



A Relativistic Correction for Boltzmann's Distribution

Dr. Lloyd G. Allred

5516 Azuza Ave, Las Vegas, NV 89122, USA

Email: lloydallred@email.com**Article History**

Received: 4 January, 2026

Revised: 27 February, 2026

Accepted: 3 March, 2026

Published: 9 March, 2026

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Abstract

The distribution of velocities of molecules in a gas is a well understood discipline of physics. The Maxwell-Boltzmann distribution describes the distribution of velocity of a molecule in a gas as a function of mass and temperature. Molecules with the same mass tend to have the same velocity. By expressing the same law in terms of energy alone, the distribution becomes independent of mass. This formulation provides a new universal gas law. In summary, as the molecules in the gas interact through collisions, the energies tend to equalize. For high-energy gases, the original law predicts velocities greater than the speed of light. If one uses Einstein's formula for energy ($E=mc^2$), then a relativistic correction is not required.

1. Introduction

The Maxwell-Boltzmann distribution function describes the distribution of velocity of a molecule in a gas as a function of mass and temperature. (See Reference [1], [2], [3]). This paper expresses the same law in terms of energy alone, independent of mass.

For smaller mass, the Maxwell-Boltzmann distribution predicts that the velocities will be much higher than for molecules with larger mass. For particles with the same energy, the velocity increase is inversely proportional to the inverse of the square root of the mass decrease. For electrons, a factor of 42.85 times faster than a proton (See Figure 1). This effect causes the electromagnetic pulse (EMP) associated with nuclear blasts.

When an atomic bomb is detonated, the electrons separate from the protons, producing enormous electromagnetic fields, sufficient to knock out electric power and damage electronic equipment.

When the Maxwell-Boltzmann distribution is expressed in terms of energy, mass disappears from the resulting probability density functions and cumulative distribution functions.

2. Results

Expressed in the energy domain, the shared probability density function of energy, E , for molecules in a gas becomes

$$f_E(E) = \frac{2}{\sqrt{\pi}} \sqrt{\frac{E}{(kT)^3}} e^{-\frac{E}{kT}} \quad (1)$$

The corresponding cumulative distribution function is

$$F_E(E) = \text{erf}\left(\sqrt{\frac{E}{kT}}\right) - \sqrt{\frac{4E}{\pi kT}} e^{-\frac{E}{kT}} \quad (2)$$

This same formulation applies to all of the species of the gas, independent of a molecule's mass. It therefore provides a unification for the Maxwell-Boltzmann distribution.

3. Background

According to definition, the cumulative distribution function of a random variable, X is defined in terms of probability (See Reference [4], [5])

$$F_X(x) = P(X < x) \quad (3)$$

Expressed verbally, for a real number x , the cumulative distribution function $F_X(x)$ is the probability that the random variable, X , is less than the real number x .

The probability density function is the derivative of the cumulative distribution (See Reference [4], [5])

$$f_E(E) = \frac{dF_E(E)}{dE} \quad (4)$$

The density function can be derived this way, but there is a simpler method. If variable y is a monotonous function of variable x , then a short interval around x of length dx will map to an interval around y of length dy . Furthermore, the probability that a random variable X lies in the interval dx will equal the probability that the mapped variable Y lies in the interval dy . From the definition of the probability density function, this can be expressed mathematically,

$$\begin{aligned} P(x < X < x + dx) &= f_x(x) dx \\ &= P(y < Y < y + dy) = f_y(y) dy \end{aligned} \quad (5)$$

Dividing by dx ,

$$f_x(x) = f_y(y) \frac{dy}{dx} \quad (6)$$

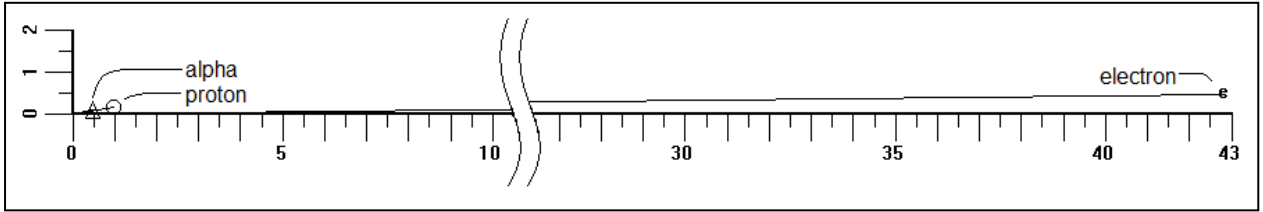


Figure-1. Relative Velocities of Particles in Plasma

Table-I. Lexicon of Special Use Terms

c	speed of light, about 3×10^8 meters per second
E	kinetic energy
E_T	total energy if a molecule or particle. $E_T = mc^2$
$erf()$	the normal (Gaussian) error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$
$f_B()$	Boltzmann density function
$F_B()$	Boltzmann distribution function
$f_E()$	Boltzmann density function as a function of kinetic energy
$F_E()$	Boltzmann distribution function as a function of kinetic energy
$f_X()$	density function of a random variable X
$F_X()$	cumulative distribution function of a random variable X
k	Boltzmann's constant 1.380649×10^{-23} Joules/ ^o Kelvin
m	mass
$P()$	P(event) is the probability of an event
T	Temperature
v	Velocity
μ_E	Average kinetic energy of molecules of a gas

4. Analysis

The traditional Boltzmann distribution, Reference [1], yields a different probability distribution function of velocity for each of the species of a gas, as a function of mass given by

$$F_B(v) = P(V < v)$$

$$\begin{aligned}
&= \operatorname{erf}\left(\frac{v}{\sqrt{\frac{2kT}{m}}}\right) - \sqrt{\frac{2}{\pi}} \frac{v e^{-\frac{mv^2}{2kT}}}{\sqrt{\frac{kT}{m}}} \\
&= \operatorname{erf}\left(\sqrt{\frac{mv^2}{2kT}}\right) - \sqrt{\frac{4mv^2}{\pi}} \frac{e^{-\frac{mv^2}{2kT}}}{\sqrt{kT}} \quad (7)
\end{aligned}$$

Before Einstein (in Boltzmann's era), kinetic energy was always computed

$$E = \frac{mv^2}{2} \quad (8)$$

By inspection, Equation (7) can be directly expressed in energy.

$$\begin{aligned}
F_E(E) &= F_B(v) \\
&= \operatorname{erf}\left(\sqrt{\frac{E}{kT}}\right) - \sqrt{\frac{4E}{\pi kT}} e^{-\frac{E}{kT}} \quad (9)
\end{aligned}$$

This establishes that the Boltzmann distribution can be expressed directly as a function of energy without having to appeal to the mass of each specie of gas.

Also from Reference [1], the probability density function of the Maxwell-Boltzmann distribution is given by

$$f_B(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2kT}} \quad (10)$$

The above function provides a finite probability that the velocity will exceed the speed of light, c . This cannot be realized in the physical universe. For extremely energetic gases, this probability becomes significant, and this classical formulation for the Boltzmann distribution is useless. To resolve this issue, the Boltzmann distribution can be expressed in terms of kinetic energy instead of velocity.

Reiterating Equation (8), before Einstein (in Boltzmann's era), kinetic energy was always computed

$$E = \frac{mv^2}{2} \quad (11)$$

Differentiating with respect to v ,

$$\frac{dE}{dv} = mv \quad (12)$$

Therefore

$$\frac{dv}{dE} = \frac{1}{mv} \quad (13)$$

From Equation (6),

$$\begin{aligned}
f_E(E) &= f_v(v) \frac{dv}{dE} \\
&= f_v(v) \frac{1}{mv} \\
&= \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2kT}} \frac{1}{mv} \quad (14)
\end{aligned}$$

Collecting terms inside the square root,

$$\begin{aligned}
f_E(E) &= 4\pi \sqrt{\left(\frac{m}{2\pi kT}\right)^3 \frac{v^2}{m^2}} e^{-\frac{mv^2}{2kT}} \\
&= 4\pi \sqrt{\frac{1}{4(\pi kT)^3} \frac{mv^2}{2}} e^{-\frac{mv^2}{2kT}} \quad (15)
\end{aligned}$$

This can be recognized as

$$f_E(E) = \frac{2}{\sqrt{\pi}} \sqrt{\frac{E}{(kT)^3}} e^{-\frac{E}{kT}} \quad (16)$$

This establishes the validity of Equation (1).

5. Kinetic Energy vs Average Temperature

It is a well-publicized conclusion that the average kinetic energy of molecules of a gas is proportional to temperature (See Reference [7]).

$$\mu_E = \frac{3}{2}kT \quad (17)$$

This conclusion is weak in comparison to the concept that different species of molecules share the same energy distribution.

6. Relativistic Considerations

If Boltzmann had expressed his law in terms of energy instead of mass and velocity, relativistic correction would not be required. Keep in mind that Boltzmann died just after Einstein published his first paper on relativity. Reiterating Equation (8),

$$E = \frac{mv^2}{2} \quad (18)$$

Einstein concluded that total energy can be expressed

$$E = mc^2 \quad (19)$$

Or more properly,

(20)

$$E_T = mc^2 \quad (21)$$

and that mass increases with energy.

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad (22)$$

where m_0 is rest mass, and v is velocity.

Total energy can be split into rest energy and kinetic energy.

$$E_T = E + m_0c^2 \quad (23)$$

Provides a formula for kinetic energy,

$$E = (m - m_0) c^2 \quad (24)$$

Solving for kinetic energy as a function of velocity,

$$E = \left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 \right) c^2 \quad (25)$$

The term m_0 can be factored out from the right side.

$$E = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \right) m_0 c^2 \quad (26)$$

This formula agrees with the classical Newtonian formula to within 0.1% for velocities less than 1/3 of the speed of light, $c/3=10^8$ meters per second.

Dividing both sides by m_0c^2 ,

$$\frac{E}{m_0c^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \quad (27)$$

Adding 1 to both sides,

$$\frac{E}{m_0c^2} + 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad (28)$$

Squaring and inverting both sides,

$$1 - \frac{v^2}{c^2} = \frac{1}{\left(\frac{E}{m_0c^2} + 1 \right)^2} \quad (29)$$

Solving for $\frac{v^2}{c^2}$,

$$\frac{v^2}{c^2} = 1 - \frac{1}{\left(\frac{E}{m_0c^2} + 1\right)^2} \quad (30)$$

Because the expression on the right is always less than 1, this formula always provides a velocity less than the speed of light, solving the conundrum with the traditional Boltzmann distribution which predicts velocities greater than the speed of light.

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