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On Bivariate Modeling of the COVID-19 Data with a New Type I Half-Logistic Inverse Weibull Distribution

Ahmed Elhassanein
Department of Mathematics, College of Science, University of Bisha, Bisha, Saudi Arabia
Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt
Email: el_hassanein@yahoo.com

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Abstract

This manuscript presents a new univariate six parameters type I half-logistic inverse Weibull distribution. Explicit expressions for the quantile function, the moments, the moment generating function and the maximum likelihood estimators are formulated. Simulation is employed to investigate the goodness of fit and to discuss the behaviour of the new model. Competitive models are compared via real data. The univariate one is used as a base line to construct a bivariate one named bivariate six parameters type I half-logistic inverse Weibull distribution. Mathematical properties of the new bivariate distrib-ution are investigated. The goodness of fit and the model performance are discussed via simulation. COVID-19 mortality data for Italy and Canada are treated as a bivariate random variable to prove the applicability of the new bivariate distribution.

Keywords: Bivariate distribution; Six parameters type I half-logistic inverse Weibull dis-tribution; Maximum likelihood estimators; Bias; Standard error.

1. Introduction

Parametric probability distributions have a lot of applications especially in data analysis, statistical learning, and image processing [1-5]. Recently there has been a grating interest in formulating new parametric probability distrib-utions. The inverse Weibull (IW) distribution is frequently considered in the literature. The suitable use of the IW model to describe the degeneration phe-nomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines has been discussed by Keller and Kamath [6]. It has been employed by Erto and Rapone [7] to model the times to break-down of the insulating fluid, subject to the action of constant tension. Akgul, *et al.* [8] considered the two-parameter IW model to investigate the wind speed. The three-parameter IW distribution has been constructed by De Gusmo, *et al.* [9]. Its properties have been discussed by Oluyede and Yang [10] and Jana and Bera [11]. Using the Marshall-Olkin method, another three-parameter IW model has been presented by Okasha, *et al.* [12]. The type I half-logistic two-parameter G family has been investigated by Cordeiro, *et al.* [13]. Alkarni , *et al.* [14] presented a new three parameter one as a generalized version. The univariate models can be applied only in the cases where there is an only one random variable or where there is a set of independent random variables.

They fail in the case where there is a set of dependent random variables. The problem of constructing bivariate models has a grate attention [15-28]. Here we aim: to introduce a new univariate six-parameter type I half logis-tic inverse Weibull (SPTIHLIW) distribution; to derive explicit mathematical expressions for its statistical quantities; to show its flexibility; to discuss the goodness of fit; to prove its superiority in comparison with a set of well-known models; to investigate its applicability to real data; to extend the univari-ate one to a bivariate one named bivariate six parameters type I half-logistic inverse Weibull (BISPTIHLIW) distribution; to derive the properties of the BISPTIHLIW distribution including, joint density function, joint cumulative function, conditional distributions, joint moments, hazard bivariate function and copula function; to investigate its performance; to emphasize the goodness of fit; to show the applicability of the BISPTIHLIW distribution for different types of data. As far as I know, the most of available distribution can only be applied for well-behaved data. The new model can be applied for ill-conditioned date including heavy tailed data. It has a joint probability density function with only one form with no singular parts. The pdf offers different shapes for different values of parameters. The hazard function has different shapes. It shows also good performance in terms of simulation study and application of real data. In addition, same algorithm was used to generate bivariate models with common properties including different shapes of the pdf, with singular part and no closed form of the maximum likelihood estimators. This paper uses a more general, easy, and different algorithm to present a bivariate dis-tribution, with absolutely continuous pdf with new marginals. The marginal are flexible with so favorable properties. Although the statistical quantities are complected they are in closed forms. The pdf shows different shapes and characteristics for different values of parameters and so its hazard function. The forthcoming part of the manuscript is organized as follows: The new univariate six-parameter type I half-logistic inverse Weibull (SPTIHLIW) dis-tribution with its properties are constructed in Section 2. Section 3 is devoted to estimate the unknown parameters. Goodness of fit is discussed via simu-lation in section 4. Section 5 gives a real data application with comparison to competitive well-known models. The bivariate six-parameter type I half logistic inverse Weibull (BISTIHLIW) distribution is formulated in section 6. Estimation of unknown parameters and goodness of fit are presented in sec-tions 7 and 8, respectively. Section 9 gives an application of the BISTIHLIW model. Concluding notes are given in section 10.

2. The SPTIHLIW Model

Definition 2.1. A one dimensional random variable Φ is said to follow the SPTIHLIW distribution if its cdf has the following form.

$$F_\Phi(\phi)=\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta. \tag{1}$$

where $\epsilon,\varepsilon,\zeta,\eta,\theta,\vartheta>0$ and $\phi>0$. It will be denoted by $\Phi\sim SPTIHLIW(\epsilon,\varepsilon,\zeta,\eta,\theta,\vartheta)$

Proposition 2.2. For the cdf (1) the pdf is derived as

$$\begin{aligned}f_\Phi(\phi)=&2\epsilon\varepsilon\zeta\eta\theta\vartheta\phi^{-\theta-1}e^{-\epsilon\phi^{-\varepsilon}}(1-e^{-\epsilon\phi^{-\varepsilon}})^{\zeta-1}[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^{\eta-1}\\&\times\frac{1}{(1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta)^2}\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1}\\&\times\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^{\vartheta-1}\end{aligned} \tag{2}$$

Proposition 2.3. An expansion of the pdf (2) can be obtained as follows.

$$f_\Phi(\phi)=\epsilon\varepsilon\sum_{i=0}^{\vartheta}\sum_{j=0}^{\theta i}\sum_{k=0}^{\eta j}\sum_{l=0}^{\infty}\sum_{m=1}^{\zeta(k+l)}(-1)^{i+j+k+l+m}\binom{\vartheta}{i}\binom{\theta i}{j}\binom{\eta j}{k}\binom{\eta j+l-1}{l}\Omega_{j,k,l,m}, \tag{3}$$

where $\Omega_{j,k,l,m}=m\binom{\zeta(k+l)}{m}\phi^{-\varepsilon-1}e^{-\epsilon m\phi^{-\varepsilon}}$, and $0<\phi<1$.

Proposition 2.4. The survival function for the cdf (1) is computed as

$$S(\phi)=1-\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta, \tag{4}$$

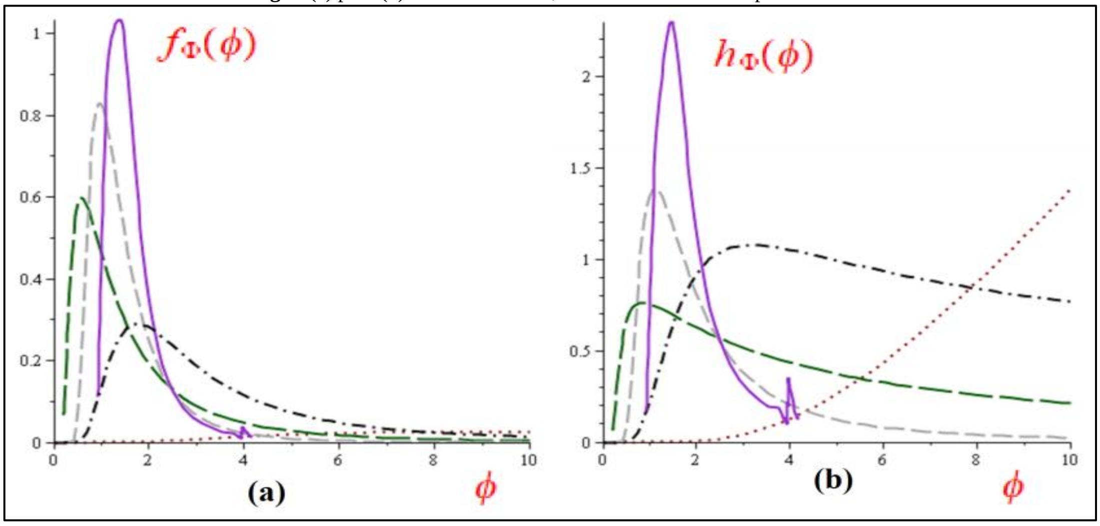
where $\phi>0$:

Proposition 2.5. From (3) and (4) the hazard function is constructed as

$$\begin{aligned}h_\Phi(\phi)=&2\epsilon\varepsilon\zeta\eta\theta\vartheta\phi^{-\theta-1}e^{-\epsilon\phi^{-\varepsilon}}(1-e^{-\epsilon\phi^{-\varepsilon}})^{\zeta-1}[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^{\eta-1}\\&\times\frac{1}{(1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta)^2}\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1}\\&\times\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^{\vartheta-1}\\&\times\left[1-\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta\right]^{-1},\end{aligned} \tag{5}$$

Figure 1, gives the pdf and the hazard function for different values of para-meters. It shows the flexibility of the pdf and changes of the hazard function according to parameters.

Fig-1. (a) pdfs (b) hazard functions, for different values of parameters



Proposition 2.6. Expressions for the s^{th} moment and the moment generating function for the pdf (2) are respectively given by.

$$m_s = \epsilon^{\frac{s}{\varepsilon}} \sum_{i=0}^{\vartheta}\sum_{j=0}^{\theta i}\sum_{k=0}^{\eta j}\sum_{l=0}^{\infty}\sum_{m=1}^{\zeta(k+l)} (-1)^{i+j+k+l+m}\binom{\vartheta}{i}\binom{\theta i}{j}\binom{\eta j}{k}\binom{\eta j+l-1}{l} F_{j,k,l,m}, \qquad (6)$$

$$M(t) = \epsilon^{\frac{h}{\varepsilon}} \sum_{i=0}^{\vartheta}\sum_{j=0}^{\theta i}\sum_{k=0}^{\eta j}\sum_{l=0}^{\infty}\sum_{m=1}^{\zeta(k+l)}\sum_{h=0}^{\infty} (-1)^{i+j+k-\eta j+l+m}\binom{\vartheta}{i}\binom{\theta i}{j}\binom{\eta j}{k}\binom{\eta j+l-1}{l} \Psi_{j,k,l,m,h}, \qquad (7)$$

where $F_{j,k,l,m} = m^{\frac{s}{\varepsilon}}\binom{\zeta(k+l)}{m}\Gamma(1-\frac{s}{\varepsilon})$ and $\Psi_{j,k,l,m,h} = m^{\frac{h}{\varepsilon}}\binom{\zeta(k+l)}{m}\Gamma(1-\frac{h}{\varepsilon})\frac{t^h}{h!}$.

Proposition 2.7. The quantile function for the cdf (1) is

$$q(u) = \left\{-\frac{1}{\epsilon}\ln\left(1-[\frac{1-\varrho(u)}{1+\varrho(u)}]^{\frac{1}{\zeta}}\right)\right\}^{-\frac{1}{\varepsilon}}, \qquad (8)$$

where $\varrho(u) = (1-(1-u^{\frac{1}{\vartheta}})^{\frac{1}{\theta}})^{\frac{1}{\eta}}$ and $u \in (0,1)$.

3. The SPTIHLIW Model: Estimation

Proposition 3.1. (I) The maximum likelihood function for the pdf (2) is com-puted as follows

$$H_\Phi(\epsilon,\varepsilon,\zeta,\eta,\theta,\vartheta) = n\ln(2\epsilon\varepsilon\zeta\eta\theta\vartheta) - (\theta+1)\sum_{i=1}^{n}\ln\phi_i - \epsilon\sum_{i=1}^{n}\phi_i^{-\varepsilon}$$

$$+(\zeta-1)\sum_{i=1}^{n}\ln(1-e^{-\epsilon\phi_i^{-\varepsilon}}) + (\eta-1)\sum_{i=1}^{n}\ln(\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^\zeta})$$

$$-2\sum_{i=1}^{n}\ln(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^\zeta) + (\theta-1)\sum_{i=1}^{n}\ln(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^\zeta}]^\eta)$$

$$+(\vartheta-1)\sum_{i=1}^{n}\ln\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right)$$

(II) The score function

$\Sigma = (\Sigma_\epsilon, \Sigma_\varepsilon, \Sigma_\zeta, \Sigma_\eta, \Sigma_\theta, \Sigma_\vartheta)'$, where $\Sigma_\epsilon = \Sigma_{\epsilon 1} + \Sigma_{\epsilon 2} + \Sigma_{\epsilon 3}$, $\Sigma_\varepsilon = \Sigma_{\varepsilon_1} + \Sigma_{\varepsilon_2} + \Sigma_{\varepsilon_3}$, $\Sigma_\zeta = \Sigma_{\zeta_1} + \Sigma_{\zeta_2} + \Sigma_{\zeta_3}$, $\Sigma_\eta = \Sigma_{\eta_1} + \Sigma_{\eta_2}$, is given by

$$\Sigma_{\epsilon 1} = \frac{n}{2\epsilon} - \sum_{i=1}^{n}\phi_i^{-\varepsilon} + (\zeta-1)\sum_{i=1}^{n}\frac{\phi_i^{-\varepsilon}e^{-\epsilon\phi_i^{-\varepsilon}}}{(1-e^{-\epsilon\phi_i^{-\varepsilon}})} - 2\zeta(\eta-1)$$

$$\times \sum_{i=1}^{n} \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{2\zeta}} - 2\zeta \sum_{i=1}^{n} \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta-1}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}},$$

$$\Sigma_{\epsilon_2} = 2\zeta\eta(\theta-1) \sum_{i=1}^{n} \left\{ \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta})^2} [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta-1} \right.$$

$$\left. \times \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{-1} \right\},$$

$$\Sigma_{\epsilon_3} = 2\zeta\eta\theta(\vartheta-1) \sum_{i=1}^{n} \left\{ \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta})^2} [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta-1} \right.$$

$$\left. \times \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1} \left(1 - \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1} \right\},$$

$$\Sigma_{\varepsilon_1} = \frac{n}{2\varepsilon} + \epsilon \sum_{i=1}^{n} \phi_i^{-\varepsilon} \ln \phi_i - \epsilon(\zeta-1) \sum_{i=1}^{n} \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} \ln \phi_i}{(1-e^{-\epsilon\phi_i^{-\varepsilon}})} + 2\zeta\epsilon(\eta-1)$$

$$\times \sum_{i=1}^{n} \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta} \ln \phi_i}{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{2\zeta}} + 2\zeta\epsilon \sum_{i=1}^{n} \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta-1} \ln \phi_i}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}$$

$$\Sigma_{\varepsilon_2} = 2\zeta\epsilon\eta(\theta-1) \sum_{i=1}^{n} \left\{ \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta-1} \ln \phi_i}{(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta})^2} [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta-1} \right.$$

$$\left. \times \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{-1} \right\},$$

$$\Sigma_{\varepsilon_3} = 2\zeta\eta\theta(\vartheta-1) \sum_{i=1}^{n} \left\{ \frac{\phi_i^{-\varepsilon} e^{-\epsilon\phi_i^{-\varepsilon}} (1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta-1} \ln \phi_i}{(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta})^2} [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta-1} \right.$$

$$\left. \times \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1} \left(1 - \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1} \right\},$$

$$\Sigma_{\zeta_1} = \frac{n}{2\zeta} + \sum_{i=1}^{n} \ln(1-e^{-\epsilon\phi_i^{-\varepsilon}}) - 2(\eta-1) \sum_{i=1}^{n} \frac{(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta} \ln(1-e^{-\epsilon\phi_i^{-\varepsilon}})}{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{2\zeta}}$$

$$-2 \sum_{i=1}^{n} \frac{(1-e^{-\epsilon\phi^{-\varepsilon}})^{\zeta} \ln(1-e^{-\epsilon\phi_i^{-\varepsilon}})}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}},$$

$$\Sigma_{\zeta_2} = 2\eta(\theta-1) \sum_{i=1}^{n} \left\{ \frac{(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta} \ln(1-e^{-\epsilon\phi^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta})^2} [\frac{1-(1-e^{-\epsilon\phi^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta-1} \right.$$

$$\left. \times \left(1 - [\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{-1} \right\},$$

$$\Sigma_{\zeta_3} = 2\eta\theta(\vartheta-1)\sum_{i=1}^{n}\left\{\frac{(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}\ln(1-e^{-\epsilon\phi_i^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta})^{2}}[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta-1}\right.$$

$$\left.\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1}\right\},$$

$$\Sigma_{\eta_1} = \frac{n}{2\eta}+\sum_{i=1}^{n}\ln(\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}})-(\theta-1)$$

$$\times\sum_{i=1}^{n}\left\{[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{-1}\right\}$$

$$\Sigma_{\eta_2} = -\theta(\vartheta-1)\sum_{i=1}^{n}\left\{[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]\right.$$

$$\left.\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}\right\},$$

$$\Sigma_{\theta} = \frac{n}{2\theta}-\sum_{i=1}^{n}\ln\phi_i+\sum_{i=1}^{n}\ln(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta})$$

$$-(\vartheta-1)\sum_{i=1}^{n}\left\{\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1}\right.$$

$$\left.\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\ln\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right\},$$

$$\text{and } \Sigma_{\vartheta} = \frac{n}{2\vartheta}+\sum_{i=1}^{n}\ln\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_i^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right).$$

4. The SPTIHLIW Model: Simulation

To investigate the goodness of fit the function (8) is used to generate samples with size (n = 20, 50, 100). Three different random variables are treated: $\Phi_1 \sim$ *SPTIHLIW* (1.3, 1.7, 0.5, 0.9, 2.1, 2.0); $\Phi_2 \sim$ *SPTIHLIW* (1.8, 0.7, 1.5, 1.3, 0.8, 1.3); $\Phi_3 \sim$ *SPTIHLIW* (0.9, 2.3, 2.1, 2.0, 1.4, 0.7). The maximum likelihood method is mployed to estimate the unknown parameters. The average esti-mates with bias and standard errors for $\Phi_1, \Phi_2, \text{and } \Phi_3$ are presented in Table 1, Table 2 and Table 3, respectively. We can observe the good performance the ML method and the goodness of fit by following the bias and standard error for estimates.

Table-1. Estimates for $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta) = (1.3, 1.7, 0.5, 0.9, 2.1, 2.0)$

Sample Size	Parameter	Average Estimate	Bias	SE
	ϵ	1.515	0.215	0.245
	ε	2.481	0.781	0.296
$n = 20$	ζ	1.112	0.612	0.121
	η	1.009	0.509	0.203
	θ	1.914	-0.186	0.294
	ϑ	1.589	-0.411	0.211
	ϵ	1.449	0.149	0.126
	ε	2.081	0.318	0.139
$n = 50$	ζ	1.086	0.586	0.101
	η	1.237	0.337	0.166
	θ	1.826	-0.274	0.192
	ϑ	1.734	-0.266	0.135
	ϵ	1.207	-0.093	0.128
	ε	1.860	0.160	0.081
$n = 100$	ζ	0.968	0.468	0.092
	η	1.291	0.391	0.270
	θ	1.916	-0.184	0.136
	ϑ	1.733	-0.267	0.118

Table-2. Estimates for $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta) = (1.8, 0.7, 1.5, 1.3, 0.8, 1.3)$.

Sample Size	Parameter	Average Estimate	Bias	SE
	ϵ	1.614	-0.186	0.146
	ε	1.017	0.317	0.175
$n = 20$	ζ	1.125	-0.475	0.126
	η	1.802	0.202	0.135
	θ	1.505	0.705	0.237
	ϑ	2.043	0.743	0.178
	ϵ	1.615	-0.185	0.130
	ε	0.869	0.169	0.049
$n = 50$	ζ	1.239	-0.361	0.093
	η	1.907	0.607	0.163
	θ	1.750	0.950	0.196
	ϑ	1.657	0.357	0.127
	ϵ	1.605	-0.195	0.106
	ε	0.8975	0.197	0.054
$n = 100$	ζ	1.118	-0.482	0.072
	η	1.476	0.176	0.141
	θ	1.700	0.900	0.229
	ϑ	1.502	0.202	0.130

Table-3. Estimates for $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta) = (0.9, 2.3, 2.1, 2.0, 1.4, 0.7)$.

Sample Size	Parameter	Average Estimate	Bias	SE
$n = 20$	ϵ	1.612	0.712	0.238
	ε	3.190	0.890	0.211
	ζ	2.366	0.266	0.424
	η	2.227	0.227	0.431
	θ	2.558	1.158	0.835
	ϑ	1.383	0.683	0.299
$n = 50$	ϵ	1.668	0.768	0.132
	ε	2.922	0.692	0.171
	ζ	2.266	0.166	0.241
	η	1.183	-0.817	0.131
	θ	1.981	0.581	0.253
	ϑ	0.924	0.224	0.243
$n = 100$	ϵ	1.465	0.565	0.117
	ε	3.005	0.705	0.153
	ζ	2.020	-0.080	0.230
	η	1.819	-0.181	0.098
	θ	1.396	-0.004	0.146
	ϑ	0.782	0.082	0.170

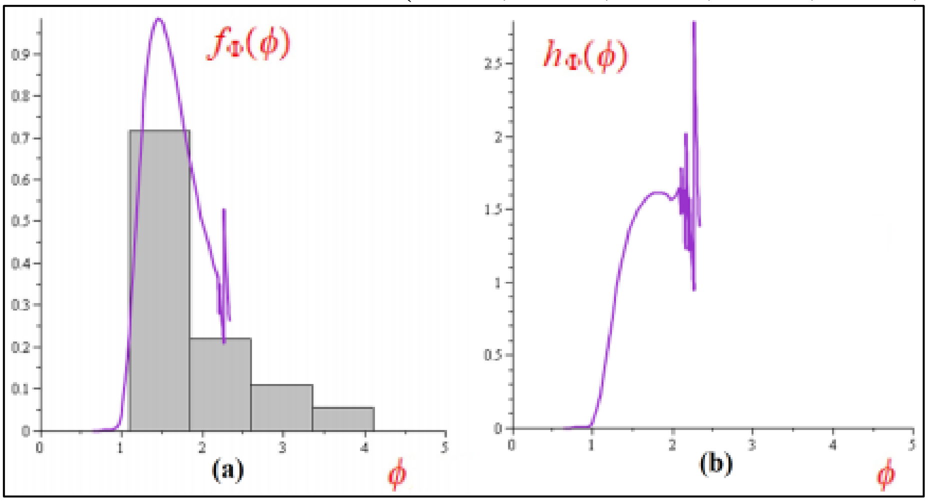
5. Relief Times Data

Analgesics are the drugs used to treat pain. The relief times data (in minutes) of 20 patients receiving an analgesic given by Gross and Clark [20] are modeled by the 10 new SPTIHLIW. The data are: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, and 2. The ML estimates for the SPTIH-LIW model are $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta)$ = (0:3892779, 8:9029837, 2:6088263, 0:1860415, 0:1630330, 8:8436032). The pdf and the hazard function are plotted in Fig. 2. The predefined competitive models are used for comparison, the BGIWGc model, BTW model, EHL-W, EG-W, and Kum-W, the results are displayed in Table 4, see [14, 29] for more details. From Table 4, the new SPTIHLIW shows good performance regarding to the value of AIC and BIC.

Table-4. The well-known models and the SPTIHLIW model with -L, AIC and BIC

Model	-L	AIC	BIC
SPTIHLIW	15.0227	42.0455	48.0199
BGIWGc	15.831	43.662	49.3639
BTW	16.5255	43.051	48.0297
EHL-W	17.113	42.226	46.2089
EG-W	17.486	42.972	46.9549
Kum-W	20.477	48.954	52.9369

Fig-2. The Ötted model for $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta)$ = (0:3892779; 8:9029837; 2:6088263; 0:1860415; 0:1630330; 8:8436032)



6. The BISPTIHLIW Model

The new bivariate six-parameter type I half logistic model is constructed using (1) as a base line distribution.

Definition 6.1. A two dimensional random variable (Φ_1, Φ_2) is said to fol - low the BISPTIHLIW distribution with parameters $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$, where $\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta > 0,\ -1 < \gamma_1 + \gamma_3 < 1, -1 < \gamma_2 + \gamma_3 < 1$ if its cumulative function is given by

$$F_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = F^1_{\Phi_1,\Phi_2}(\phi_1, \phi_2) F^2_{\Phi_1,\Phi_2}(\phi_1, \phi_2) \qquad (9)$$

where $F^1_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = 1 + (\gamma_1 + \gamma_3)\left(1 - \left\{1 - \left(1 - [\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta\right) +$
$(\gamma_2 + \gamma_3)\left(1 - \left\{1 - \left(1 - [\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta\right)$, and

$F^2_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = \left\{1 - \left(1 - [\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta \left\{1 - \left(1 - [\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta$,
for $\phi_1 > 0$ and $\phi_2 > 0$. It will be denoted by $(\Phi_1, \Phi_2) \sim BISPTIHLIW(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$.

Proposition 6.2. The jpdf for $(\Phi_1, \Phi_2) \sim BISPTIHLIW(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ has the following form,

$$f_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = f^1_{\Phi_1,\Phi_2}(\phi_1, \phi_2) f^2_{\Phi_1,\Phi_2}(\phi_1, \phi_2) f^3_{\Phi_1,\Phi_2}(\phi_1, \phi_2) f^4_{\Phi_1,\Phi_2}(\phi_1, \phi_2) \qquad (10)$$

where $f^1_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = (2\epsilon\varepsilon\zeta\eta\theta\vartheta)^2(\phi_1\phi_2)^{-\theta-1} e^{-\epsilon(\phi_1^{-\varepsilon}+\phi_2^{-\varepsilon})}(1 - e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta-1}(1 - e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta-1} \frac{1+\Upsilon(\gamma_1,\phi_1)+\Upsilon(\gamma_2,\phi_2)}{(1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta)^2(1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta)^2}$,

$f^2_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = [\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^{\eta-1} [\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^{\eta-1}$,

$f^3_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = \left(1 - [\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1} \left(1 - [\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1}$,

$$f^4_{\Phi_1,\Phi_2}(\phi_1,\phi_2) = \left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^{\vartheta-1}\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^{\vartheta-1},$$

and

$$\Upsilon(\gamma,x) = (\gamma+\gamma_3)\left(1-2\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon x^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon x^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta\right). \quad (11)$$

Proposition 6.3. The marginals for $(\Phi_1,\Phi_2) \sim BISPTIHLIW(\epsilon,\varepsilon,\zeta,\eta,\theta,\vartheta,\gamma_1,\gamma_2,\gamma_3)$ are derived as

$$F_{\Phi_1}(\phi_1) = \left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta \times\left\{1+(\gamma_1+\gamma_3)\left(1-\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta\right)\right\}, \quad (12)$$

$$F_{\Phi_2}(\phi_2) = \left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta \times\left\{1+(\gamma_2+\gamma_3)\left(1-\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^\vartheta\right)\right\}, \quad (13)$$

$$f_{\Phi_1}(\phi_1) = 2\epsilon\varepsilon\zeta\eta\theta\vartheta\phi_1^{-\theta-1}e^{-\epsilon\phi^{-\varepsilon}}(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta-1} \times[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1} \times\frac{1+\Upsilon(\gamma_1,\phi_1)}{(1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta)^2}\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^{\vartheta-1} \quad (14)$$

and

$$f_{\Phi_2}(\phi_2) = 2\epsilon\varepsilon\zeta\eta\theta\vartheta\phi_2^{-\theta-1}e^{-\epsilon\phi^{-\varepsilon}}(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta-1} \times[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1} \times\frac{1+\Upsilon(\gamma_2,\phi_2)}{(1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta)^2}\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right\}^{\vartheta-1} \quad (15)$$

Where $\Upsilon(\gamma,x)$ is given by (11).

Proposition 6.4. Expressions for the conditional densities for $(\Phi_1, \Phi_2) \sim BISPTIHLIW(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ are

$$f_{\Phi_1/\Phi_2}(\phi_1/\phi_2) = \frac{2\epsilon\varepsilon\zeta\eta\theta\vartheta\phi_1^{-\theta-1}e^{-\epsilon\phi_1^{-\varepsilon}}(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta})^2(1+\Upsilon(\gamma_2,\phi_2))}[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}]^{\eta-1} \times(1+\Upsilon(\gamma_2,\phi_2)+\Upsilon(\gamma_1,\phi_1))\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1} \times\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right\}^{\vartheta-1} \quad (16)$$

$$f_{\Phi_2/\Phi_1}(\phi_2/\phi_1) = \frac{2\epsilon\varepsilon\zeta\eta\theta\vartheta\phi_2^{-\theta-1}e^{-\epsilon\phi_2^{-\varepsilon}}(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta})^2(1+\Upsilon(\gamma_1,\phi_1))}[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}]^{\eta-1} \times(1+\Upsilon(\gamma_2,\phi_2)+\Upsilon(\gamma_1,\phi_1))\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1} \times\left\{1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right\}^{\vartheta-1} \quad (17)$$

Where $\Upsilon(\gamma, x)$ is given by (11).

Proposition 6.5. The conditional moments for $(\Phi_1, \Phi_2) \sim BISPTIHLIW(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ are calculated for (16) and (17) as

$$m^s_{\Phi_1/\Phi_2}(\phi_2) = E(\Phi_1^r/\Phi_2=\phi_2) = (1+\frac{(\gamma_1+\gamma_3)}{1+\Upsilon(\gamma_2,\phi_2)})\epsilon^{\frac{s}{\varepsilon}} \times\sum_{i=0}^{\vartheta}\sum_{j=0}^{\theta i}\sum_{k=0}^{\eta j}\sum_{l=0}^{\infty}\sum_{m=1}^{\zeta(k+l)}(-1)^{i+j+k-\eta j+l+m}\binom{\vartheta}{i}\binom{\theta i}{j}\binom{\eta j}{k}\binom{\eta j+l-1}{l}F^s_{j,k,l,m} -\frac{2(\gamma_1+\gamma_3)}{1+\Upsilon(\gamma_2,\phi_2)}\epsilon^{\frac{s}{\varepsilon}} \times\sum_{i_{1,2}=0}^{\vartheta}\sum_{j_{1,2}=0}^{\theta}\sum_{k_{1,2}=0}^{\eta i_{1,2}}\sum_{l_{1,2}=0}^{\infty}\sum_{m_{1,2}=1}^{\zeta(k_{1,2}+l_{1,2})} m_2\xi^s_{i_{1,2},j_{1,2},k_{1,2},l_{1,2},m_{1,2}}, \quad (18)$$

and

$$m^s_{\Phi_2/\Phi_1}(\phi_1) = E(\Phi_2^s/\Phi_1=\phi_1) = (1+\frac{(\gamma_2+\gamma_3)}{1+\Upsilon(\gamma_1,\phi_1)})\epsilon^{\frac{s}{\varepsilon}} \times\sum_{i=0}^{\vartheta}\sum_{j=0}^{\theta i}\sum_{k=0}^{\eta j}\sum_{l=0}^{\infty}\sum_{m=1}^{\zeta(k+l)}(-1)^{i+j+k-\eta j+l+m}\binom{\vartheta}{i}\binom{\theta i}{j}\binom{\eta j}{k}\binom{\eta j+l-1}{l}F^s_{j,k,l,m} +\frac{2(\gamma_2+\gamma_3)}{1+\Upsilon(\gamma_1,\phi_1)}\epsilon^{\frac{s}{\varepsilon}} \times\sum_{i_{1,2}=0}^{\vartheta}\sum_{j_{1,2}=0}^{\theta}\sum_{k_{1,2}=0}^{\eta i_{1,2}}\sum_{l_{1,2}=0}^{\infty}\sum_{m_{1,2}=1}^{\zeta(k_{1,2}+l_{1,2})} m_2\xi^s_{i_{1,2},j_{1,2},k_{1,2},l_{1,2},m_{1,2}}, \quad (19)$$

where

$$F^{\alpha}_{j,k,l,m} = m^{\alpha}\binom{\zeta(k+l)}{m}\Gamma(1-\frac{\alpha}{\varepsilon}), \quad (20)$$

$$\xi^{\alpha}_{i_{1,2},j_{1,2},k_{1,2},l_{1,2}} = (-1)^{i_1+i_2+j_1+j_2+k_1+k_2+l_1+l_2+m_1+m_2}\binom{\vartheta}{i_1}\binom{\vartheta}{i_2} \times\binom{\theta i_2}{j_2}\binom{\eta j_1}{k_1}\binom{\eta j_2}{k_2}\binom{\theta i_1}{j_1}\binom{\eta j_1+l-1}{l_1}\binom{\eta j_2+l-1}{l_2} \times\binom{\zeta(k_1+l)}{m_1}\binom{\zeta(k_2+l)}{m_2}\frac{\Gamma(1-\frac{\alpha}{\varepsilon})}{(m_1+m_2)^{1-\frac{\alpha}{\varepsilon}}}, \quad (21)$$

Where $\Upsilon(\gamma, x)$ is given by (11).

Proposition 6.6. The joint moments for $(\Phi_1, \Phi_2) \sim BISPTIHLIW\ (\epsilon, \varepsilon, \zeta, \eta,\ \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ are

$$m_{r,s} = E(\Phi_1^r \Phi_2^s) = m_{r,s}^1 - m_{r,s}^2 - m_{r,s}^3 \qquad (22)$$

where $m_{r,s}^1 =$

$$(1+\gamma_1+\gamma_2+2\gamma_3)\epsilon^{\frac{r}{\varepsilon}+\frac{s}{\varepsilon}} \sum_{i=0}^{\vartheta} \sum_{j=0}^{\theta i} \sum_{k=0}^{\eta j} \sum_{l=0}^{\infty} \sum_{m=1}^{\zeta(k+l)} (-1)^{i+j+k+l+m} \binom{\vartheta}{i} \binom{\theta i}{j} \binom{\eta j}{k} \binom{\eta j+l-1}{l} F_{j,k,l,m}^{r}$$

$$\times \sum_{i=0}^{\vartheta} \sum_{j=0}^{\theta i} \sum_{k=0}^{\eta j} \sum_{l=0}^{\infty} \sum_{m=1}^{\zeta(k+l)} (-1)^{i+j+k+l+m} \binom{\vartheta}{i} \binom{\theta i}{j} \binom{\eta j}{k} \binom{\eta j+l-1}{l} F_{j,k,l,m}^{s},$$

$m_{r,s}^2 =$

$$2(\gamma_1+\gamma_3)\epsilon^{\frac{r}{\varepsilon}+\frac{s}{\varepsilon}} \sum_{i=0}^{\vartheta} \sum_{j=0}^{\theta i} \sum_{k=0}^{\eta j} \sum_{l=0}^{\infty} \sum_{m=1}^{\zeta(k+l)} (-1)^{i+j+k+l+m} \binom{\vartheta}{i} \binom{\theta i}{j} \binom{\eta j}{k} \binom{\eta j+l-1}{l} F_{j,k,l,m}^{s}$$

$$\times \sum_{i_{1,2}=0}^{\vartheta} \sum_{j_{1,2}=0}^{\theta i_{1,2}} \sum_{k_{1,2}=0}^{\eta j_{1,2}} \sum_{l_{1,2}=1}^{\infty} \sum_{m_{1,2}=1}^{\zeta(k_{1,2}+l_{1,2})} m_2 \xi_{i_{1,2},j_{1,2},k_{1,2},l_{1,2},m_{1,2}}^{r},$$

$m_{r,s}^3 =$

$$2(\gamma_2+\gamma_3)\epsilon^{\frac{r}{\varepsilon}+\frac{s}{\varepsilon}} \sum_{i=0}^{\vartheta} \sum_{j=0}^{\theta i} \sum_{k=0}^{\eta j} \sum_{l=0}^{\infty} \sum_{m=1}^{\zeta(k+l-\eta j)} (-1)^{i+j+k+l+m} \binom{\vartheta}{i} \binom{\theta i}{j} \binom{\eta j}{k} \binom{\eta j+l-1}{l} F_{j,k,l,m}^{r}$$

$$\times \sum_{i_{1,2}=0}^{\vartheta} \sum_{j_{1,2}=0}^{\theta i_{1,2}} \sum_{k_{1,2}=0}^{\eta j_{1,2}} \sum_{l_{1,2}=1}^{\infty} \sum_{m_{1,2}=1}^{\zeta(k_{1,2}+l_{1,2})} m_2 \xi_{i_{1,2},j_{1,2},k_{1,2},l_{1,2},m_{1,2}}^{s},$$

$F_{j,k,l,m}^{\alpha}$, and $\xi_{i_{1,2},j_{1,2},k_{1,2},l_{1,2},m_{1,2}}^{\alpha}$ are given by (20) and (21), respectively.

Proposition 6.7. The bivariate reliability function for $(\Phi_1, \Phi_2) \sim\ BISPTIHLIW(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$, is

$$r(\phi_1, \phi_2) = \frac{(\Upsilon(\gamma_1, \phi_1) + \gamma_1 + \gamma_3)(\Upsilon(\gamma_2, \phi_2) + \gamma_2 + \gamma_3)}{8(\gamma_1 + \gamma_3)} \times \frac{(\Upsilon(\gamma_1, \phi_1) + \Upsilon(\gamma_2, \phi_2) - \gamma_1 - \gamma_2 - 2\gamma_3 + 2)}{(\gamma_2 + \gamma_3)} \qquad (23)$$

Proposition 6.8. The bivariate hazard rate function for

$(\Phi_1, \Phi_2) \sim BISPTIHLIW(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$, is

$$h_{\Phi_1,\Phi_2}(\phi_1, \phi_2) = h_{\Phi_1,\Phi_2}^1(\phi_1, \phi_2) h_{\Phi_1,\Phi_2}^2(\phi_1, \phi_2) h_{\Phi_1,\Phi_2}^3(\phi_1, \phi_2) h_{\Phi_1,\Phi_2}^4(\phi_1, \phi_2) \quad (24)$$

where,

$$h_{\Phi_1,\Phi_2}^1(\phi_1, \phi_2) = \frac{32(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\epsilon\varepsilon\zeta\eta\theta\vartheta)^2(\phi_1\phi_2)^{-\theta-1}}{(1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta})^2(1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta})^2} \frac{e^{-\epsilon(\phi_1^{-\varepsilon}+\phi_2^{-\varepsilon})}(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta-1}(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta-1}}{(\Upsilon(\gamma_1,\phi_1)+\gamma_1+\gamma_3)(\Upsilon(\gamma_2,\phi_2)+\gamma_2+\gamma_3)},$$

$$h_{\Phi_1,\Phi_2}^2(\phi_1, \phi_2) = \left[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\zeta}}\right]^{\eta-1}\left[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\zeta}}\right]^{\eta-1} \frac{1}{(\Upsilon(\gamma_1,\phi_1)+\Upsilon(\gamma_2,\phi_2)-\gamma_1-\gamma_2-2\gamma_3+2)},$$

$$h^3_{\Phi_1,\Phi_2}(\phi_1,\phi_2) = \left(1-[\tfrac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1}\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1},$$

and

$$h^4_{\Phi_1,\Phi_2}(\phi_1,\phi_2) = \left\{1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta}\right\}^{\vartheta-1}\left\{1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta}\right\}^{\vartheta-1}$$

Proposition 6.9. The copula function for $(\Phi_1,\Phi_2)\sim BISPTIHLIW(\epsilon,\varepsilon,\zeta,\ \eta,\theta,\vartheta,\gamma_1,\gamma_2,\gamma_3)$, is

$$c(x,y) = \frac{1+\Upsilon(\gamma_1,x)+\Upsilon(\gamma_2,y)}{(1+\Upsilon(\gamma_1,x))(1+\Upsilon(\gamma_2,y))}, \qquad (25)$$

where $\Upsilon(\varepsilon,x)$ is given by (11), [30, 31]

7. The BISPTIHLIW Model: Estimation

Proposition 7.1. Let $(\Phi_{11},\Phi_{21}),(\Phi_{12},\Phi_{22}),\dots,(\Phi_{1n},\Phi_{2n})$ be a random sample of size n from a random variable $(\Phi_1,\Phi_2)\sim BISPTIHLIW(\epsilon,\varepsilon,\zeta,\eta,\theta,\vartheta,\gamma_1,\gamma_2,\gamma_3)$.
Then:

(I) The maximum log-likelihood function is given by $H_{\Phi_1,\Phi_2}(\Theta) = H_1(\Theta) + H_2(\Theta)+H_3(\Theta)+H_4(\Theta)$, where $\Theta = (\epsilon,\varepsilon,\zeta,\eta,\theta,\vartheta,\gamma_1,\gamma_2,\gamma_3)$,

$$H_1(\Theta) = 2n\ln(2\epsilon\varepsilon\zeta\eta\theta\vartheta) - (\theta+1)\sum_{i=1}^{n}(\ln\phi_{1i}+\ln\phi_{2i}) - \epsilon\sum_{i=1}^{n}(\phi_{1i}^{-\varepsilon}+\phi_{2i}^{\varepsilon})$$

$$+(\zeta-1)\sum_{i=1}^{n}\left(\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})+\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})\right),$$

$$H_2(\Theta) = (\eta-1)\sum_{i=1}^{n}\left(\ln(\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta})+\ln(\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta})\right)$$

$$-2\sum_{i=1}^{n}\left(\ln(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta)+\ln(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta)\right),$$

$$H_3(\Theta) = (\theta-1)\sum_{i=1}^{n}\left\{\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}]^\eta)+\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}]^\eta)\right\}$$

$$+\sum_{i=1}^{n}\ln(1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})),$$

$$\text{and } H_4(\Theta) = (\vartheta-1)\sum_{i=1}^{n}\left\{\ln(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\zeta}]^\eta\right)^\theta)\right.$$

$$+\ln(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\zeta}]^\eta\right)^\theta)\Bigg\}.$$

(II) The score function $F=(F_\epsilon,F_\varepsilon,F_\zeta,F_\eta,F_\theta,F_\vartheta,F_{\gamma_1},F_{\gamma_2},F_{\gamma_3})'$ is given by $F_\epsilon=F_{\epsilon1}+F_{\epsilon2}+F_{\epsilon3}+F_{\epsilon4}$, $F_\varepsilon=F_{\varepsilon1}+F_{\varepsilon2}+F_{\varepsilon3}+F_{\varepsilon4}$, $F_\zeta=F_{\zeta_1}+F_{\zeta_2}+F_{\zeta_3}+F_{\zeta_4}$, $F_\eta=F_{\eta_1}+F_{\eta_2}+F_{\eta_3}$, and $F_\theta=F_{\theta_1}+F_{\theta_2}+F_{\theta_3}$, $F_\vartheta=F_{\vartheta_1}+F_{\vartheta_2}$, F_{γ_1}, F_{γ_2}, and F_{γ_3}, where

$$F_{\epsilon1}=\frac{n}{\epsilon}-\sum_{i=1}^n(\phi_{1i}^{-\varepsilon}+\phi_{2i}^{-\varepsilon})+(\zeta-1)\sum_{i=1}^n\left(\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}}{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}}{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}\right)$$

$$-2\zeta(\eta-1)\sum_{i=1}^n\left(\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{2\zeta}}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{2\zeta}}\right)$$

$$-2\zeta\sum_{i=1}^n\left(\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta-1}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta-1}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}\right),$$

$$F_{\epsilon2}=2\zeta\eta(\theta-1)\sum_{i=1}^n\Bigg\{\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta)^2}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}]^{\eta-1}$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}]^\eta\right)^{-1}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta)^2}$$

$$\times[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}]^\eta\right)^{-1}\Bigg\},$$

$$F_{\epsilon3}=2\zeta\eta\theta\vartheta\sum_{i=1}^n\frac{1}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})}\Bigg\{\frac{(\gamma_1+\gamma_3)\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta)^2}$$

$$\times[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right)^{\vartheta-1}$$

$$+\frac{(\gamma_2+\gamma_3)\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta)^2}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}]^{\eta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}]^\eta\right)^\theta\right)^{\vartheta-1}$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\zeta}]^\eta\right)^{\theta-1}\Bigg\},$$

$$F_{\epsilon4}=2\zeta\eta\theta(\vartheta-1)\sum_{i=1}^n\Bigg\{\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta-1}}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\zeta)^2}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\gamma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\gamma}]^{\eta-1}$$

$$\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta-1}$$

$$+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma-1}}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma})^{2}}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta-1}$$

$$\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{-1}\Bigg\},$$

$$F_{\varepsilon_1}=\frac{n}{\varepsilon}+\epsilon\sum_{i=1}^{n}(\phi_{1i}^{-\varepsilon}\ln\phi_{1i}+\phi_{2i}^{-\varepsilon}\ln\phi_{2i})+\epsilon(\varsigma-1)\sum_{i=1}^{n}\left(\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}\ln\phi_{1i}}{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}\ln\phi_{2i}}{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}\right),$$

$$F_{\varepsilon_2}=2\epsilon\varsigma(\eta-1)\sum_{\times i=1}^{n}\left\{\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}\ln\phi_{1i}}{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{2\varsigma}}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}\ln\phi_{2i}}{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{2\varsigma}}\right\}$$

$$+2\epsilon\varsigma\sum_{i=1}^{n}\left\{\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma-1}\ln\phi_{1i}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma-1}\ln\phi_{2i}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}\right\},$$

$$F_{\varepsilon_3}=2\epsilon\varsigma\eta(\theta-1)\sum_{i=1}^{n}\left\{\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma-1}\ln\phi_{1i}}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma})^{2}}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta-1}\right.$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{-1}+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma-1}\ln\phi_{2i}}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma})^{2}}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta-1}$$

$$\left.\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{-1}\right\},$$

$$F_{\varepsilon_4}=2\varsigma\eta\theta\vartheta\sum_{i=1}^{n}\left\{\frac{(\gamma_1+\gamma_3)\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma-1}\ln\phi_{1i}}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma})^{2}}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta-1}\right.$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}$$

$$+\frac{(\gamma_2+\gamma_3)\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma-1}\ln\phi_{2i}}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma})^{2}}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta-1}$$

$$\left.\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta-1}\right\}\frac{1}{1+\Upsilon(\gamma_1,\phi_{1i})+F(\gamma_2,\phi_{2i})},$$

$$F_{\varepsilon_5}=2\varsigma\eta\theta(\vartheta-1)\sum_{i=1}^{n}\left\{\frac{\phi_{1i}^{-\varepsilon}e^{-\epsilon\phi_{1i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma-1}}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma})^{2}}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\gamma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\gamma}}]^{\eta-1}\right.$$

$$\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^\varsigma}]^\eta\right)^\theta\right)^{-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^\eta\right)^{\theta-1}$$

$$+\frac{\phi_{2i}^{-\varepsilon}e^{-\epsilon\phi_{2i}^{-\varepsilon}}(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma-1}}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma)^2}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}]^\eta\right)^{\theta-1}$$

$$\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^\varsigma}]^\eta\right)^\theta\right)^{-1}\Bigg\},$$

$$F_{\varsigma_1}=\frac{n}{\varsigma}+\sum_{i=1}^{n}\left(\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})+\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})\right)$$

$$-2(\eta-1)\sum_{i=1}^{n}\left\{\frac{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\gamma\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{2\varsigma}}+\frac{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{2\varsigma}}\right\},$$

$$F_{\varsigma_2}=-2\sum_{i=1}^{n}\left\{\frac{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\gamma\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}+\frac{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}\right\},$$

$$F_{\varsigma_3}=2\eta(\theta-1)\sum_{i=1}^{n}\Bigg\{\frac{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\gamma\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma})^2}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-ee^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^\theta\right)^{-1}$$

$$+\frac{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma})^2}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma}]^{\theta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\gamma}]^\theta\right)^{-1}\Bigg\},$$

$$F_{\varsigma_4}=2\eta\theta(\vartheta-1)\sum_{i=1}^{n}\Bigg\{\frac{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma})^2}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^{\eta-1}$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^\eta\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^\eta\right)^\theta\right)^{-1}$$

$$+\frac{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma})^2}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}]^{\eta-1}$$

$$\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}]^\eta\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^\varsigma}]^\eta\right)^\theta\right)^{-1}\Bigg\},$$

$$F_{\varsigma_5}=2\eta\theta\vartheta\sum_{i=1}^{n}\Bigg\{(\gamma_1+\gamma_3)\frac{(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma\ln(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma})^2}[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^{\eta-1}$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^\theta\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^\varsigma}]^\eta\right)^\theta\right)^{\vartheta-1}$$

$$+(\gamma_2+\gamma_3)\frac{(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}\ln(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})}{(1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta})^2}[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta-1}\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}$$

$$\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}\Bigg\}\frac{1}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})},$$

$$\digamma_{\eta_1}=\frac{n}{\eta}+\sum_{i=1}^{n}\left(\ln(\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}})+\ln(\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}})\right),$$

$$\digamma_{\eta_2}=-(\theta-1)\sum_{i=1}^{n}\Bigg\{[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{-1}$$

$$+[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{-1}\Bigg\},$$

$$\digamma_{\eta_3}=\theta\vartheta\sum_{i=1}^{n}\Bigg\{[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}$$

$$\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1}+[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]$$

$$\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{-1}\Bigg\},$$

$$\digamma_{\eta_4}=2\theta\vartheta\sum_{i=1}^{n}\Bigg\{(\gamma_1+\gamma_3)[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]$$

$$\times\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}$$

$$+(\gamma_2+\gamma_3)[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\ln[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta-1}$$

$$\times\left(1-\left(1-[\frac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\zeta}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}\Bigg\}\frac{1}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})},$$

$$\digamma_{\theta_1}=\frac{n}{\theta}-\sum_{i=1}^{n}(\ln\phi_{1i}+\ln\phi_{2i})+\sum_{i=1}^{n}\Bigg\{\ln(1-[\frac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\zeta}}]^{\eta})$$

$$+\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta})\Bigg\},$$

$$F_{\theta_2}=2(\vartheta-1)\sum_{i=1}^{n}\Bigg\{\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{-1}\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}$$

$$\times\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta})+\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{-1}$$

$$\times\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta})\Bigg\},$$

$$F_{\theta_3}=2\vartheta\sum_{i=1}^{n}\Bigg\{(\gamma_1+\gamma_3)\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}$$

$$\times\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{i1}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{i1}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{i1}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{i1}^{-\varepsilon}})^{\varsigma}}]^{\eta})$$

$$+(\gamma_2+\gamma_3)\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{\vartheta-1}$$

$$\times\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\ln(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta})\Bigg\}\frac{1}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})},$$

$$F_{\vartheta_1}=\sum_{i=1}^{n}\Bigg\{\ln(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_1^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta})+\ln(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_2^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta})\Bigg\},$$

$$F_{\vartheta_2}=2\sum_{i=1}^{n}\Bigg\{(\gamma_1+\gamma_3)\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{\vartheta}$$

$$\times\ln\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{1i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)+(\gamma_2+\gamma_3)\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)^{\vartheta}$$

$$\times\ln\left(1-\left(1-[\tfrac{1-(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}{1+(1-e^{-\epsilon\phi_{2i}^{-\varepsilon}})^{\varsigma}}]^{\eta}\right)^{\theta}\right)\Bigg\}\frac{1}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})},$$

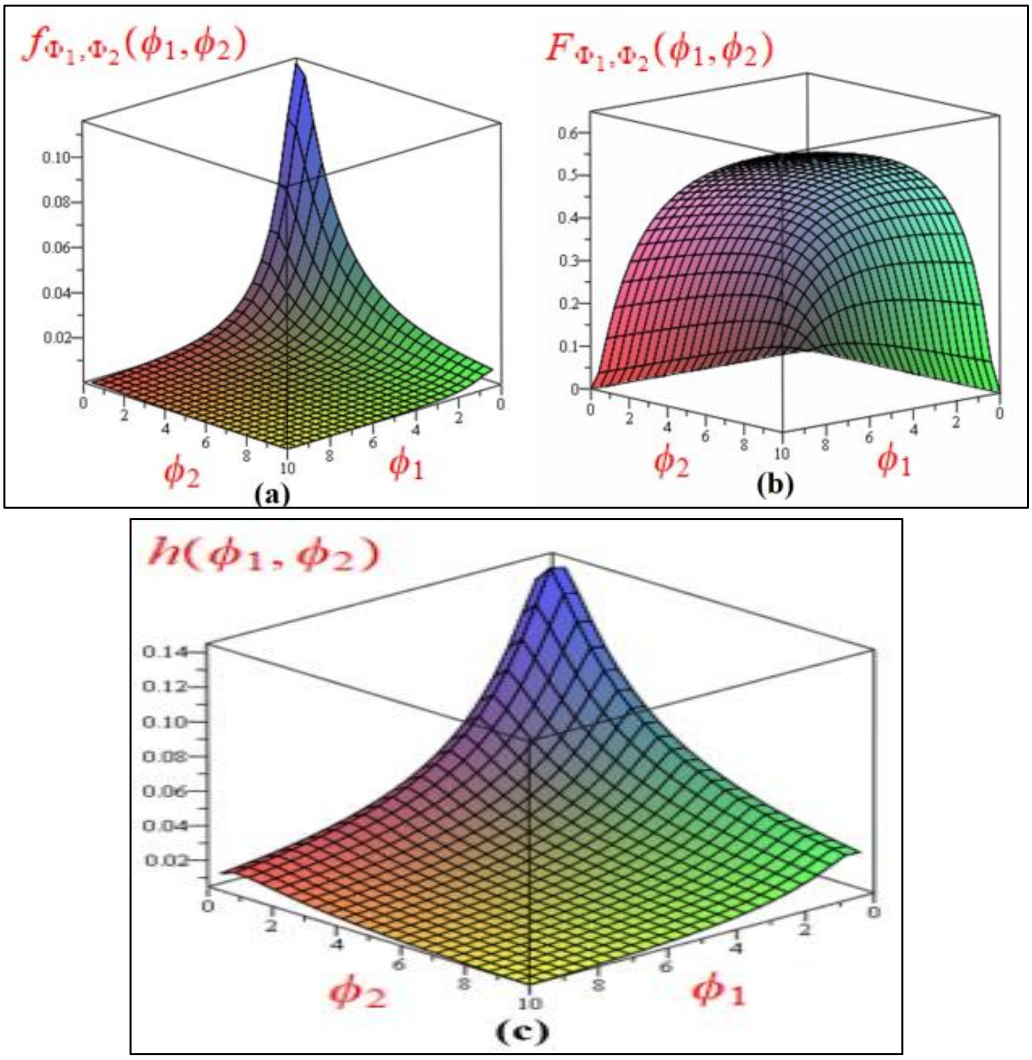
$$F_{\gamma_1}=2\sum_{i=1}^{n}\frac{\Upsilon(\gamma_1,\phi_{1i})}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})},\quad F_{\gamma_2}=2\sum_{i=1}^{n}\frac{\Upsilon(\gamma_2,\phi_{2i})}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})},$$

And $F_{\gamma_3}=2\sum_{i=1}^{n}\frac{\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})}{1+\Upsilon(\gamma_1,\phi_{1i})+\Upsilon(\gamma_2,\phi_{2i})}$, where $\Upsilon(\gamma,x)$ is given by (11).

8. The BISPTIHLIW Model: Simulation

Here we discuss numerically the characteristics of the new BISPTILIW distribution then use simulation to investigate the goodness of fit. We con-sider three different random variables. The first one is the BISPTIHLIW (1.3, 0.9, 2.5, 0.9, 0.3, 0.8, 0.2, 0.9, 0.6) whose jpdf, jcdf, marginals, and cop-ula are plotted in Fig. 3. We can observe the right skewness and unimodal-ity of the jpdf and marginals in Fig. 3 (a), 3(f), and 3(g). The jcdf $\longrightarrow$ 0.6, Fig. 3 (b), and cdfs $\longrightarrow$ 0.5, Figs. 3 (g) and 3(i), where (Φ_1, Φ_2) (10, 10), that suggests applying this model for the data with measurements scale in the interval (0,20) $\times$ (0,20). The hazard function is a decreasing function that approaches zero where $(\Phi_1, \Phi_2) \longrightarrow$ (10,10), Fig. 3 (c). The second one is the BISPTIHLIW (2.1, 1.1, 0.7, 1.2, 0.9, 1.2, 0.6, 0.4, 0.8), its correspond-ing functions are displayed in Fig. 4. In this case the jcdf $\longrightarrow$ 0.2, Fig. 4 (b), and cdfs $\longrightarrow$ 0.9, Figs. 4 (g) and 4(i), where $(\Phi_1, \Phi_2) \longrightarrow$ (10, 10), that sug-gests applying this model for the data with measurements scale in the interval (0,80) $\times$ (0,80): The hazard function changes its behaviour from increasing to decreasing for $(\phi_1, \phi_2) \in$ (10,10), Fig. 4 (c). The right skewness and unimodal-ity will be clear for higher measurements scale, Figs. 4 (a), 4(f), 4(h). The third one is the BISPTIHLIW (0.9, 1.7, 1.9, 1.6, 1.3, 1.6, 0.6, 0.7, 0.8) Its related functions are given in Fig. 5. The jcdf $\longrightarrow$ 1:0, Fig. 3 (b), and cdfs $\longrightarrow$ 1.0, Figs. 3 (g) and 3(i), where $(\Phi_1, \Phi_2) \longrightarrow$ (4, 4), that suggests applying this model for the data with measurements scale in the interval (0,4) $\times$ (0, 4): Also grantee the applicability of the new model of ratio data. The hazard function changes its behaviour from increasing to decreasing for $(\phi_1, \phi_2) \in$ (10, 10), Fig. 5 (c). Comparison between Figs. 3(d), 3(e), 4(d), 4(e) and 5(d), 5(e) shows the different levels of correlation between different set of variables. The goodness of fit is investigated via Monte Carlo simulation. Samples are generated with sizes (n=30, n=50, n=120) for different three sets of parameters. The average estimates with bias and standard errors are displayed in Table 5, Table 6, and Table 7. Using bias and standard errors as a comparison criteria, we can see that the maximum likelihood method gives good estimates for the unknown parameters. We consider $\gamma_1, \gamma_2,$ and γ_3 as chosen parameters.

Fig-3. Functions corresponding to $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ = (1.3, 0.9, 2.5, 0.9, 0.3, 0.8, 0.2, 0.9, 0.6).



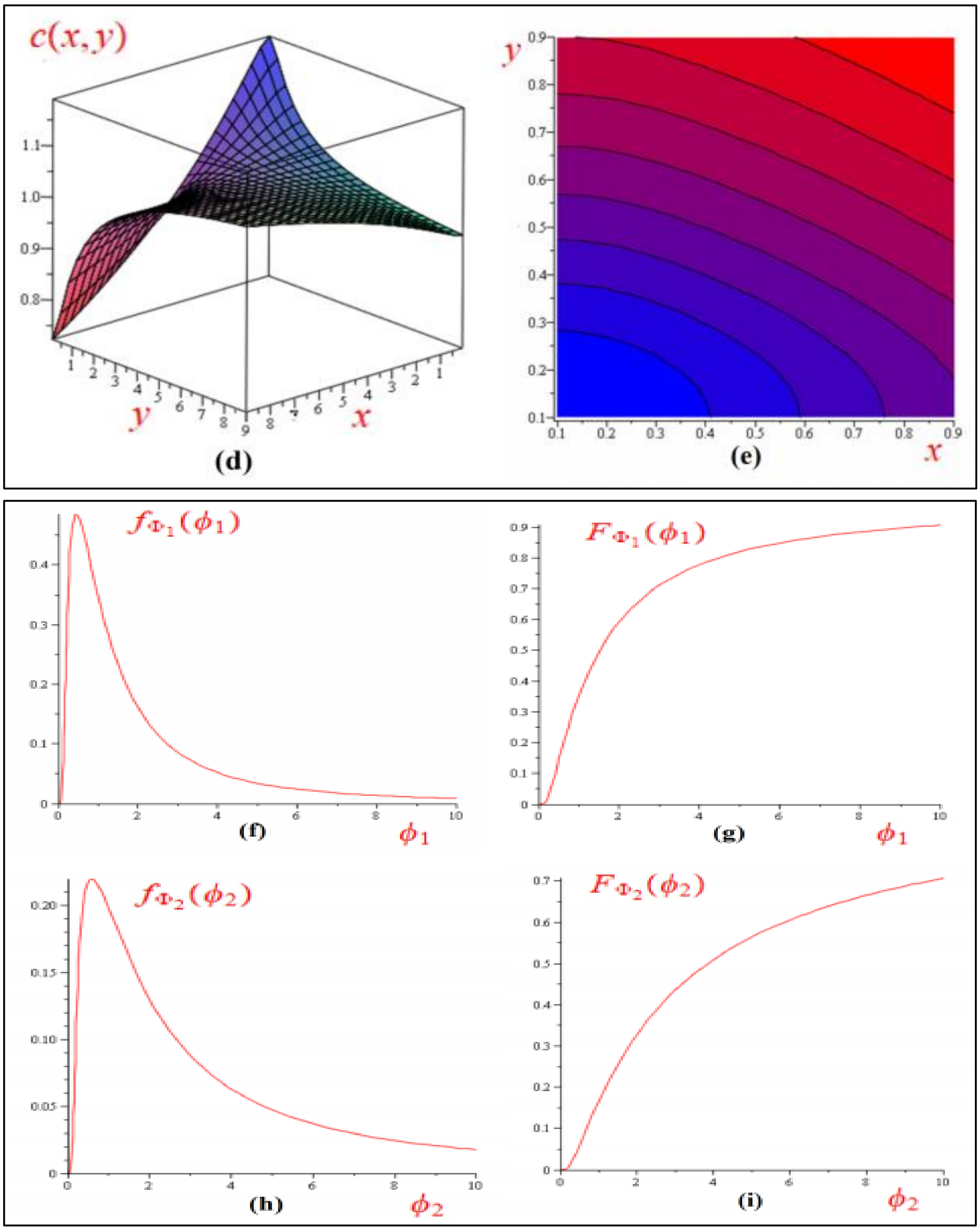
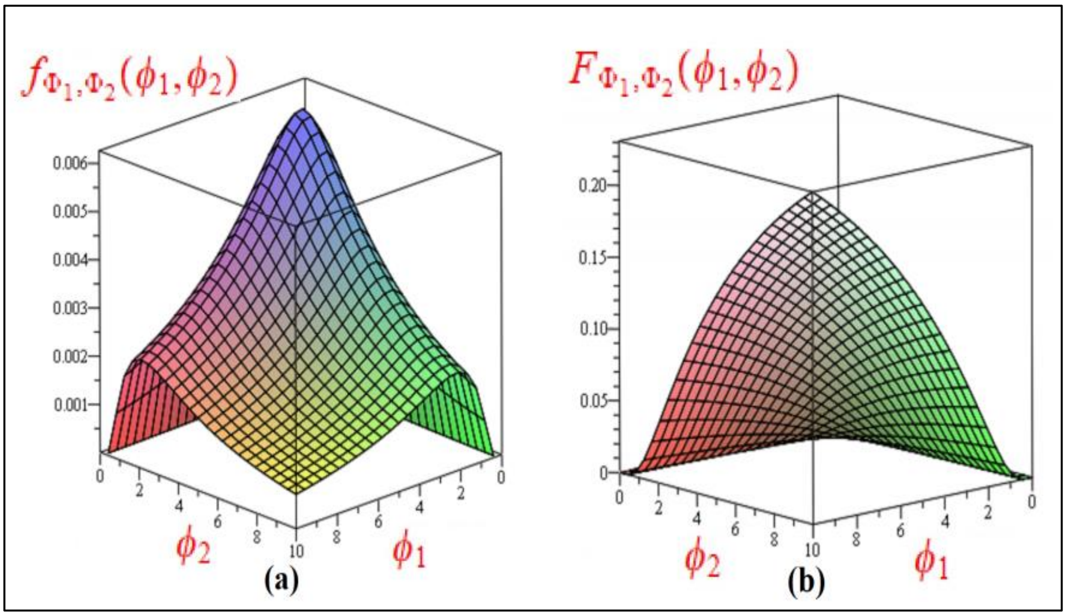


Fig-4. Functions corresponding to $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ = (2:1; 1:1; 0:7; 1:2;0:9; 1:2; 0:6; 0:4; 0:8)



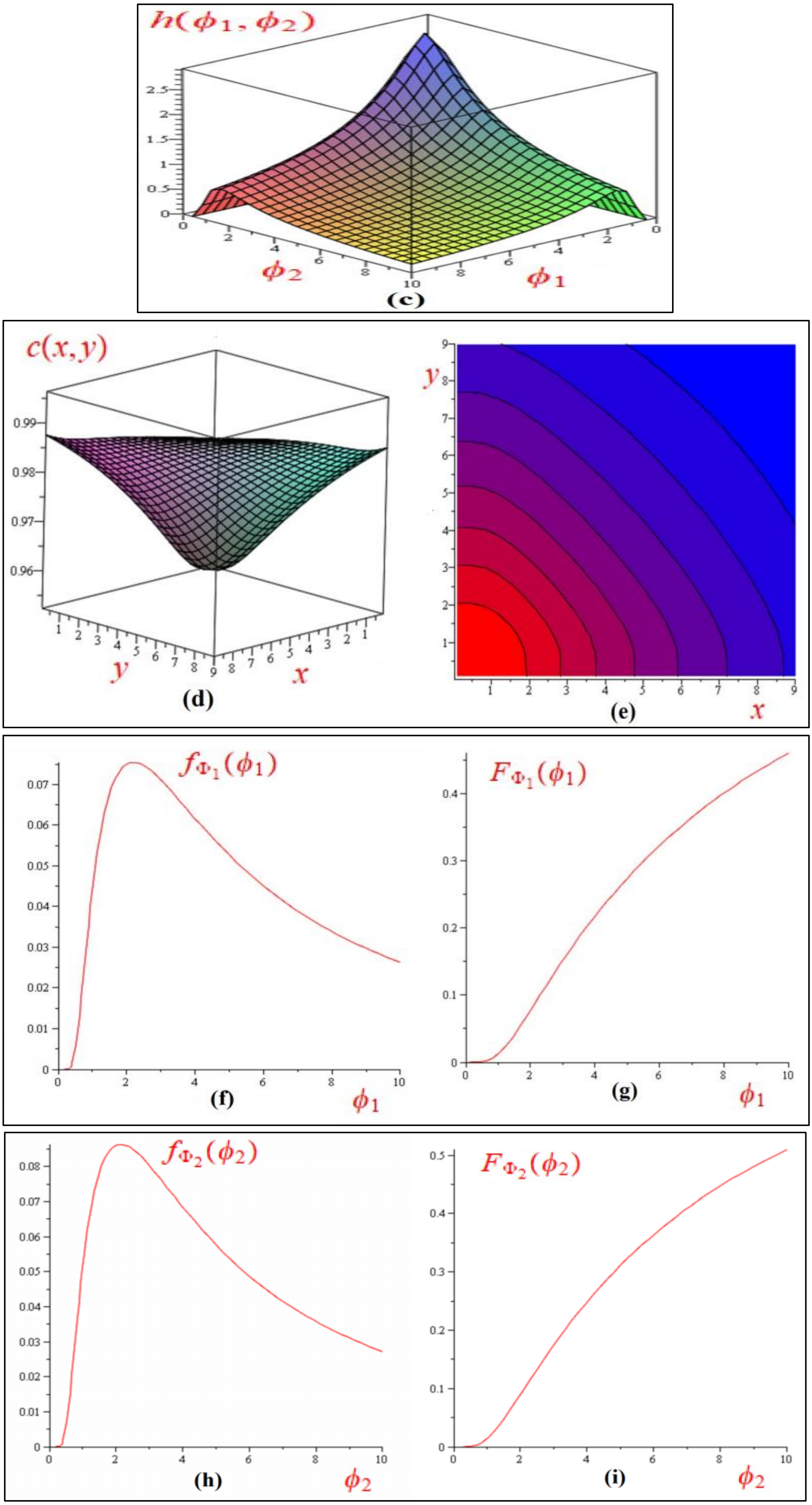
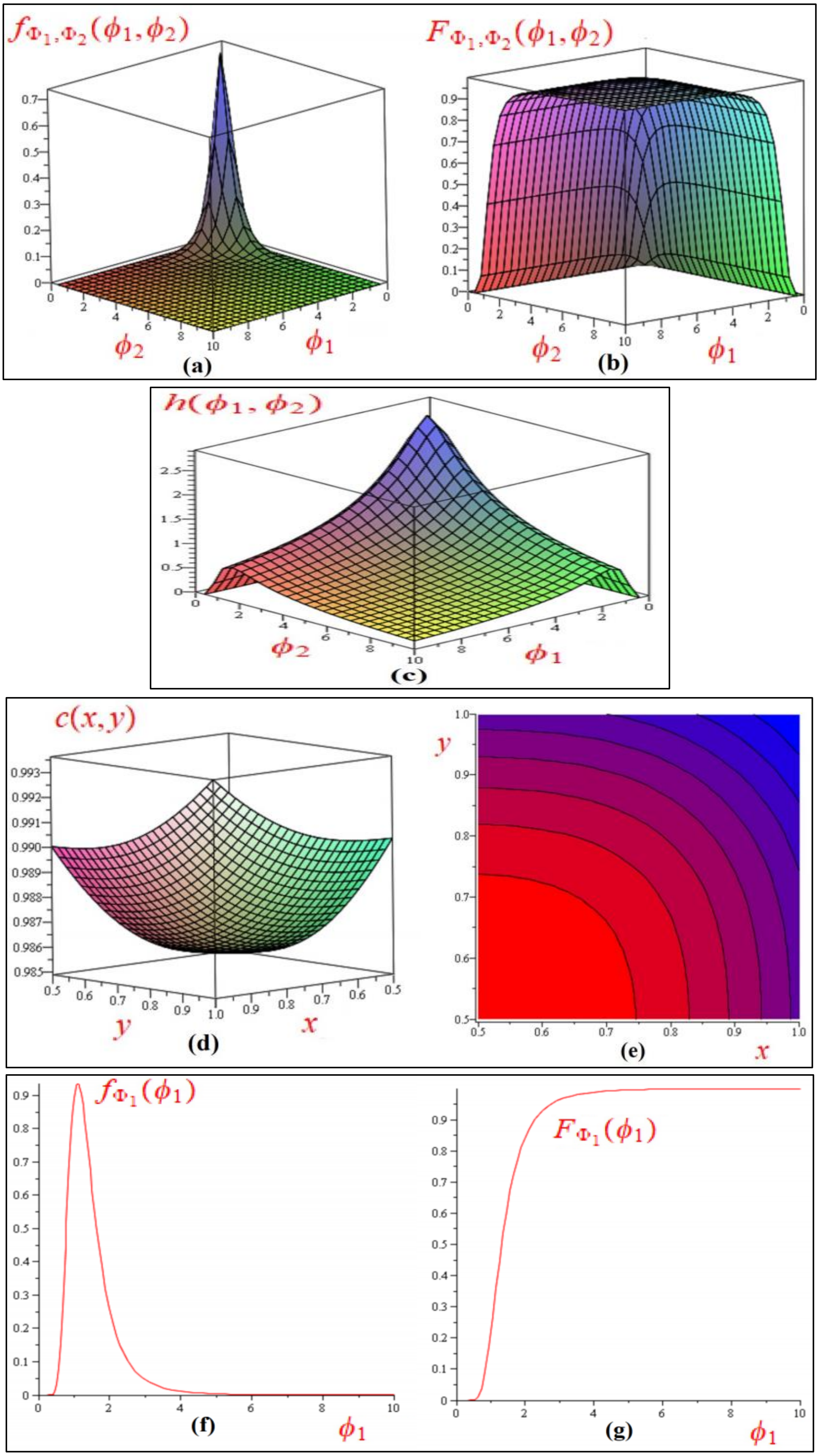

h(φ1, φ2)
φ2
φ1
(c)
c(x,y)
y
x
(d)
(e)
fΦ1(φ1)
(f)
FΦ1(φ1)
(g)
fΦ2(φ2)
(h)
FΦ2(φ2)
(i)

Fig-5. Functions corresponding to $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ = (0.9, 1.7, 1.9, 1.6, 1.3, 1.6, 0.6, 0.7, 0.8.



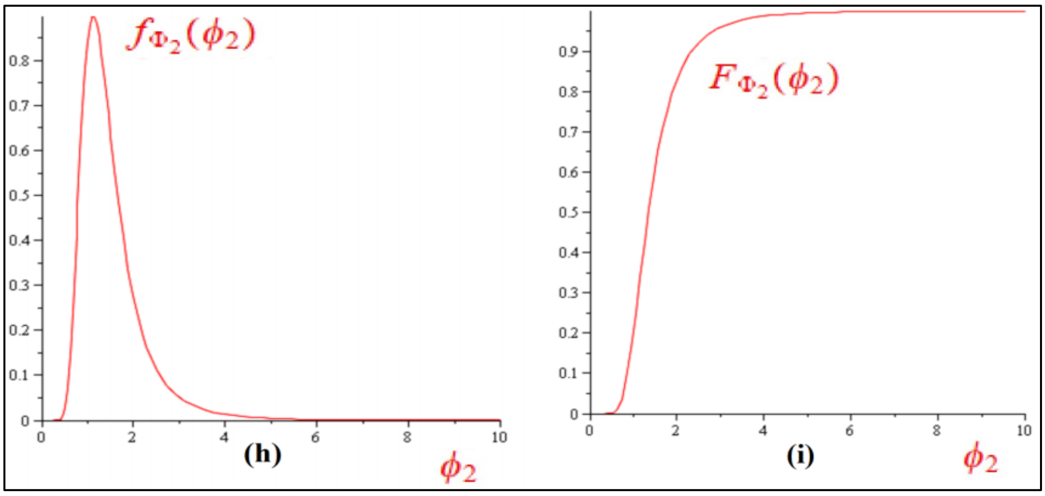


Table 5. Estimated parameters corresponding to $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta)$ = (3.0, 2.0, 1.0, 4.0, 3.0, 2.0).

Sample Size	Parameter	Average Estimate	Bias	SE
$n = 30$	ϵ	3.53	0.530	1.01
	ε	3.76	1.76	1.02
	ζ	0.378	-0.622	0.0687
	η	4.004	0.004	0.421
	θ	2.958	0.042	0.220
	ϑ	3.21	1.21	1.25
$n = 50$	ϵ	2.541	-0.495	0.441
	ε	3.37	1.37	1.26
	ζ	0.695	-0.305	0.150
	η	4.344	0.344	0.760
	θ	2.623	-0.377	0.292
	ϑ	2.95	0.950	1.93
$n = 120$	ϵ	2.854	-0.146	0.387
	ε	2.371	0.371	0.870
	ζ	0.953	-0.047	0.243
	η	4.001	0.001	0.604
	θ	3.342	-0.342	0.372
	ϑ	2.398	0.398	0.453

Table-6. Estimated parameters corresponding to $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta)$ = (1.4, 1.5, 1.8, 1.4, 1.8, 1.2).

Sample Size	Parameter	Average Estimate	Bias	SE
$n = 30$	ϵ	1.565	0.165	0.273
	ε	2.214	0.714	0.238
	ζ	2.554	0.754	0.659
	η	1.578	0.178	0.244
	θ	1.598	-0.202	0.179
	ϑ	1.484	0.284	0.285
$n = 50$	ϵ	1.271	-0.129	0.216
	ε	2.009	0.509	0.422
	ζ	2.248	0.448	0.328
	η	1.860	0.460	0.208
	θ	2.156	0.356	0.138
	ϑ	2.977	1.777	0.274
$n = 120$	ϵ	1.421	0.021	0.249
	ε	1.718	0.218	0.196
	ζ	2.077	0.277	0.296
	η	1.645	0.254	0.169
	θ	1.639	-0.161	0.047
	ϑ	1.284	-0.084	0.225

Table-7. Estimated parameters corresponding to $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta)$ = (2:5; 1:7; 0:9;2:6; 2:7; 0:8).

Sample Size	Parameter	Average Estimate	Bias	SE
$n = 30$	ϵ	2.002	-0.498	0.501
	ε	2.105	0.405	0.424
	ζ	1.558	0.658	0.423
	η	3.897	1.297	0.901
	θ	2.122	-0.578	0.357
	ϑ	1.378	0.587	0.460
$n = 50$	ϵ	2.435	-0.065	0.500
	ε	1.748	0.048	0.370
	ζ	1.942	1.042	0.400
	η	3.480	0.880	0.840
	θ	2.504	-0.196	0.302
	ϑ	1.290	0.490	1.470
$n = 120$	ϵ	2.550	0.050	0.447
	ε	1.793	0.093	0.342
	ζ	0.646	-0.254	0.422
	η	2.973	0.373	0.320
	θ	2.786	0.186	0.245
	ϑ	0.972	0.172	0.474

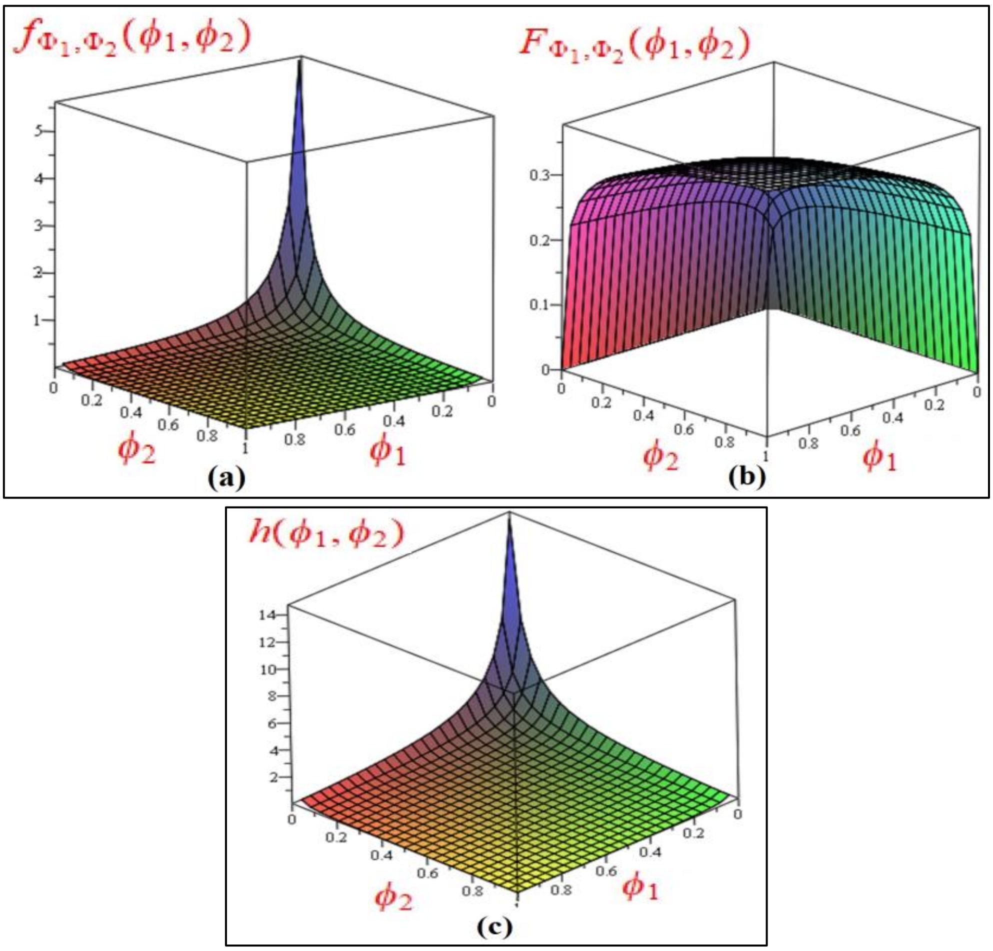
9. Mortality COVID-19 Data

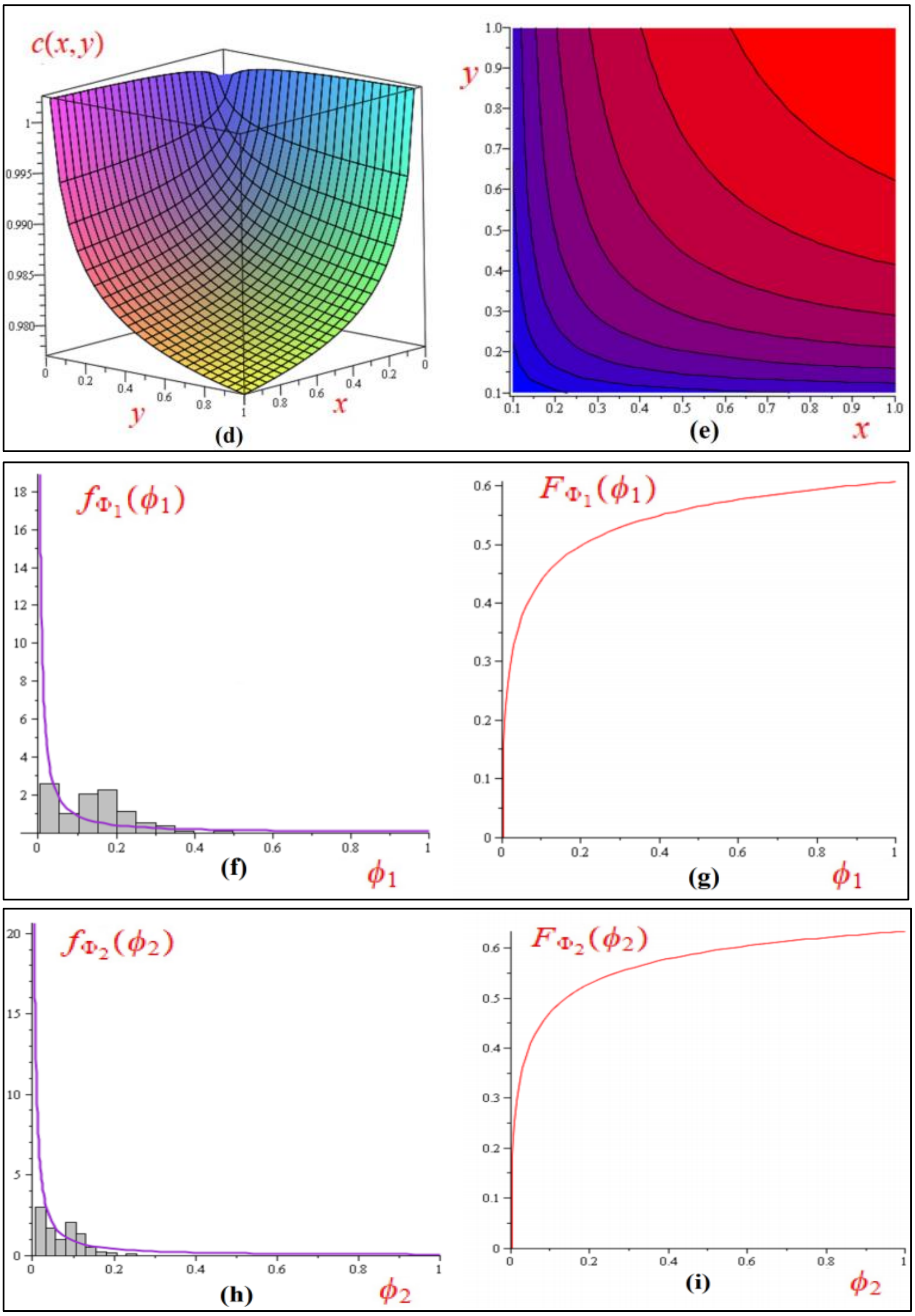
Here the new *BISPTIHLIW* $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$ is used to model the mortality COVID-19 data for Italy and Canada in the period from 1 April to 21 August 2020. Data is available at https://github.com/CSSEGISandData/ COVID-19/ and in [32, 33]. We consider the two dimensional random variable (Φ_1, Φ_2), with observed values the mortality COVID-19 data. The estimated parameters are given in Table 8. Its corresponding functions are plotted in Fig. 6.

Table-8. Estimated parameters for the $(\epsilon, \varepsilon, \zeta, \eta, \theta, \vartheta, \gamma_1, \gamma_2, \gamma_3)$.

Parameter	Estimate	AIC	BIC
ϵ	4.66303844		
ε	0.38218062		
ζ	0.50162001		
η	0.33702847	-133.2667	-106.7279
θ	0.26290366		
ϑ	0.10324411		
γ_1	0.07491866		
γ_2	0.20259496		
γ_3	-0.57018218		

Fig-6. Statistical quantities for the random variable (U,V) $\sim$ *BIETIHLIW* (3.4884764, 0.5391180, 1.0062037, 2.9732696, 10.3121449, 0.2651493, 0.9940035, 0.6297572)





10. Conclusion

In this study a new univariate six-parameter type I half-logistic inverse Weibull distribution has been introduced, that is a generalization of the model intro-duced by Elhassanein [18]. Statistical properties of the new model including the pdf, the cdf, the s^{th} moment and the moment generating function have been computed in explicit forms. The flexibility of the model has been proved for different values of parameters. The results of simulation study showed the good performance of the model in terms of maximum likelihood method. The model showed good results via comparison with competitive models and ap-plicability for different types of data. The bivariate extension BISPTIHLIW model has been formulated. That has an absolutely continuous pdf and its statistical quantities are available in explicit forms. Simulation showed a good performance of the model with respect to the goodness of fit. The applicability of the bivariate model has been proved for different types of data using the COVID-19 mortality data for Italy and Canada that are treated as a bivariate random variable.

Conflict of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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