (12)$$

$$[(1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta - 1} = G^{\theta - 1}(x)[(1 + \lambda) - 2\lambda G(x) + \lambda G^2(x)]^{\theta - 1} \quad (13)$$

Applying the binomial series expansion given (12) to (13), we have

$$[(1 + \lambda) - 2\lambda G(x) + \lambda G^2(x)]^{\theta - 1} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} \binom{\theta - 1}{i}\binom{i}{j}\binom{j}{k}(-2)^j \lambda^i \left(\frac{1}{2}\right)^k [G(x)]^{j+k}$$

Consequently,

$$[(1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta - 1} = M_{ijk}^{\theta}[G(x)]^{\theta + j + k - 1}$$

Where,

$$M_{ijk}^{\theta} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} \binom{\theta - 1}{i}\binom{i}{j}\binom{j}{k}(-2)^j \lambda^i \left(\frac{1}{2}\right)^k$$

Finally, the pdf of GECT family of distribution can be represented as

$$w(x) = \theta(1 + \lambda)M_{ijk}^{\theta} g(x)[G(x)]^{\theta + j + k - 1} - 4a\lambda M_{ijk}^{\theta} g(x)[G(x)]^{\theta + j + k}$$
$$+3a\lambda M_{ijk}^{\theta} g(x)[G(x)]^{\theta + j + k + 1} \quad (14)$$

2. Exponentiated Cubic Transmuted Weibull Distribution

By putting (1) in (9), we obtain the cdf of the Exponentiated Cubic Transmuted Weibull$(ECTW)$ distribution given by

$$W(x) = \left[(1 + \lambda)(1 - e^{-\alpha x^{\beta}}) - 2\lambda(1 - e^{-\alpha x^{\beta}})^2(x) + \lambda(1 - e^{-\alpha x^{\beta}})^3(x)\right]^{\theta} \quad (15)$$

And its associated pdf is given as

$$w(x) = \theta\alpha\beta x^{\beta} e^{-\alpha x^{\beta}} \left(1 + \lambda - 4\lambda(1 - e^{-\alpha x^{\beta}}) + 3\lambda(1 - e^{-2\alpha x^{\beta}})^2\right)$$
$$\times \left[(1+\lambda)\left(1 - e^{-\alpha x^{\beta}}\right) - 2\lambda(1 - e^{-\alpha x^{\beta}})^2 + \lambda(1 - e^{-\alpha x^{\beta}})^3\right]^{\theta-1} \quad (16)$$

Corresponding, using the property given in (14), we can re-write (16) as

$$w(x) = \theta(1+\lambda)M_{ijk}^{\theta}\alpha\beta x^{\beta} e^{-\alpha x^{\beta}} \left[1 - e^{-\alpha x^{\beta}}\right]^{\theta+j+k-1} - 4a\lambda M_{ijk}^{\theta}\alpha\beta x^{\beta} e^{-\alpha x^{\beta}} \left[1 - e^{-\alpha x^{\beta}}\right]^{\theta+j+k}$$
$$+3a\lambda M_{ijk}^{\theta}\alpha\beta x^{\beta} e^{-\alpha x^{\beta}} \left[1 - e^{-\alpha x^{\beta}}\right]^{\theta+j+k+1} \quad (17)$$

Andthe expressions for the Survival and Hazard function are respectively given by

$$S(x) = 1 - \left[(1+\lambda)\left(1 - e^{-\alpha x^{\beta}}\right) - 2\lambda(1 - e^{-\alpha x^{\beta}})^2 + \lambda(1 - e^{-\alpha x^{\beta}})^3\right]^{\theta} \quad (18)$$

and

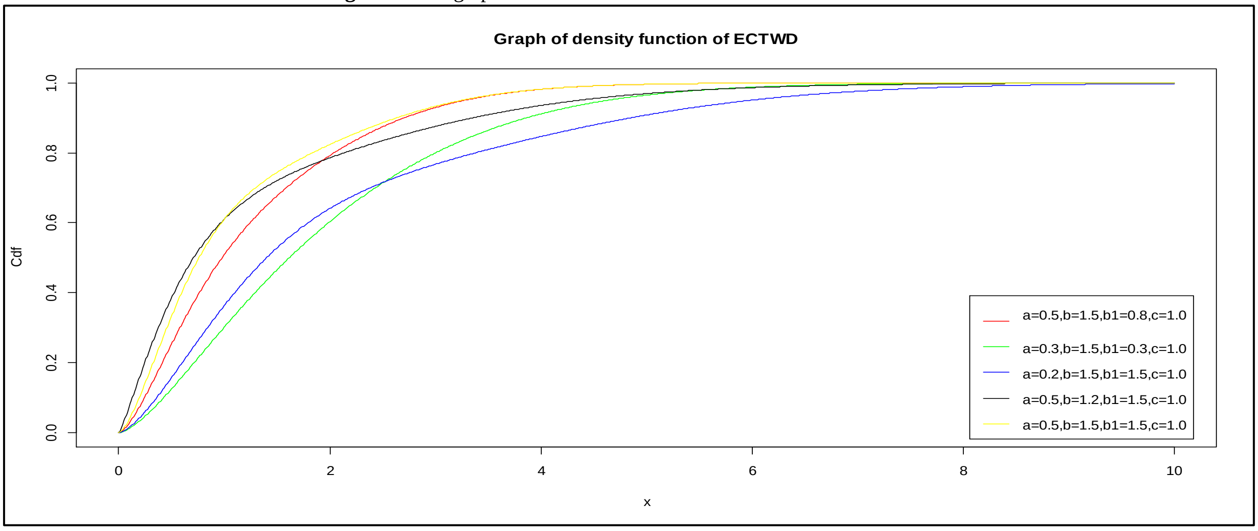
$$h(x) = \frac{H^{*}\left[(1+\lambda)\left(1 - e^{-\alpha x^{\beta}}\right) - 2\lambda(1 - e^{-\alpha x^{\beta}})^2 + \lambda(1 - e^{-\alpha x^{\beta}})^3\right]^{\theta-1}}{1 - \left[(1+\lambda)e^{-\alpha x^{\beta}} - 2\lambda(1 - e^{-\alpha x^{\beta}})^2 + \lambda(1 - e^{-\alpha x^{\beta}})^3\right]^{\theta}} \quad (19)$$

where

$$H^{*} = \theta\alpha\beta x^{\beta} e^{-\alpha x^{\beta}} \left(1 + \lambda - 4\lambda\left(1 - e^{-\alpha x^{\beta}}\right) + 3\lambda\left(1 - e^{-\alpha x^{\beta}}\right)^2\right)$$

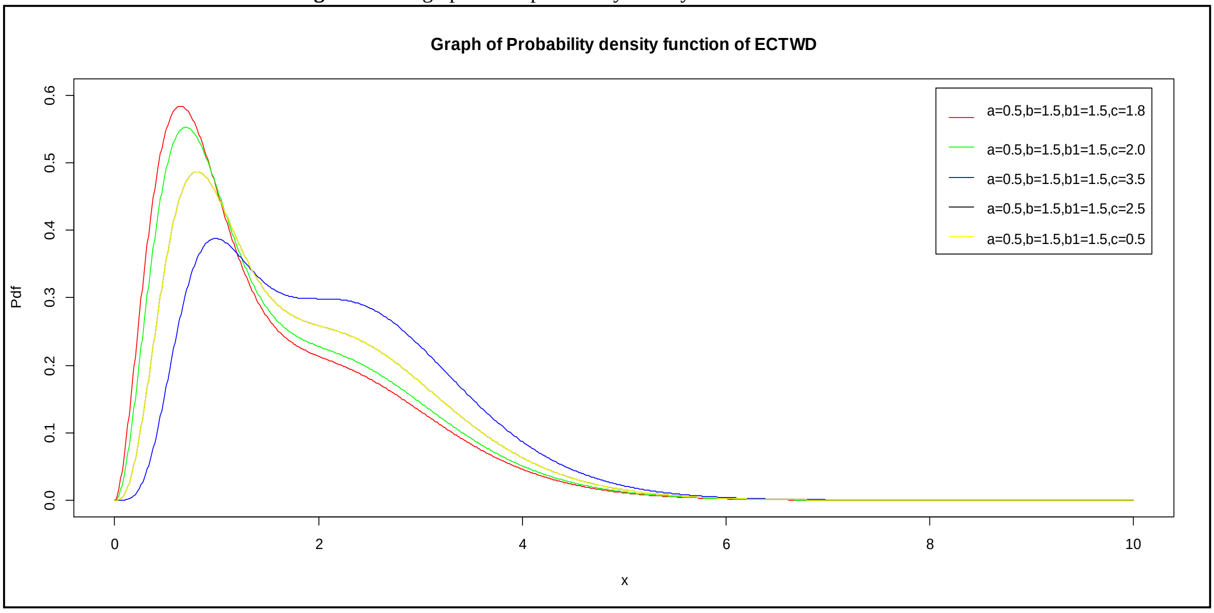
The graph of the distribution, density, survival and the hazard functions are given below in figure (1, 2, 3 and 4).

Figure-1. The graph of the distribution function of $ECTW$ distribution



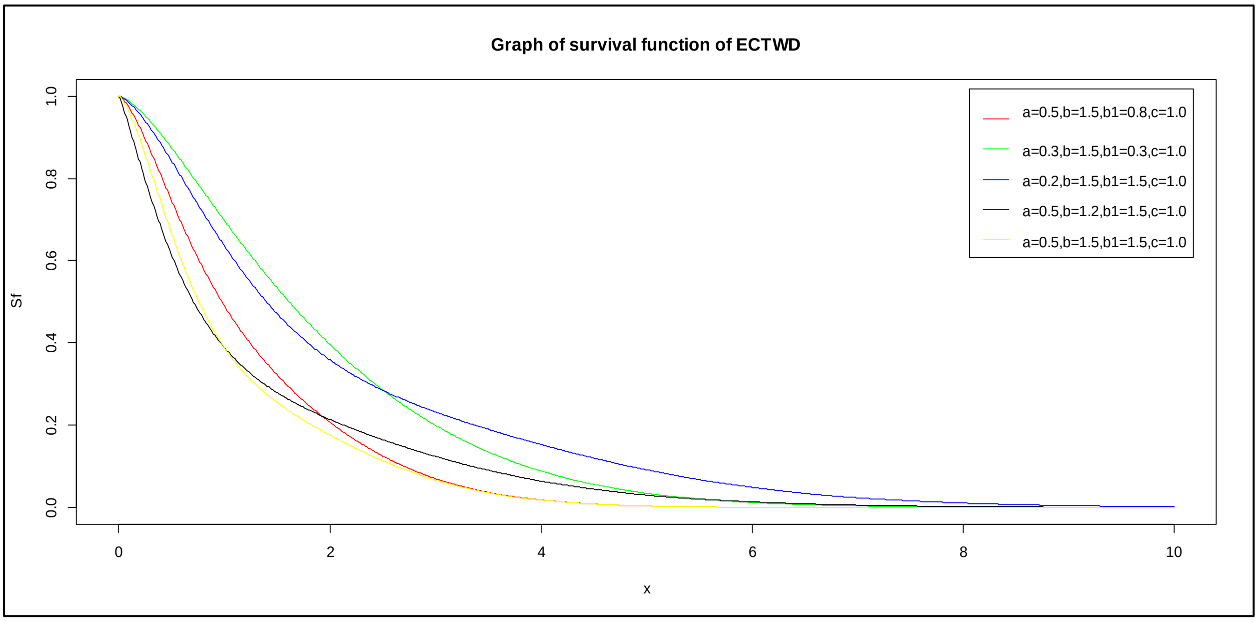
- ✓ The figure 1 drawn above indicates that the $ECTW$ distribution has a proper probability density function.

Figure-2. The graph of the probability density function of $ECTW$ distribution



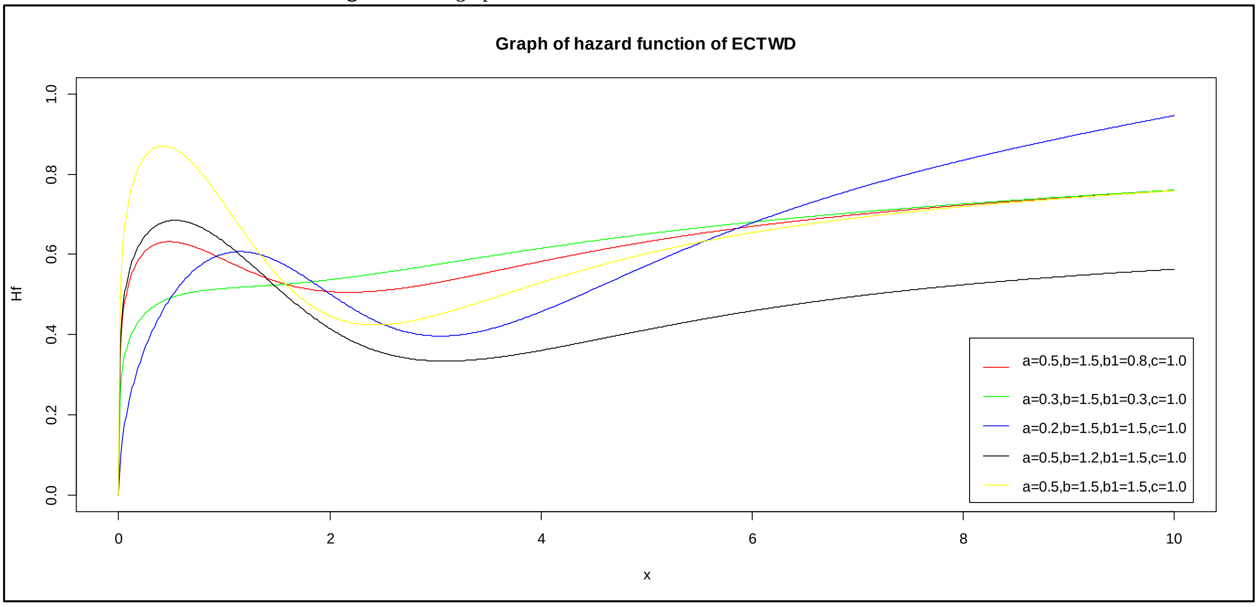
- ✓ Figure 2 drawn above indicates that the pdf of $ECTW$ distribution is positively skewed.

Figure-3. The graph of the Survival function of the $ECTW$ distribution



- ✓ Figure 3 drawn above indicates that the survival probability of $ECTW$ distribution decreases as survival time increases.

Figure-4. The graph of the Hazard function of the ECTW distribution



- ✓ Figure 4 drawn above indicates that the hazard function of ECTW distribution is increasing, decreasing, constant and reversible.
- ✓ This property enables the distribution to be used to model real life data most especially those that possesses non-monotone failure rate (Bathtub- shape failure rate).

3. Statistical Properties of ECTW Distribution

In this section, emphases are on some statistical properties of Exponentiated Cubic Transmuted Weibull distribution given in (16). These properties include moment, moment generating function, quantile function, Random number generation and Renyl entropy.

3.1. Moment

In statistical analysis, moments are very useful tool in describing certain properties of the distributions. An expression for raw moment of exponentiated cubic transmuted Weibull distribution is given in the following Lemma.

Lemma 1 Suppose that the random variable X follows the Exponentiated cubic transmuted Weibull distribution, then the r^{th} moment of X is given as

$$E(X^r) = \theta M_{ijk}^{\theta} \alpha^{-\frac{\alpha}{\beta}} \Gamma\left(1+\frac{r}{\beta}\right)\left\{(1+\lambda)\sum_{m=0}^{\infty}(-1)^m\binom{\theta+j+k-1}{m}(m+1)^{-\left(\frac{r+\beta}{\beta}\right)} - 4\lambda\sum_{l=0}^{\infty}(-1)^l\binom{\theta+j+k}{l}(l+1)^{-\left(\frac{r+\beta}{\beta}\right)} + 3\lambda\sum_{p=0}^{\infty}(-1)^p\binom{\theta+j+k-1}{p}(p+1)^{-\left(\frac{r+\beta}{\beta}\right)}\right\} \quad (20)$$

Proof. The r^{th}moment of a distribution is given as

$$E(X^r) = \int_{-\infty}^{\infty} x^r w(x)dx \quad (21)$$

Using the density function of exponentiated cubic transmuted Weibull distribution, given in (17), in (21) and by simplifying the expression we obtain the rth moment of exponentiated cubic transmuted Weibull distribution given in (20). The mean and the variance can easily be obtained by taking $r = 1,2$ in (20).

3.2. Moment Generating Function

Moment generating function is a very useful function that can be used to describe certain properties of the distribution. It can be used to obtain moments of a distribution. The moment generating function of exponentiated cubic transmuted Weibull distribution is given in the following lemma.

Lemma 2. Let X follows the exponentiated cubic transmuted Weibull distribution, then the moment generating function, $M_X(t)$ is

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\theta M_{ijk}^{\theta} \alpha^{-\frac{\alpha}{\beta}} \Gamma\left(1+\frac{r}{\beta}\right)\left\{(1+\lambda)\sum_{m=0}^{\infty}(-1)^m\binom{\theta+j+k-1}{m}(m+1)^{-\left(\frac{r+\beta}{\beta}\right)} - 4\lambda\sum_{l=0}^{\infty}(-1)^l\binom{\theta+j+k}{l}(l+1)^{-\left(\frac{r+\beta}{\beta}\right)} + 3\lambda\sum_{p=0}^{\infty}(-1)^p\binom{\theta+j+k-1}{p}(p+1)^{-\left(\frac{r+\beta}{\beta}\right)}\right\}\right] \quad (22)$$

Where $t \in \mathbb{R}$

Proof the moment generating function of a random variable X is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} w(x)dx \quad (23)$$

Where $w(x)$ is given in (17). Using series expansion for e^{tX} given by

$$e^{tX} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r \quad (24)$$

Using (24), we can re-write equation (23) as follows

$$M_X(t) = \int_{-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x)dt = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r) \quad (25)$$

Using $E(X^r)$ from (20) in (25), we have (22).

It can be observed from the series expansion of (25) that moments are the coefficients of $\frac{t^r}{r!}$.

Table 1 below gives the first four moment of $ECTW$ distribution, Variance(μ_2), Coefficient of Skewness(S_k) and Coefficient of Kurtosis (K_u)for arbitrary values of the parameters of $ECTW$ distribution taking a fixed value of $\lambda = 0.9$ and $\theta = 1.5$.

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$S_k = \frac{\mu_3}{(\sigma^2)^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^2}{\sqrt[\frac{3}{2}]{\mu_2' - (\mu_1')^2}}$$

$$K_u = \frac{\mu_4}{(\sigma^2)^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3(\mu_1')^2}{\{\mu_2' - (\mu_1')^2\}^2}$$

Table-1. the first four moments, Skewness and Kurtosis of ECTW distribution

$\alpha = 0.3, \beta = 0.1$	$\alpha = 0.3, \beta = 0.1$	$\alpha = 0.6, \beta = 0.3$	$\alpha = 2.5, \beta = 1.5$
μ_1'	0.3906	0.8069	0.7297
μ_2'	0.3906	0.8069	0.7297
μ_3'	0.3906	0.8069	0.7297
μ_4'	0.3906	0.8069	0.7297
μ_2	0.2380	0.1558	0.1972
S_k	−1.5956	−35.7239	−18.7771
K_u	1.2010	3.4209	2.0700

From table 1 drawn above we can conclude that, for any parameter values of the ECTW distribution, the first four moments is the same, the distribution is negatively skewed and can be used to model data with varying degree of kurtosis.

3.3. Quantile Function

The quantile function x_q of the exponentiated Cubic transmuted Weibull distribution is obtained by solving (9) for x and is given as,

$$x_q = \left(-\frac{\ln y}{\alpha}\right)^{\frac{1}{\beta}} \qquad (26)$$

Where,

$$y = \frac{-b}{3a} - \frac{2^{\frac{1}{3}}V_1}{3a\left(V_2 + \sqrt{4V_1^3 + V_2^2}\right)^{\frac{1}{3}}} + \frac{\left(V_2 + \sqrt{4V_1^3 + V_2^2}\right)^{\frac{1}{3}}}{3\left(2^{\frac{1}{3}}\right)a}$$

$$V_1 = -\boldsymbol{b}^2 + 3\boldsymbol{ac},\ V_2 = -2b^3 + 9abc - 27a^2d, a = -\lambda, b = \lambda, c = -1, d = 1 - q^{\frac{1}{\theta}}$$

Equivalently,

$$y = \frac{1}{3} + \frac{2^{\frac{1}{3}}V_1}{3\lambda\left(V_2 + \sqrt{4V_1^3 + V_2^2}\right)^{\frac{1}{3}}} - \frac{\left(V_2 + \sqrt{4V_1^3 + V_2^2}\right)^{\frac{1}{3}}}{3\left(2^{\frac{1}{3}}\right)\lambda} \qquad (27)$$

The three quartiles, lower quartile (q_1), middle quartile (q_2) and the upper quartile (q_3) can be obtained by taking q to be 0.25, 0.5 and 0.75 in (26) respectively.

3.4. Random Number Generation

Random number can be generated from the Exponentiated Cubic transmuted Weibull distribution by equating the cdf of the ECTW distribution with a uniform random number and inverting the expression. More so, the random number from ECTW distribution is obtained by solving

$$\left\{\left(1 - e^{-\alpha x^\beta}\right)\left[(1+\lambda) - 2\lambda\left(e^{-\alpha x^\beta}\right) + \lambda\left(e^{-\alpha x^\beta}\right)\right]\right\}^{\boldsymbol{\theta}} = \boldsymbol{u} \qquad (28)$$

for x. the random sample for ECTW distribution can also be expressed as

$$\boldsymbol{X} = \left(-\frac{\ln y}{\alpha}\right)^{\frac{1}{\beta}}$$

Where y is given in equation (27); with $d = 1 - u$ and $u \sim \cup\,(0,1)$

3.5. Renyi Entropy

The Renyi entropy of a random variable X represents a measure of uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Renyi [20], introduced the Renyi entropy defined as

$$H_\phi(T) = \frac{1}{1-\phi} log \int_{-\infty}^{\infty} w(x)^\phi dx\ , \qquad \phi > 0 \text{ and } \phi \neq 1 \qquad (29)$$

Inserting (16) in (29), then we have

$$H_\phi(T) = \frac{1}{1-\phi} log \int_{-\infty}^{\infty} \left\{\alpha\beta x^\beta e^{-\alpha x^\beta}\left(1 + \lambda - 4\lambda(1 - e^{-\alpha x^\beta}) + 3\lambda(1 - e^{-\alpha x^\beta})^2\right)\left[(1+\lambda)\left(1 - e^{-\alpha x^\beta}\right) - 2\lambda(1 - e^{-\alpha x^\beta})^2 + \lambda(1 - e^{-\alpha x^\beta})^3\right]^{\theta-1}\right\}^\phi dx$$

With simple mathematical operation, we have

$$H_\phi(T) = \frac{1}{1-\phi} log\left\{\left[\beta^{\phi-1}\phi^{\frac{1-\phi(\beta+1)}{\beta}}\alpha^{\frac{1-\phi}{\beta}}\Gamma\left(\frac{(\beta+1)(\phi-1)}{\beta}\right)\right] \times \int_{-\infty}^{\infty} L^* dx\right\} \qquad (30)$$

where

$$L^* = \left(1+\lambda-4\lambda\left(1-e^{-\alpha x^\beta}\right) + 3\lambda\left(1-e^{-\alpha x^\beta}\right)^2\right)\left[(1+\lambda)\left(1-e^{-\alpha x^\beta}\right) - 2\lambda\left(1-e^{-\alpha x^\beta}\right)^2 + \lambda\left(1-e^{-\alpha x^\beta}\right)^3\right]^{\theta-1}$$

3.6. Order Statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. The pdf $f_{i:n}(x)$ of the ith order statistic for a random sample $x_1, x_2, \ldots x_n$ from the ECTW distribution id given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} w_{ECTW}(x)[W_{ECTW}(x)]^{i-1}[1-W_{ECTW}(x)]^{n-i} \tag{31}$$

That is,

$$f_{i:n}(x) = \sum_{p=0}^{n-i} \frac{n!\,(-1)^p}{(i-1)!\,(n-i-p)!\,p!} w_{ECTW}(x)[W_{ECTW}(x)]^{i+p-1}$$

$$\begin{aligned} f_{i:n}(x) = {} & G^*\theta\alpha\beta x^\beta e^{-\alpha x^\beta}\left(1+\lambda-4\lambda(1-e^{-\alpha x^\beta})+3\lambda(1-e^{-2\alpha x^\beta})^2\right) \\ & \times\left[(1+\lambda)\left(1-e^{-\alpha x^\beta}\right)-2\lambda(1-e^{-\alpha x^\beta})^2+\lambda(1-e^{-\alpha x^\beta})^3\right]^{\theta-1} \\ & \times\left[(1+\lambda)\left(1-e^{-\alpha x^\beta}\right)-2\lambda(1-e^{-\alpha x^\beta})^2+\lambda(1-e^{-\alpha x^\beta})^3\right]^{\theta(i+p-1)} \end{aligned} \tag{32}$$

Where

$$G^* = \sum_{p=0}^{n-i} \frac{n!\,(-1)^p}{(i-1)!\,(n-i-p)!\,p!}$$

From (32) we can obtain the first order statistics by taking $i = 1$ and the n^{th} order statistics by taking $i = n$

4. Estimation of the Parameters

The likelihood function of ECTW distribution is given by

$$L(\Omega) = \prod_{i=1}^{n}\left\{\theta\alpha\beta x^\beta e^{-\alpha x^\beta}\left(1+\lambda-4\lambda\left(1-e^{-\alpha x^\beta}\right) + 3\lambda\left(1-e^{-\alpha x^\beta}\right)^2\right)\left[(1+\lambda)\left(1-e^{-\alpha x^\beta}\right)-2\lambda\left(1-e^{-\alpha x^\beta}\right)^2 + \lambda\left(1-e^{-\alpha x^\beta}\right)^3\right]^{\theta-1}\right\} \tag{33}$$

The log-likelihood function $l(\Omega) = log(L(\Omega))$ of the ECTW distribution is given by

$$\begin{aligned} l = {} & nlog(\alpha\beta\theta) + (\beta-1)\sum_{i=0}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left(1-4\lambda\left(1-e^{-\alpha x_i^\beta}\right)+3\lambda\left(1-e^{-\alpha x_i^\beta}\right)^2\right) \\ & -\alpha\sum_{i=1}^{n} x_i^\beta + (\theta-1)\sum_{i=0}^{n} log\left[(1+\lambda)\left(1-e^{-\alpha x_i^\beta}\right)-2\lambda(1-e^{-\alpha x_i^\beta})^2+\lambda(1-e^{-\alpha x_i^\beta})^3\right] \end{aligned} \tag{34}$$

Equation (34) can be maximized by solving the nonlinear likelihood equations obtained by differentiating this with respect to α, β, λ and θ. The components o the score vector are given by

$$\begin{aligned} \frac{\partial l}{\partial \alpha} = {} & \frac{n}{\alpha} + \sum_{i=0}^{n}\frac{-4\lambda x^\beta e^{-\alpha x_i^\beta} + 6\lambda x^\beta e^{-\alpha x_i^\beta}\left(1-e^{-\alpha x_i^\beta}\right)}{\left(1-4\lambda(1-e^{-\alpha x_i^\beta})+3\lambda(1-e^{-\alpha x_i^\beta})^2\right)} - \sum_{i=0}^{n} x_i^\beta \\ & \sum_{i=0}^{n}\frac{(1+\lambda)x^\beta e^{-\alpha x_i^\beta} - 4\alpha\lambda x^\beta e^{-\alpha x_i^\beta}\left(1-e^{-\alpha x_i^\beta}\right) + 3\lambda x^\beta e^{-\alpha x_i^\beta}\left(1-e^{-\alpha x_i^\beta}\right)^2}{\left[(1+\lambda)(1-e^{-\alpha x_i^\beta})-2\lambda(1-e^{-\alpha x_i^\beta})^2+\lambda(1-e^{-\alpha x_i^\beta})^3\right]} \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta} = {} & \frac{n}{\beta} + \sum_{i=0}^{n}\log(x_i) + \sum_{i=0}^{n}\frac{-4\lambda\alpha x^\beta lnx e^{-\alpha x_i^\beta} + 6\lambda x^\beta e^{-\alpha x_i^\beta} lnx\left(1-e^{-\alpha x_i^\beta}\right)}{\left(1-4\lambda(1-e^{-\alpha x_i^\beta})+3\lambda(1-e^{-\alpha x_i^\beta})^2\right)} - \alpha\sum_{i=0}^{n} x_i^\beta lnx \\ & \sum_{i=0}^{n}\frac{(1+\lambda)\alpha x^\beta lnx e^{-\alpha x_i^\beta} - 4\alpha\lambda x^\beta lnx e^{-\alpha x_i^\beta}\left(1-e^{-\alpha x_i^\beta}\right) + 3\lambda\alpha x^\beta lnx e^{-\alpha x_i^\beta}\left(1-e^{-\alpha x_i^\beta}\right)^2}{\left[(1+\lambda)(1-e^{-\alpha x_i^\beta})-2\lambda(1-e^{-\alpha x_i^\beta})^2+\lambda(1-e^{-\alpha x_i^\beta})^3\right]} \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial l}{\partial \lambda} = {} & \sum_{i=0}^{n}\frac{\left(-4\left(1-e^{-\alpha x_i^\beta}\right)+3\left(1-e^{-\alpha x_i^\beta}\right)^2\right)}{\left(1-4\lambda(1-e^{-\alpha x_i^\beta})+3\lambda(1-e^{-\alpha x_i^\beta})^2\right)} \\ & + \sum_{i=0}^{n}\frac{\left(1-e^{-\alpha x_i^\beta}\right)-2(1-e^{-\alpha x_i^\beta})^2+(1-e^{-\alpha x_i^\beta})^3}{\left[(1+\lambda)(1-e^{-\alpha x_i^\beta})-2\lambda(1-e^{-\alpha x_i^\beta})^2+\lambda(1-e^{-\alpha x_i^\beta})^3\right]} \end{aligned} \tag{37}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=0}^{n} log\left[(1+\lambda)\left(1-e^{-\alpha x_i^{\beta}}\right) - 2\lambda\left(1-e^{-\alpha x_i^{\beta}}\right)^2 + \lambda\left(1-e^{-\alpha x_i^{\beta}}\right)^3\right] \quad (38)$$

By setting the above partial equations above to zero, the equations obtained are not in closed form and values of the parameters $\alpha, \beta, \lambda\ and\ \theta$ must be found by iterative methods. The maximum likelihood estimates of the parameters, denoted by $\widehat{\Omega}$ is obtained by solving nonlinear equation $\left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \theta}\right)^T = 0$,using a numerical method. The fisher information matrix is given by $I(\Omega) = \left[I_{\Upsilon_I \Upsilon_J}\right]_{5\times 5} = E\left(-\frac{\partial^2 l}{\partial \Upsilon_I \Upsilon_J}\right), i, j = 1, 2, 3, 4,$ can be numerically obtained by R or MATLAB software. The Fisher information matrix $n\boldsymbol{I}(\Omega)$can be approximated by

$$J(\widehat{\Omega}) \approx \left[-\frac{\partial^2 l}{\partial \Upsilon_I \Upsilon_J}\Big|_{\Omega=\widehat{\Omega}}\right]_{4\times 4}, \quad i, j = 1, 2, 3, 4 \quad (39)$$

To obtain numerical solution for the values of the estimates of ECTW distribution we may employ software such as R, Maple, OX Program etc.

5. Applications

In this section, we illustrate the usefulness and applicability of the ECTW distribution by fitting it to tworeal life data sets. We fit the density function of Exponentiated Cubic Transmuted Weibull (ECTW), Cubic Transmuted Weibull (CTW), distribution and the Weibull (W) distributions. To demonstrate the tractability/ flexibility of the new model proposed model, we consider the following measures of fits: Anderson-Darling (A^*), Probability Value (P-Value), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Consistent Akaike Information Criteria (CAIC) and Komogorov Smirnoff (KS) Statistic. The best model among the competing model will be the one having the smallest AIC, BIC, CAIC and A^* and the highest P-Value. The first data set consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang [21]. This data have been used by Rahman, *et al.* [19] to fit Kumaraswamy exponentiated Burr XII and generalized transmuted log-logistic distribution. The starting point of the iterative processes for the cancer patient's data set is (1:0; 0:009; 10:0; 0:1; 0:1). The data point is given as:0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Exploratory Data Analysis of the data is given in Table 2, Table 3 gives the estimates of the parameters of ECTW, CTW, W distribution (standard error in parentheses) and Table 4 contains the measures of goodness of fit for the cancer data. Figure 5 provides the Total Time on Test Plot and the Box plot for the cancer data.The graphs show the data is unimodal and skewed to the right.

Figure-5. Total Time on Test Plot and the Box plot for the data

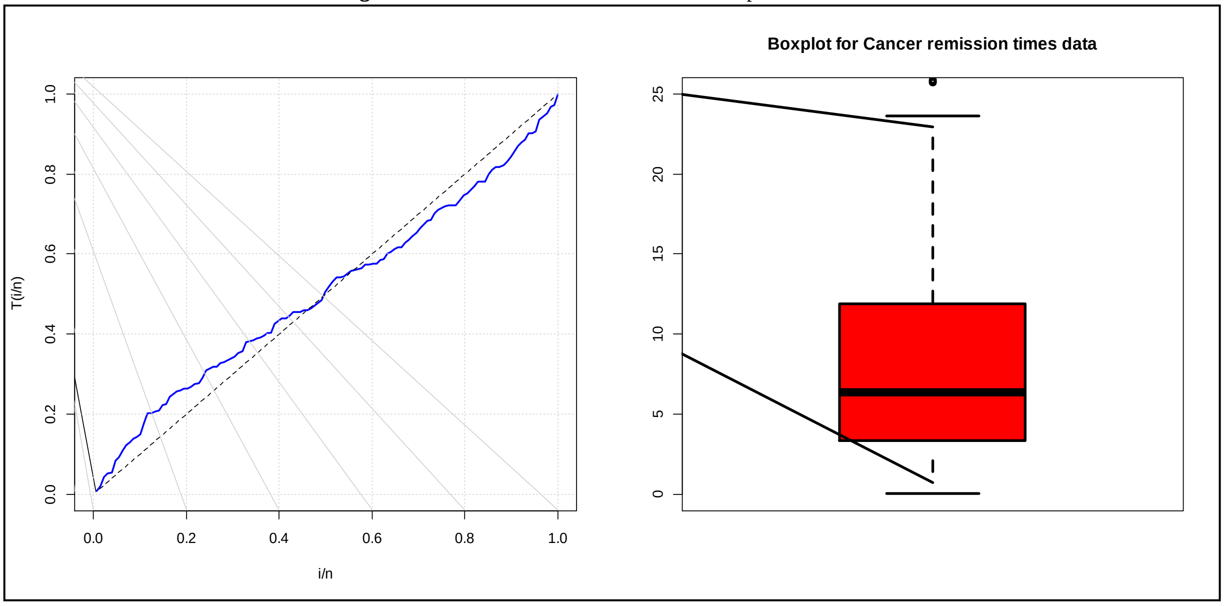


Table-2. Exploratory Data Analysis of The Cancer Data

min	Q_1	$median$	$mean$	Q_3	max	$kurtosis$	$skew.$	$var.$	$range$
0.08	3.348	6.395	9.366	11.840	79.05	16.154	3.326	110.425	78.85

Table-3. MLEs, standard errors (in parenthesis) for the Cancer Data

Model	α	β	λ	θ
$ECTW$	0.0328(0.0151)	0.1337(0.1510)	0.1370(0.2101)	0.9913(0.0180)
CTW	0.1916(0.0393)	0.8321(0.0735)	−0.9444(0.1715)	−
W	0.1537(0.0284)	0.2442(0.0215)	−	−

Table-4. Measures of goodness-of-fit for the Cancer Data

Model	$-l$	AIC	BIC	$CAIC$	KS	A^*	$P-Value$
$ECTW$	410.437	826.873	835.429	827.067	0.0435	0.2526	0.9686
CTW	415.670	837.340	837.661	837.534	0.0539	0.4582	0.8507
W	549.420	1102.839	1108.543	1102.935	0.6679	3.2936	$2.2e-16$

The inverse of the Hessian matrix of the MLSs of ECTW distribution is computed as

$$J(\widehat{\Omega}) = \begin{vmatrix} 0.0002 & 0.0022 & -0.0026 & -0.0004 \\ 0.0022 & 0.0228 & -0.0263 & -0.0012 \\ -0.0026 & -0.02625 & 0.0442 & 0.0032 \\ -0.00038 & -0.0012 & 0.0032 & 0.0003 \end{vmatrix}$$

The 95% confidence interval for α, β, λ and θ are (0.0032, 0.0624), (-0.1623,0.4297), (-0.2748, 0.5488) and (0.9560, 1.0266)

The second data set is obtained from Smith and Naylor [22] and consists of strengths of 1.5 cm glass fibers measured at the National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13,1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. Exploratory Data Analysis of the data is given in Table 5. Table 6 gives the estimates of the parameters of ECTW, CTW, W distribution (standard error in parentheses) and Table 7 contains the measures of goodness of fit for the glass fibre data. Figure 6 provides the Total Time on Test Plot and the Box plot for the data. The graph shows the data exhibits increasing failure rate and skewed to the left.

Table-5. EXPLORATORY DATA ANALYSIS OF THE FAILURE DATA

min	Q_1	$median$	$mean$	Q_3	max	$kurtosis$	$skew.$	$range$
0.55	1.375	1.590	1.51	1.69	2.24	1.103	−0.922	1.69

Figure-6. Total Time on Test Plot and the Box plot for the glass fibre data

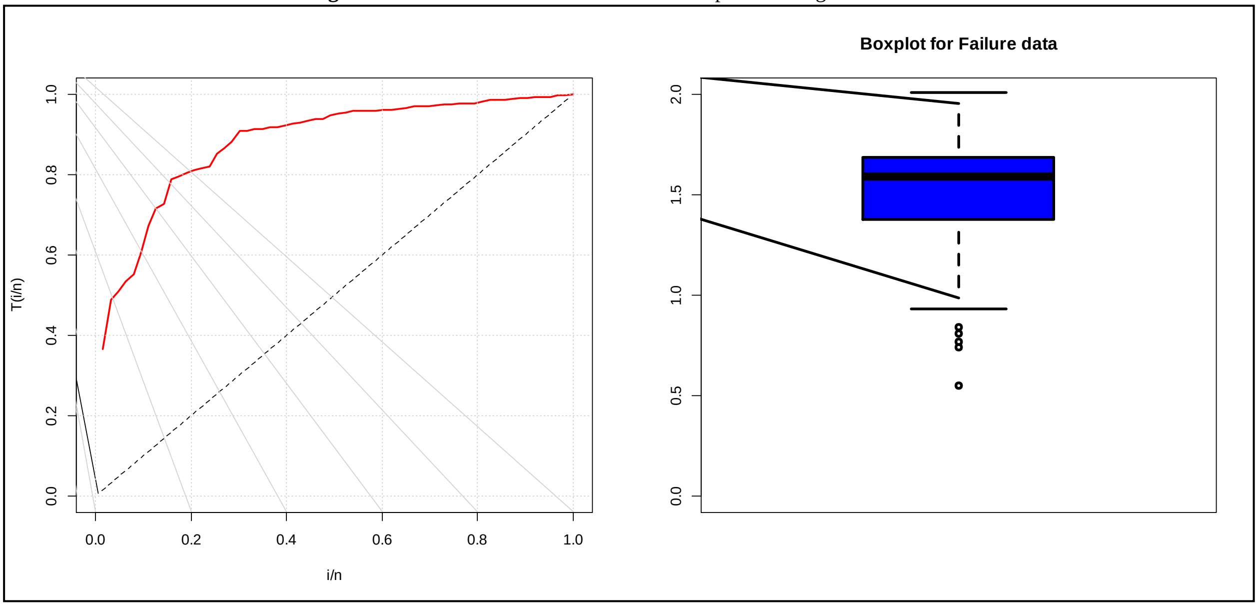


Table-6. MLEs, standard errors (in parenthesis) for the Glass fibre Data

Model	α	β	λ	θ
$ECTW$	$4.286e-06$ $(1.105e-07)$	$1.037e-02$ $(1.038e-02)$	$1.096e-02$ $(1.274e-01)$	$9.835e-01$ $(2.922e-03)$
CTW	$1.232e-01$ $(1.448e-02)$	$1.000e+00$ $(1.695e-01)$	$4.287e+04$ $(2.970e+00)$	$-$ $-$
W	0.5715809 (0.0982)	0.7823089 (0.1129)	$-$ $-$	$-$ $-$

Table-7. Measures of goodness-of-fit for glass fibre Data

Model	$-l$	AIC	BIC	KS	A^*	$P-Value$
$ECTW$	14.305	36.611	37.301	0.1240	1.0168	0.2876
CTW	19.758	45.516	44.306	0.1393	1.2205	0.1734
W	135.231	274.462	278.748	0.3569	3.2936	$2.15e-07$

The inverse of the Hessian matrix of the MLSs of ECTW distribution is computed as

$$J(\hat{\Omega}) = \begin{vmatrix} 1.220208\text{e}-14 & 1.393646\text{e}-09 & 1.116355\text{e}-13 & -7.569296\text{e}-08 \\ 1.393646\text{e}-09 & 1.076414\text{e}-04 & 6.810689\text{e}-09 & -1.569720\text{e}-07 \\ 1.116355\text{e}-13 & 6.810689\text{e}-09 & 1.622147\text{e}-02 & 3.039228\text{e}-07 \\ -7.569296\text{e}-08 & -1.569720\text{e}-07 & 3.039228\text{e}-07 & 8.538101\text{e}-06 \end{vmatrix}$$

The 95% confidence interval for α, β, λ and θ are (-7.2367, 4.5378), (-6.7114,4.3492), (-6.8084, 6.7669), (7.1664, 38.3022)

6. Conclusion

In this study, a four-parameter model called Exponentiated Cubic Transmuted Weibull distribution is proposed and its statistical properties are derived. The estimators for the parameters of the distribution are developed using maximum-likelihood estimation. The applications of the Exponentiated Cubic Transmuted Weibull distribution are demonstrated by using two real life datasets. The performance of the distribution was compared with other distribution regards to providing good fit to the data sets is assessed. The results show that the Exponentiated Cubic Transmuted Weibull distribution provides a more reasonable parametric fits to the data sets.

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