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A Note on the Approximate Formula for the Associated Legendre Function

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Abstract

The approximate formula for the associated Legendre function has been used for the determination of the wavelength and phase velocity of free oscillations of the Earth and long-period surface waves. The approximate formula is defined for several mathematical parameters. One of them is $n >> 1/\varepsilon$, where n is the angular degree and ε is a parameter related to the colatitude angle θ. In the present study the relationship between n and ε is defined as $n_0 = [\ 1/\varepsilon\ ] +1$, where the notation [ ] denotes the Gaussian symbol. If a condition $n \geqq n_0$ $(n>1/\varepsilon)$ is assumed, the approximate formula may be reasonably applied for the angular degrees $n \geqq 2$ for $\pi/6 \leqq \theta \leqq 5\pi/6$. This condition is in harmony with the practical usage implemented conventionally.

Keywords: Approximate formula; Associated legendre function; Angular degree; Seismic surface waves.

1. Introduction

Theoretical computations of the free oscillation of a heterogeneous Earth model were studied by Pekeris and Jarosh [1] and Alterman, *et al.* [2]. The equation of wave motion for free oscillation of the Earth for the colatitude component can be represented by the associated Legendre function. Brune, *et al.* [3], theoretically determined the phase velocities of an Earth model using the approximate formula of the associated Legendre function and compared those with the observed phase velocities from spheroidal oscillations caused by the Chilean earthquake of May 22, 1960. The approximate formula has been conventionally implemented for the study of seismic surface waves for small and large angular degrees [4-7].

In the early years of wave motion studies, the threshold size of the angular degree n was discussed by Matsumoto and Sato [8] and Saito [9]. However, as far as the author is aware, no study of this threshold size has been published yet. The present study aims only to give a simple mathematical condition for the approximate formula, which is accordant with practical usage for small and large angular degrees.

2. Associated Legendre Function

The equation of wave motion in the spherical coordinate system [5, 10, 11] is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial S}{\partial \theta}\right) + \frac{1}{r^2 sin\theta}\frac{\partial^2 S}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2}, \qquad (1)$$

where r is the radius of the Earth, θ and ϕ are respectively the colatitude and azimuthal angles, and c is the wave speed. Introducing the constants m^2 and $n(n+1)$ into the wave equation for the colatitude angle yields the following:

$$\frac{d}{d\vartheta}\left(sin\vartheta\,\frac{d\Theta}{d\vartheta}\right) - \left[\frac{m^2}{sin^2\vartheta} - n(n+1)\right](sin\vartheta)\Theta = 0. \qquad (2)$$

For $m \neq 0$ the solutions are given by the associated Legendre functions $P_n^m(x)$ and $Q_n^m(x)$ [12]:

$$\Theta = P_n^m(x) = (1-x^2)^{m/2}\, d^m P_n(x)/dx^m$$
$$\Theta = Q_n^m(x) = (1-x^2)^{m/2}\, d^m Q_n(x)/dx^m, \qquad (3)$$

where $\cos\theta = x$. Equation (2) is the associated Legendre differential equation. In Eq. (3), the first kind of the associated Legendre function $P_n^m(x)$ and the second kind of the associated Legendre function $Q_n^m(x)$ give independent solutions for $-1 < x < 1$.

The solutions of the free oscillations of the Earth have the following form Brune, *et al.* [13]; Saito [9]:

$$u = U_n(r)\, Y_n(\theta, \phi)\, e^{i\omega t}, \qquad (4)$$

where $U_n(r)$ is the displacement function which depends on r and n, ω is the angular frequency, and $Y_n(\theta, \phi)$ represents a surface spherical function of degree n such that

$$Y_n(\theta,\phi) = P_n^m(\cos\theta)\, e^{im\phi}. \quad (5)$$

The parameters n and m are the angular and azimuthal degrees, respectively. The former is the parameter related to the node in the latitudinal direction and the latter is related to the node in the longitudinal direction.

3. Approximate Formula

When n is much greater and satisfies n >> m, with n and m being positive real numbers, the associated Legendre function is approximated as Moriguchi, *et al.* [12]:

$$P_n^m(\cos\theta) \sim e^{m\pi i}\, n^m\, (2/n\pi \sin\theta)^{1/2} \cos[(n+1/2)\theta + m\pi/2 - \pi/4] \quad (6)$$

$$[\varepsilon \leqq \theta \leqq \pi - \varepsilon,\ \varepsilon > 0,\ n \gg m,\ n >> 1/\varepsilon]. \quad (7)$$

The corresponding wavelength λ and phase velocity C are given by

$$\lambda = 2\pi a/(n+1/2) \quad (8)$$

$$C = \omega a/(n+1/2) = 2\pi a/(n+1/2)T, \quad (9)$$

where a is the radius of the Earth (a=6371 km) and T is the eigenperiod. Equation (8) is called the formula of Jeans [14]. In the present study, a spherically symmetric Earth is assumed and the parameter m=0 is assumed.

The approximate formula Eq. (6) is derived from the first term of the infinite series of the associated Legendre function for mathematical conditions n>>1 and n>>m [12]. Brune, *et al.* [13], derived an asymptotic form similar to Eq. (6) from asymptotic expansions of $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$. The asymptotic form may apply when

$$n >> 1,\ \varepsilon < \theta < \pi - \varepsilon,\ 0 < \varepsilon << \pi/6. \quad (10)$$

By an application of Stirling's formula, Ben-Menahem and Singh [4] obtained an asymptotic expression similar to Eq. (6) for

$$n >> m,\ \varepsilon \leqq \theta \leqq \pi - \varepsilon,\ \varepsilon : \text{a positive fixed number.} \quad (11)$$

By an application of the JWKB approximation, Dahlen and Tromp [5] derived an asymptotic representation similar to Eq. (6) for.

$$n >> 1,\ n >> m,\ 0 << \theta << \pi. \quad (12)$$

Kunimasa [15], derived an approximate formula similar to Eq. (6) through Schlăfli's integral, Goursat's theorem, Laplace's integral and Saddle point method , for

$$n \to \infty,\ n >> m. \quad (13)$$

Although the derivation processes are different, the above mathematical conditions are approximately close to each other.

To clarify the characteristics of Eq. (6), the relationship between ε and θ is shown in Table 1. Here, the integer n_0 is defined as

$$n_0 = [1/\varepsilon] + 1, \quad (14)$$

where the notation [ ] denotes the Gaussian symbol. The number n_0 may be useful to evaluate the range n >> 1/ ε in Eq. (7). However, as shown in the column (C) in Table 1, we can not determine the integer n which satisfies the inequality equation n >>1/ε , because the distinction in symbols of inequality between >> and > seems to be not rigorously defined [16]. Table 1 shows that the relationship $n \geqq n_0$ can be satisfied if n > 1/ε ; but not if n >> 1/ ε . By introducing parameter n_0, we can explicitly determine the angular degree n.

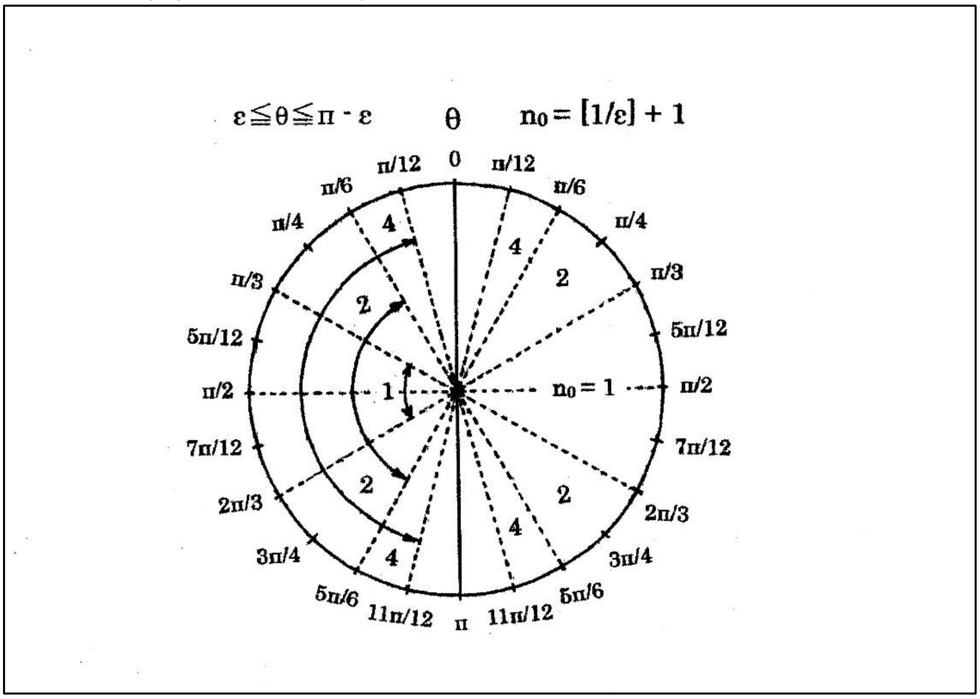
Table-1. Relationship between the colatitude angle θ and the angular degree n of $P_n^m(\cos\theta)$. Columns (A), (B) and (C) are parameters which are defined on the basis of Eq. (7). Columns (D) and (E) are parameters defined in the present study

(A)	(B)	(C)	(D)	(E)
ε	$\varepsilon \leqq \theta \leqq \Pi - \varepsilon$	$n >> 1/\varepsilon$	$n_0 = [1/\varepsilon] + 1$	$n \geqq n_0$
Π / 12	Π/12≦θ≦11Π/12	n > > 3.819	4	n≧4
Π / 6	Π / 6≦θ≦ 5Π/ 6	n > > 1.909	2	n≧2
Π / 4	Π / 4≦θ≦ 3Π/ 4	n > > 1.273	2	n≧2
Π / 3	Π / 3≦θ≦ 2Π/ 3	n > > 0.954	1	n≧1
5Π / 12	5Π/12≦θ≦7Π/12	n > > 0.763	1	n≧1
Π / 2	Π / 2 ≦θ≦ Π / 2	n > > 0.636	1	n≧1

4. Discussion

The area of n_0 given in Table 1 is shown in Figure 1. The θ-area related to parameter n_0 depends on parameter. In Table 1, parameter $\varepsilon = \pi/12$ is adopted, but if $\varepsilon = \pi/24$ is adopted, then $1/\varepsilon \fallingdotseq 7.6$. Namely, n_0 becomes n_0 = [7.6]+1 = 8. Therefore, the inequality equation $n \geqq n_0$ is becomes $n \geqq 8$ for $\pi/24 \leqq \theta \leqq 23\pi/24$. For other θ-area, the relationship between ε and θ shown in Table 1 is preserved. Figure 1 also shows that if $n \geqq n_0$ (n>1/ε) , the computation of the wavelength and phase velocity may be admissible for small angular degrees $n \geqq 2$ for $\pi/6 \leqq \theta \leqq 5\pi/6$. The inequality equation $n \geqq 1$ in the column (E) in Table 1 is not in harmony with the mathematical conditions n >> 1 and n >> m for which the approximate formula Eq. (6) is derived as mentioned earlier.

Figure-1. Relationship between the colatitude angle θ and the integer n_0 defined by $n_0 = [1/\varepsilon] + 1$, where ε is a parameter related to the colatitude angle. The areas belonging to n_0 are indicated by arcs with arrows



Equation (6) is correct to terms of order $(1/n)^{1/2}$ for m≠0 and order $(1/n)^{3/2}$ for m=0 [8, 13]. Thus, the error increases with decreasing n. As mentioned in Section 3, Eq. (6) is derived from the first term of the infinite series of the associated Legendre function. The effect of higher terms of the infinite series on Eq. (6) will be studied in a future work.

Near the pole and antipode, there is a phase advance of $\pi/2$ due to a polar phase shift when surface waves travel those regions [13]. The wavelength and phase velocity near the pole and antipode have been investigated by several authors [17-23]. However, a discussion of these works is beyond the scope of the present study.

5. Conclusion

A mathematical condition for the approximate formula of the associated Legendre function was considered from a point of practical usage. For the determination of the wavelength and phase velocity, a condition $n \geqq n_0$ ($n>1/\varepsilon$) may be reasonable and practical rather than $n>>1/\varepsilon$. If $n \geqq n_0$, the computation of wavelength and phase velocity can be admissible for the angular degrees $n \geqq 2$ for $\pi/6 \leqq \theta \leqq 5\pi/6$.

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