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Stable Numerical Differentiation Algorithms Based on the Fourier Transform in Frequency Domain

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Abstract

A class of stable numerical differential algorithms is constructed based on the Fourier transform. The instability of numerical differentiation problem is over came by modifying the integral "kernel" in frequency domain. The convergence of the approximate derivatives is ensured based on some reasonable assumptions of the modified "kernel" function. The *a-posteriori* choice strategy of the regularization parameter is considered. Moreover, the convergence analysis and error estimate of the approximate derivatives are also given.

Keywords: Numerical differentiation; Fourier transform; ill-posedness; regularization parameter.

1. Introduction

Numerical differentiation aims to compute the derivative of a function approximately, which has been used extensively in image feature detection [1], magnetic resonance imaging [2], neural networks [3] and so on. The analytical derivative method always cannot be used in practical issues because we only know the measured data of the given function.

When the measured data contains some noise, the error of the calculated derivatives by finite difference method maybe huge. In order to overcome the instability, some stabilization methods should be introduced. There have been many works concerning on how to construct the stable numerical differentiation algorithms, such as the stable difference method [4, 5], the regularization method [6-8], the Lanczos integral method [9-11], the mollification method [12, 13], the method based on direct and inverse problems of pdes [14, 15] and so on.

Let $f(x)\in L^2(R)$ be a real-valued function, $\hat{f}(\xi)$ be its Fourier transform, i.e.,

$$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\xi x}dx. \tag{1.1}$$

For the k-order derivative $f^{(k)}(x)(k=1,2,\cdots)$ of $f(x)$, its Fourier transform is

$\widehat{f^{(k)}}(\xi)=(i\xi)^k\hat{f}(\xi).$

Taking the inverse Fourier transform, we have

$$f^{(k)}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(i\xi)^k\hat{f}(\xi)e^{i\xi x}d\xi. \tag{1.2}$$

In the digital signal processing, a function $f(x)$ can be represented as a weighted sum of signals $\hat{f}(\xi)$. Consider the numerical differentiation problem, the term $(i\xi)^k$ can be viewed as an integral "kernel" for calculating $f^{(k)}(x)$, and the weights of the high-frequency components can be amplified by $(i\xi)^k$. Consider the noisy data $f^{\delta}\in L^2(R)$ satisfying

$$\|f^{\delta}-f\|\le\delta, \tag{1.3}$$

where $\|\cdot\|$ is the L^2-norm and the constant $\delta > 0$ represents the noise level. The noise of $f(x)$ in the high frequency components can also be amplified by the term $(i\xi)^k$, and a natural way to construct stable algorithms is to filter the high-frequency components.

In this paper, a class of stable numerical differential algorithms is constructed based on filtering the high-frequency components in (1.2). The convergence of the approximate derivatives is ensured based on some reasonable assumptions of the kernel function. The a-posteriori choice strategy of the regularization parameter is considered, and the convergence analysis of the approximate derivatives is also given.

The paper is organized as follows. In section 2, we construct a class of stable numerical differential algorithms based on Fourier transform in frequency domain, and some different choices of the "kernel" function are also given. The a-posteriori choice strategy of the regularization parameter and the convergence analysis of the approximate derivatives are given in Section 3.

2. Numerical Differentiation in Frequency Domain

In this section, the approximate *k*-order derivatives are constructed by

$$R_\alpha^k f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q(\alpha,\xi)(i\xi)^k \hat{f}(\xi) e^{i\xi x} d\xi,\ f \in L^2(R). \tag{2.1}$$

In order to ensure the convergence of $R_\alpha^k f(x)$, we give the following conclusion.

Theorem 2.1 Assume that

$$q(\alpha,\xi): R^+ \times R \to R$$

is monotone decreasing with respect to α, and satisfies the following conditions.

(1) $|q(\alpha,\xi)| \le 1$ for all $\alpha > 0$ and $\xi \in R$;

(2) there exists a function $c(\alpha)$ satisfying

$$|q(\alpha,\xi)| \cdot |\xi^k| \le c(\alpha)$$

for all $\xi \in R$ and every $\alpha > 0$;

(3a) $\lim\limits_{\alpha\to 0} q(\alpha,\xi) = 1$ for every $\xi \in R.$

Then it has $\lim\limits_{\alpha\to 0} R_\alpha^k f(x) = f^{(k)}(x)$ and $\| R_\alpha^k \| \le c(\alpha)$. Moreover, it has

$$\lim_{\delta\to 0} R_{\alpha(\delta)}^k f^\delta(x) = f^{(k)}(x)$$

if the choice $\alpha = \alpha(\delta)$ satisfies $\alpha(\delta) \to 0$ and $c(\alpha)\delta \to 0$ as $\delta \to 0$.

Proof: The operator R_α^k is bounded since it has

$$\| R_\alpha^k f\|^2 = \int_{-\infty}^{+\infty} |q(\alpha,\xi)(i\xi)^k \hat{f}(\xi)|^2 \, d\xi \le c(\alpha)^2 \|\hat{f}\|^2 = c(\alpha)^2 \|f\|^2,$$

i.e., $\| R_\alpha^k \| \le c(\alpha)$.

From $\| f^{(k)}\|^2 = \| \hat{f}^{(k)}\|^2 = \int_{-\infty}^{+\infty} |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi$ we know that there exists $M \in N$ satisfying

$$\max\{\int_{-\infty}^{-M} |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi, \int_M^{+\infty} |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi\} \le \frac{\varepsilon^2}{4}$$

for any $\varepsilon > 0$. There exists $\alpha_0 > 0$ such that

$$|q(\alpha,\xi) - 1|^2 \le \frac{\varepsilon^2}{2\| f^{(k)}\|^2},$$

for all $\xi \in C$ and $0 < \alpha < \alpha_0$ from the assumption (3). Thus, it has

$$\begin{aligned}
\| R_\alpha^k f - f^{(k)}\|^2 &= \int_{-\infty}^{+\infty} |q(\alpha,\xi) - 1|^2 |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi = \int_{-\infty}^{-M} |q(\alpha,\xi) - 1|^2 |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi \\
&\quad + \int_{-M}^{M} |q(\alpha,\xi) - 1|^2 |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi + \int_M^{+\infty} |q(\alpha,\xi) - 1|^2 |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi \\
&\le \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{2\| f^{(k)}\|^2} \int_{-M}^{M} |f^{(k)}(x)|^2 \, dx + \frac{\varepsilon^2}{4} \le \varepsilon^2,
\end{aligned}$$

and then it has $\lim_{\alpha\to 0} R_\alpha^k f(x) = f^{(k)}(x)$.

By the standard triangle inequality, it has

$$\| R_\alpha^k f^\delta - f^{(k)} \| \le \| R_\alpha^k f^\delta - R_\alpha^k f \| + \| R_\alpha^k f - f^{(k)} \| \le c(\alpha)\delta + \| R_\alpha^k f - f^{(k)} \|, \tag{2.2}$$

thus $\lim_{\delta\to 0} R_{\alpha(\delta)}^k f^\delta(x) = f^{(k)}(x)$ if $\alpha(\delta)\to 0$ and $c(\alpha)\delta \to 0$ as $\delta \to 0$.

Theorem 2.2 Assume that $\| f \|_p \le M$, $p > k$, the assumptions (1) and (2) of the theorem 2.1 hold and (3a) be replaced by

(3b) For all $\alpha > 0$ and $\xi \in R$, there exists $c_1 > 0$ satisfying

$$|(q(\alpha,\xi)-1)\xi^k| \le c_1 \alpha^{p-k}(1+\xi^2)^{p/2}.$$

Then it has

$$\| R_\alpha^k f - f^{(k)} \| \le c_1 \alpha^{p-k} \| f \|_p, \tag{2.3}$$

and

$$\| R_\alpha^k f^\delta - f^{(k)} \| \le c(\alpha)\delta + c_1 M \alpha^{p-k}.$$

Proof: From equations (1.2) and (2.1) we know that

$$\begin{aligned}
\| R_\alpha^k f - f^{(k)} \|^2 &= \int_{-\infty}^{+\infty} |q(\alpha,\xi)-1|^2 |(i\xi)^k \hat{f}(\xi)|^2 \, d\xi \\
&= \int_{-\infty}^{+\infty} |q(\alpha,\xi)-1|^2 (\xi)^{2k} |\hat{f}(\xi)|^2 \, d\xi \\
&= \int_{-\infty}^{+\infty} \Big(\frac{(q(\alpha,\xi)\xi^k)}{(1+\xi^2)^{\frac{p}{2}}}\Big)^2 (1+\xi^2)^p |\hat{f}(\xi)|^2 \, d\xi \\
&\le c_1^2 \alpha^{2(p-k)} \| f \|_p^2.
\end{aligned}$$

Thus, it has

$$\| R_\alpha^k f - f^{(k)} \| \le c_1 \alpha^{p-k} \| f \|_p.$$

From equations (2.2) and (2.3) we know that

$$\| R_\alpha^k f^\delta - f^{(k)} \| \le \| R_\alpha^k f^\delta - R_\alpha^k f \| + \| R_\alpha^k f - f^{(k)} \| \le c(\alpha)\delta + c_1 M \alpha^{p-k}.$$

Theorem 2.3 The following functions $q(\alpha,\xi)$ satisfy the assumptions (1), (2), (3a) and (3b) respectively:

(a)

$$q(\alpha,\xi) = \begin{cases} 1, \alpha|\xi| \le 1; \\ 0, \alpha|\xi| > 1. \end{cases}$$

In this case, (2) holds with $c(\alpha) = \frac{1}{\alpha^k}$, (3b) holds with $c_1 = 1$.

(b)

$$q(\alpha,\xi) = \frac{1}{1+(\alpha\xi)^{2k}},$$

In this case, (2) holds with $c(\alpha) = \frac{1}{2\alpha^k}$, (3b) holds with $c_1 = 1$, when $p < 3k$.

(c)

$$q(\alpha,\xi) = \frac{1}{1+(\alpha|\xi|)^k},$$

In this case, (2) holds with $c(\alpha) = \frac{1}{\alpha^k}$, (3b) holds with $c_1 = 1$, when $p < 2k$.

(d)

$$q(\alpha,\xi) = e^{-(\alpha|\xi|)^k},$$

In this case, (2) holds with $c(\alpha) = \frac{1}{\alpha^k}$, (3b) holds with $c_1 = 1$, when $p < 2k$.

Proof: (a) Consider the assumption (2), it is sufficient to consider the case $\alpha|\xi|\le 1$, where $|q(\alpha,\xi)\xi^k|=|\xi|^k\le\frac{1}{\alpha^k}$. For the assumption (3b), we only need to consider the case $\alpha|\xi|>1$, it has

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}}=\frac{|\xi|^k}{(1+\xi^2)^{p/2}}\le|\xi|^{k-p}\le\alpha^{p-k}.$$

(b) The assumption (2) can be obtained by

$$|q(\alpha,\xi)\xi^k|=\frac{|\xi|^k}{1+(\alpha\xi)^{2k}}\le\frac{1}{2\alpha^k}.$$

For the assumption (3b), we consider in both cases $\alpha|\xi|>1$ and $\alpha|\xi|\le 1$. In the case $\alpha|\xi|>1$, it has

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}}=\frac{\alpha^{2k}|\xi|^{3k}}{(1+(\alpha\xi)^{2k})(1+\xi^2)^{p/2}}\le|\xi|^{k-p}\le\alpha^{p-k}.$$

In the case $\alpha|\xi|\le 1$, it has

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}}\le\frac{\alpha^{2k}|\xi|^{3k}}{(1+\xi^2)^{p/2}}\le\alpha^{p-k}|\alpha\xi|^{3k-p}\le\alpha^{p-k}$$

when $3k-p>0$. Thus, the assumption (3b) holds with $c_1=1$.

(c) The assumption (2) can be obtained by

$$|q(\alpha,\xi)\xi^k|=\frac{|\xi|^k}{1+(\alpha|\xi|)^k}\le\frac{1}{\alpha^k}.$$

For the assumption (3b), we consider in both cases $\alpha|\xi|>1$ and $\alpha|\xi|\le 1$. In the case $\alpha|\xi|>1$, it has

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}}=\frac{\alpha^k|\xi|^{2k}}{(1+(\alpha|\xi|)^k)(1+\xi^2)^{p/2}}\le\alpha^k|\xi|^{2k-p}\le\alpha^{p-k}|\alpha\xi|^{2k-p}\le\alpha^{p-k}$$

when $2k-p>0$. Thus, the assumption (3b) holds with $c_1=1$.

(d) The assumption (2) can be obtained by

$$|q(\alpha,\xi)\xi^k|=\frac{|\xi|^k}{e^{(\alpha|\xi|)^k}}\le\frac{|\xi|^k}{|\alpha\xi|^k}\le\frac{1}{\alpha^k}.$$

For the assumption (3b), we similarly consider in both cases $\alpha|\xi|>1$ and $\alpha|\xi|\le 1$. In the case $\alpha|\xi|>1$, it has

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}}\le\frac{|\xi|^k}{(1+\xi^2)^{p/2}}\le|\xi|^{k-p}\le\alpha^{p-k}.$$

In the case $\alpha|\xi|\le 1$, by means of inequalities $e^x\ge 1+x$ and $e^x\le\frac{1}{1-x}(x<1)$ it has

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}}\le\frac{\alpha^k|\xi|^{2k}}{(1-(\alpha|\xi|)^{2k})(1+\xi^2)^{p/2}}\le\alpha^k|\xi|^{2k-p}\le\alpha^{p-k}|\alpha\xi|^{2k-p}\le\alpha^{p-k}$$

when $2k-p>0$. Thus, the assumption (3b) holds with $c_1=1$.

Theorem 2.4 Assume that $Q(x):R^+\to R$ satisfies $0\le Q(x)\le 1$, $\lim_{x\to 0}Q(x)=1$ and

$$\begin{cases}Q(x)\le\frac{1}{x},x>1;\\ Q(x)\ge 1-x,x\le 1.\end{cases}$$

Denote $q(\alpha,\xi):=Q\{(\alpha|\xi|)^k\}$, then it satisfies the assumption (1) (2) (3a) and (3b) when $p<2k$.

Proof: The assumptions (1) and (3a) are obviously true. We only need to prove (2) and (3b) in both cases $\alpha|\xi|>1$ and $\alpha|\xi|\le 1$. When $\alpha|\xi|>1$, it has

$$|q(\alpha,\xi)\xi^k|=|Q\{(\alpha|\xi|)^k\}\xi^k|\le\frac{1}{\alpha^k}$$

and

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}} \le |\xi|^{k-p} \le \alpha^{p-k}.$$

When $\alpha|\xi|\le 1$ and $2k-p>0$, it has

$$|q(\alpha,\xi)\xi^k| = |Q\{(\alpha|\xi|)^k\}\xi^k| \le \frac{|\xi|^k}{|\alpha\xi|^k} \le \frac{1}{\alpha^k}$$

and

$$\frac{|(q(\alpha,\xi)-1)\xi^k|}{(1+\xi^2)^{p/2}} \le \frac{(1-Q\{(\alpha|\xi|)^k\})|\xi|^k}{(1+\xi^2)^{p/2}} \le (\alpha|\xi|)^k|\xi|^{k-p} = \alpha^{p-k}|\alpha\xi|^{2k-p} \le \alpha^{p-k}.$$

3. The Selection Strategy of Regularization Parameter

Consider the modified function $q(\alpha,\xi)$ defined in the Theorem 2.4, we need to give the selection strategy of regularization parameter α. Similar to the Morozov discrepancy principle, an $a-posteriori$ choice strategy of the parameter $\alpha(\delta)$ is considered by solving the following nonlinear equation

$$G(\alpha) := \| M_\alpha f^\delta - f^\delta \| = \|(q(\alpha,\xi)-1) f^\delta(\xi)\| = C\delta^\gamma, \tag{3.1}$$

where $C>0$, $0<\gamma\le 1$, $C\delta^\gamma < \| f^\delta \|$ and

$$M_\alpha f^\delta(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} q(\alpha,\xi) f^\delta(\xi) e^{i\xi x} dx.$$

It can be proven that the equation (3.1) has a unique solution as follows. In fact, it has

$$G^2(\alpha) = \int_{-M}^{+M} (q(\alpha,\xi)-1)^2 (f^\delta(\xi))^2 d\xi + \int_{R\backslash[-M,M]} (q(\alpha,\xi)-1)^2 (f^\delta(\xi))^2 d\xi,$$

There exists $M>0$ satisfying

$$\int_{R\backslash[-M,M]} (q(\alpha,\xi)-1)^2 (f^\delta(\xi))^2 d\xi \le 2\int_{R\backslash[-M,M]} (f^\delta(\xi))^2 d\xi \le \epsilon$$

for any $\epsilon>0$. When α is small enough, it has

$$\int_{-M}^{+M} (q(\alpha,\xi)-1)^2 (f^\delta(\xi))^2 d\xi \le \epsilon,$$

and then $G^2(\alpha)\le 2\epsilon$, i.e., $\lim_{\alpha\to 0} G(\alpha)=0$. Since

$$\begin{aligned} G^2(\alpha) - \| f^\delta \|^2 &= \int_R ((q(\alpha,\xi)-1)^2 - 1)(f^\delta(\xi))^2 d\xi \\ &= \int_{-M}^{+M} ((q(\alpha,\xi)-1)^2 - 1)(f^\delta(\xi))^2 d\xi \\ &\quad + \int_{R\backslash[-M,+M]} ((q(\alpha,\xi)-1)^2 - 1)(f^\delta(\xi))^2 d\xi, \end{aligned}$$

it has $\lim_{\alpha\to+\infty} G(\alpha) = \| f^\delta \|$ based on $\lim_{\alpha\to+\infty} q(\alpha,\xi)=0$ and the similar discussion above. In addition, it has

$$(G^2(\alpha))' = \frac{d}{d\alpha}\int_{-\infty}^{+\infty} (q(\alpha,\xi)-1)^2 (f^\delta(\xi))^2 d\xi = \int_{-\infty}^{+\infty} 2\frac{\partial q(\alpha,\xi)}{\partial\alpha}(q(\alpha,\xi)-1)(f^\delta(\xi))^2 d\xi > 0.$$

Hence, the equation (3.1) has a unique solution.

In the following, we consider two different choices of $q(\alpha,\xi) = \frac{1}{1+(\alpha\xi)^{2k}}$ and $q(\alpha,\xi) = \frac{1}{1+(\alpha\xi)^{k}}$ given in Theorem 2.3. When $q(\alpha,\xi) = \frac{1}{1+(\alpha\xi)^{2k}}$, it has

Theorem 3.1 Assume that $f^\delta \in L^2(R)$ satisfies $\| f^\delta \| > \delta$, the regularization parameter $\alpha=\alpha(\delta)$ is chosen by solving $\|(q(\alpha,\xi)-1) f^\delta(\xi)\| = \delta$, then it has

(1) $\lim_{\delta\to 0} R^k_{\alpha(\delta)} f^\delta(x) = f^{(k)}(x)$;

(2) $\| R_\alpha^k f^\delta - f^{(k)} \| = O(\delta^{\frac{1}{2}})$, if $f \in L^{2k}(R)$ and $\| f^{(2k)} \| \le M$.

Proof: Denote

$$J_\alpha(g) := \| \hat{g}(\xi) - f^\delta(\xi) \|^2 + \alpha^{2k} \| (i\xi)^k \hat{g}(\xi) \|^2,$$

then it has

$$\begin{aligned}
J_\alpha(g) - J_\alpha(M_\alpha f^\delta) &= \| \hat{g}(\xi) - f^\delta(\xi) \|^2 + \alpha^{2k} \| (i\xi)^k \hat{g}(\xi) \|^2 - (\| (q(\alpha,\xi)-1) f^\delta(\xi) \|^2 \\
&\quad + \alpha^{2k} \| q(\alpha,\xi)(i\xi)^k f^\delta(\xi) \|^2) \\
&= \| \hat{g}(\xi) - q(\alpha,\xi) f^\delta(\xi) \|^2 + 2\Re((q(\alpha,\xi)-1) f^\delta(\xi), \hat{g}(\xi) - q(\alpha,\xi) f^\delta(\xi)) \\
&\quad + 2\alpha^{2k} \Re(q(\alpha,\xi)(i\xi)^k f^\delta(\xi), (i\xi)^k \hat{g}(\xi) - q(\alpha,\xi)(i\xi)^k f^\delta(\xi)) \\
&\quad + \alpha^{2k} \| (i\xi)^k \hat{g}(\xi) - q(\alpha,\xi)(i\xi)^k \hat{f}^\delta(\xi) \|^2 \\
&= \| \hat{g}(\xi) - q(\alpha,\xi) \hat{f}^\delta(\xi) \|^2 + \alpha^{2k} \| (i\xi)^k \hat{g}(\xi) - q(\alpha,\xi)(i\xi)^k \hat{f}^\delta(\xi) \|^2 \\
&\quad + 2\Re((q(\alpha,\xi)-1) \hat{f}^\delta(\xi) + q(\alpha,\xi)(\alpha\xi)^{2k} \hat{f}^\delta(\xi), \hat{g}(\xi) - q(\alpha,\xi) \hat{f}^\delta(\xi)).
\end{aligned}$$

Hence, it has $J_\alpha(g) \ge J_\alpha(M_\alpha f^\delta)$, $\forall g \in L^2(R)$, and then

$$\begin{aligned}
J_\alpha(M_{\alpha(\delta)} f^\delta) &= \delta^2 + \alpha^{2k} \| R_\alpha^k f^\delta \|^2 \le J_\alpha(f) \\
&\le \| \hat{f}(\xi) - f^\delta(\xi) \|^2 + \alpha^{2k} \| (i\xi)^k \hat{f}(\xi) \|^2 \\
&\le \delta^2 + \alpha^{2k} \| f^{(k)} \|^2,
\end{aligned}$$

i.e., $\| R_\alpha^k f^\delta \| \le \| f^{(k)} \|$. We have

$$\begin{aligned}
\| R_\alpha^k f^\delta - f^{(k)} \|^2 &= \| q(\alpha,\xi)(i\xi)^k f^\delta(\xi) - (i\xi)^k \hat{f}(\xi) \|^2 \\
&= \| q(\alpha,\xi)(i\xi)^k f^\delta(\xi) \| - 2\Re(q(\alpha,\xi)(i\xi)^k f^\delta(\xi), (i\xi)^k \hat{f}(\xi)) + \| (i\xi)^k \hat{f}(\xi) \|^2 \\
&\le 2\Re((i\xi)^k \hat{f}(\xi) - q(\alpha,\xi)(i\xi)^k f^\delta(\xi), (i\xi)^k \hat{f}(\xi)).
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary constant, since $H^k(R)$ is dense in $L^2(R)$, there exists $z \in H^k(R)$ such that $\| z - f^{(k)} \| \le \frac{\epsilon}{3}$, then it has

$$\begin{aligned}
\| R_\alpha^k f^\delta - f^{(k)} \|^2 &\le 2\Re((i\xi)^k \hat{f}(\xi) - q(\alpha,\xi)(i\xi)^k f^\delta(\xi), (i\xi)^k \hat{f}(\xi) - \hat{z}(\xi)) \\
&\quad + 2\Re((i\xi)^k \hat{f}(\xi) - q(\alpha,\xi)(i\xi)^k f^\delta(\xi), \hat{z}(\xi)) \\
&\le \frac{2\epsilon}{3} \| R_\alpha^k f^\delta - f^{(k)} \| + 2 \| z^{(k)} \| \cdot (\| \hat{f}(\xi) - f^\delta(\xi) \| + \| (1 - q(\alpha,\xi)) f^\delta(\xi) \|) \\
&\le \frac{2\epsilon}{3} \| R_\alpha^k f^\delta - f^{(k)} \| + 4\delta \| z^{(k)} \|.
\end{aligned}$$

The above inequality can be rewritten as

$$(\| R_\alpha^k f^\delta - f^{(k)} \| - \frac{\epsilon}{3})^2 \le \frac{\epsilon^2}{9} + 4\delta \| z^{(k)} \|. \tag{3.2}$$

When $\delta > 0$ is so small that $\delta \| z^{(k)} \| < \frac{\epsilon^2}{9}$ in (3.2), it has $\| R_\alpha^k f^\delta - f^{(k)} \| \le \epsilon$, and then $\lim_{\delta \to 0} R_{\alpha(\delta)}^k f^\delta = f^{(k)}$.

In addition, from $J_\alpha(M_\alpha f^\delta) \le J_\alpha(f)$ we know that

$$\|(q(\alpha,\xi)-1)f^{\delta}\|^2+\alpha^{2k}\| R_{\alpha}^{k}f^{\delta}-f^{(k)}\|^2$$
$$\le \| f(\xi)-f^{\delta}(\xi)\|^2+\alpha^{2k}\|(i\xi)^k f(\xi)\|^2-\alpha^{2k}\| R_{\alpha}^{k}f^{\delta}(\xi)\|+\alpha^{2k}\|R_{\alpha}^{k}f^{\delta}(\xi)-(i\xi)^k f(\xi)\|^2$$
$$\le \delta^2+2\alpha^{2k}\|(i\xi)^k\hat{f}(\xi)\|^2-2\alpha^{2k}\Re((i\xi)^k\hat{f}(\xi),q(\alpha,\xi)(i\xi)^k\hat{f}^{\delta}(\xi))$$
$$=\delta^2+2\alpha^{2k}\Re((i\xi)^k\hat{f}(\xi),(i\xi)^k\hat{f}(\xi)-q(\alpha,\xi)(i\xi)^k\hat{f}^{\delta}(\xi))$$
$$\le \delta^2+2\alpha^{2k}(\|(i\xi)^{2k}\hat{f}(\xi)\|\cdot(\|\hat{f}(\xi)-\hat{f}^{\delta}(\xi)\|+\|(1-q(\alpha,\xi))\hat{f}^{\delta}\|)$$
$$\le \delta^2+4\delta\alpha^{2k}\| f^{(2k)}\|.$$

Thus, it has

$$\| R_{\alpha}^{k}f^{\delta}-f^{(k)}\|^2\le 4\delta\| f^{(2k)}\|,$$

i.e., $\| R_{\alpha}^{k}f^{\delta}-f^{(k)}\|=O(\delta^{\frac{1}{2}})$. □

When $q(\alpha,\xi)=\dfrac{1}{1+(\alpha\xi)^k}$, it has

Theorem 3.2 Assume that $\| f\|_p\le M$ where $k<p<2k$, $f^{\delta}\in L^2(R)$ satisfies $C\delta^{\gamma}<\| f^{\delta}\|$ where $C>0$ and $0<\gamma<1$, the regularization parameter $\alpha=\alpha(\delta)$ is chosen by solving (3.1), then it has

$$\| R_{\alpha}^{k}f^{\delta}-f^{(k)}\|=O(\delta^{\min\{1-\gamma,\frac{p-k}{k}\gamma\}}).$$

Proof: From equation (3.1) we know that

$$\|(q(\alpha,\xi)-1)f^{\delta}\|=\alpha^k\|\frac{|\xi|^k f^{\delta}(\xi)}{1+(\alpha|\xi|)^k}\|=\alpha^k\| R_{\alpha}^{k}f^{\delta}\|=C\delta^{\gamma}.$$

Moreover, we have

$$\| R_{\alpha}^{k}f^{\delta}\|\le\| R_{\alpha}^{k}f^{\delta}-R_{\alpha}^{k}f\|+\| R_{\alpha}^{k}f\|\le\frac{\delta}{\alpha^k}+\| f^{(k)}\|=\frac{\| R_{\alpha}^{k}f^{\delta}\|}{C}\delta^{1-\gamma}+\| f^{(k)}\|,$$

i.e., $\| R_{\alpha}^{k}f^{\delta}\|\le\dfrac{C}{C-\delta^{1-\gamma}}\| f^{(k)}\|.$ Then, it has

$$\frac{\delta}{\alpha^k}=\frac{\| R_{\alpha}^{k}f^{\delta}\|}{C}\delta^{1-\gamma}\le\frac{\| f^{(k)}\|}{C-\delta^{1-\gamma}}\delta^{1-\gamma}.$$

Notice that

$$\| f^{\delta}\|-C\delta^{\gamma}=\| f^{\delta}\|-\|(q(\alpha,\xi)-1)f^{\delta}\|\le\| q(\alpha,\xi)f^{\delta}\|.$$

Denote the inverse Fourier transform of the regularizing solution $R_{\alpha}^{k}f^{\delta}$ as $R_{\alpha}^{k}f^{\delta}$, we define

$$A_{\alpha}(\xi):=\frac{R_{\alpha}^{k}f^{\delta}}{(i\xi)^k}=q(\alpha,\xi)f^{\delta}(\xi),\xi\ne 0,$$

then it has

$$\| A_{\alpha}(\xi)\|=\| q(\alpha,\xi)(i\xi)^k f^{\delta}(\xi)\|$$

and $A_{\alpha}(0)=f^{\delta}(0)=\int_{-\infty}^{+\infty}f^{\delta}(x)dx.$

For a suitable constant $\epsilon>0$, it has

$$\| q(\alpha,\xi)f^{\delta}(\xi)\|^2=\int_{-\epsilon}^{+\epsilon}(\frac{R_{\alpha}^{k}f^{\delta}(\xi)}{\xi^k})^2d\xi+\int_{R\backslash[-\epsilon,+\epsilon]}(\frac{R_{\alpha}^{k}f^{\delta}(\xi)}{\xi^k})^2d\xi$$
$$\le 2\epsilon(f^{\delta}(0)^2+1)+\frac{1}{\epsilon^{2k}}\|R_{\alpha}^{k}f^{\delta}\|^2$$
$$\le 2\epsilon(f^{\delta}(0)^2+1)+\frac{1}{\epsilon^{2k}}(\frac{C\delta^{\gamma}}{\alpha^k})^2.$$

Thus, it has

$$(\| f^{\delta}\| - C\delta^{\gamma})^2 \le \| q(\alpha,\xi) f^{\delta}(\xi)\|^2 \le 2\epsilon(f^{\delta}(0)^2 + 1) + \frac{1}{\epsilon^{2k}}(\frac{C\delta^{\gamma}}{\alpha^k})^2,$$

i.e.,

$$\alpha^k \le \frac{C\delta^{\gamma}}{\epsilon^k\sqrt{(\| f^{\delta}\| - C\delta^{\gamma})^2 - 2\epsilon(f^{\delta}(0)^2 + 1) + \frac{1}{\epsilon^{2k}}(\frac{C\delta^{\gamma}}{\alpha^k})^2}}.$$

From Theorem 2.2 we know that

$$\| R_{\alpha}^k f^{\delta} - f^{(k)}\| \le \frac{\delta}{\alpha^k} + M\alpha^{p-k} = O(\delta^{\min\{1-\gamma,\frac{p-k}{k}\gamma\}}).$$

4. Conclusions

A class of stable numerical differential algorithms is constructed by modifying the integral "kernel" based on the Fourier transform in frequency domain. Some different choices of the modified kernel function are given. The a-posteriori choice strategy of the regularization parameter and the convergence analysis of the approximate derivatives are also given.

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