Scientific Review
ISSN(e): 2412-2599, ISSN(p): 2413-8835
Vol. 4, Issue. 12, pp: 95-98, 2018
URL: https://arpgweb.com/journal/journal/10
DOI: https://doi.org/10.32861/sr.412.95.98

Academic Research Publishing Group

Original Research **Open Access**

On the Fractional Optimal Control Problems With Singular and Non–Singular Derivative Operators: A Mathematical Derive

Tim Chen
Department of Mathematics, Cankaya University, 06530 Ankara, Turkey
Department of Electrical Engineering, University of Bojnord, Bojnord, Iran

Bunnitru Daleanu
Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Institute of Space Sciences, Magurele–Bucharest, Romania

J. C.-Y. Chen*
NAAM Research Group, King Abdulaziz University, Jeddah 21589, Saudi Arabia
St Petersburg Univ, Dept Math & Mech, Univ Skii 28, St Petersburg 198504, Russia

Abstract

The aim of this paper is to design an efficient numerical method to solve a class of time fractional optimal control problems. In this problem formulation, the fractional derivative operator is consid- ered in three cases with both singular and non–singular kernels. The necessary conditions are derived for the optimality of these problems and the proposed method is evaluated for different choices of derivative operators. Simulation results indicate that the suggested technique works well and pro- vides satisfactory results with considerably less computational time than the other existing methods. Comparative results also verify that the fractional operator with Mittag–Leffler kernel in the Caputo sense improves the performance of the controlled system in terms of the transient response compared to the other fractional and integer derivative operators.

Keywords: Fractional derivative; Optimal control; Necessary condition; Non–singular kernel; Iterative method.

1. Introduction

The fractional calculus (FC), as a branch of mathematical analysis, investigates the extension of deriva- tives and integrals to non–integer orders [1-3]. Nowadays, the new aspects of the FC are growing fast and its applications are found in different areas like chaos synchronization [4], diffusion equation [5], biology [6], control [7] and economic [8]. Additionally, the utility of the FC in the optimal control problems (OCPs) has attracted the attention of many researchers. Agrawal [9] approximated the solution of the fractional optimal control prob- lems (FOCPs) in the Riemann–Liouville sense as a truncated series. Biswas and Sen [10] applied the Grünwald–Letnikov (GL) formula to convert the FOCP with Caputo derivative into a system of algebraic equations. Frederico and Torres [11] applied a Noether–type theorem for the FOCPs in the Caputo sense. Almeida and Torres [12] approximated the FOCP by a new integer one and used a finite dif- ference method to solve it. Sweilam and Al-Mekhlafi [13] investigated the FOCPs via an iterative optimal control scheme together with a generalized Euler method. Ejlali and Hosseini [14] employed a new framework on the basis of the direct pseudospectral method for solving the FOCPs. Jahanshahi and Torres [15] rewrote the FOCP as a classical static optimization problem by using known formulas for the fractional derivative (FD) of polynomials, and then, they solved the latter problem by the Ritz method. In Bhrawy, *et al.* [16]; Rabiei, *et al.* [17], the approximate solutions of the FOCPs were investigated by using the Boubaker polynomials and Chebyshev-Legendre operational technique, respec- tively. Lotfi [18] applied a combined penalty and variational methods for the OCPs in fractional sense. Zaky [19] employed a Legendre collocation method for distributed–order FOCPs. [20] suggested an approximation scheme to deal with the FOCPs by hybrid functions. In a recent study by Sahu and Ray [21], a comparison was done between orthonormal wavelets to solve the OCPs in fractional sense. More recently, a new iterative algorithm was examined by Jajarmi, *et al.* [22] for the nonlinear FOCPs with external persistent disturbances.

The FC includes several definitions of the fractional operators. This aspect can be considered as an advantage namely for a given complex dynamic we can choose an adequate fractional operator with or without singularity. In addition, for the control of complex systems, there is still a need of introducing new methods and techniques. Another important issue which is appeared in the control theory is to develop an appropriate strategy to control the state trajectories both in the transient and steady–state responses. Motivated by the above discussion, the aim of this paper is to design an efficient numerical technique to solve the FOCPs with both singular and non–singular operators. We derive the necessary conditions for the optimality of these problems and evaluate the performance of the new method for different cases of FDs. Simulation results verify that the suggested technique works well with low computational effort compared to the recent methods available in the literature. In addition, the performance of the controlled system in terms of the transient response is improved via the FD with Mittag–Leffler (ML) kernel as compared to the other fractional and integer derivative operators.

*Corresponding Author

The rest of this paper is structured as follows. In the next section, we give a brief introduction regard- ing the fractional operators. Next, we formulate a FOCP and derive its necessary optimality conditions. In Sect. 4, we suggest an efficient iterative method, which solves the state and costate equations forward and backward in time, respectively. Our numerical findings are reported in Sect. 5, which indicate the effectiveness of the proposed approach. Finally, the paper is finished by some concluding remarks.

2. Notations and Preliminary Results

Some notations and preliminary results regarding the fractional derivatives and integrals are given in this section. Several definitions of the FDs have been presented so far. Following [23], here we define the general forms of the left– and right–sided FDs (in the Caputo sense) as follows

$$ {}_0D_t^\rho y(t) = a_l(\rho) \int_0^t \dot{y}(\zeta) K_l(t-\zeta,\rho) d\zeta, \tag{1} $$

$$ {}_tD_T^\rho y(t) = a_r(\rho) \int_t^T \dot{y}(\zeta) K_r(\zeta-t,\rho) d\zeta, \tag{2} $$

respectively, where y : (0,T) →R is a time-dependent function, 0 < ρ < 1 denotes the fractional order of derivative, al,ar are constant coefficients for a given parameter ρ and Kl,Kr are nonnegative integrable kernel functions. The left– and right–sided fractional integrals (FIs) corresponding the definitions (1)–(2) are respectively expressed as

$$ {}_0I_t^\rho y(t) = b_l(\rho) y(t) + c_l(\rho) \int_0^t y(\zeta) M_l(t-\zeta,\rho) d\zeta, \tag{3} $$

$$ {}_tI_T^\rho y(t) = b_r(\rho) y(t) + c_r(\rho) \int_t^T y(\zeta) M_r(\zeta-t,\rho) d\zeta, \tag{4} $$

where bl,br,cl,cr are given constant coefficients and Ml,Mr are the kernel functions as before.

The equations mentioned above include the conventional Caputo, CF and AB–Caputo fractional operators as their particular cases; hence, according to the above–mentioned notations, we can state the following definitions for these operators, respectively.

Definition 2.1. [1] *For y : (0,T) → AC(0,T) and 0 < ρ < 1, the left– and right–sided Caputo FDs of order ρ are respectively described by Eqs. (1) and (2), where al(ρ) = −ar(ρ) = 1 Γ(1−ρ) , Kl(t−ζ,ρ) = (t−ζ)−ρ and Kr(ζ −t,ρ) = (ζ −t)−ρ. Also, the corresponding left– and right–sided FIs are respectively given by Eqs. (3) and (4), where bl(ρ) = br(ρ) = 0, cl(ρ) = cr(ρ) = 1 Γ(ρ), Ml(t−ζ,ρ) = (t−ζ)ρ−1 and Mr(ζ −t,ρ) = (ζ −t)ρ−1.*

Definition 2.2. [24] *For y : (0,T) → H1(0,T) and 0 < ρ < 1, the left– and right–sided CF FDs of order ρ are respectively described by Eqs. (1) and (2), where al(ρ) = −ar(ρ) = 1 1−ρ , Kl(t−ζ,ρ) = exp[− ρ 1−ρ (t−ζ)]and Kr(ζ−t,ρ) = exp[− ρ 1−ρ (ζ −t)]. Also, the corresponding left– andright–sided FIs are respectively given by Eqs. (3) and (4), where bl(ρ) = br(ρ) = 1−ρ, cl(ρ) = cr(ρ) = ρand Ml(t−ζ,ρ) = Mr(ζ −t,ρ) = 1.*

Definition 2.3.[25] *For y : (0,T) → H1(0,T) and 0 < ρ < 1, the left– and right–sided AB–Caputo FDs of order ρ are respectively described by Eqs. (1) and (2), where al(ρ) = −ar(ρ) = A(ρ) 1−ρ , Kl(t−ζ,ρ) = Eρ[− ρ 1−ρ (t−ζ)ρ], Kr(ζ −t,ρ) = Eρ[− ρ 1−ρ (ζ −t)ρ], A(ρ) is thenormalization function such that A(0) = A(1) = 1 and Eρ is the ML function. Also, the corresponding left– and right–sided FIs are respectively given by Eqs. (3) and (4), where bl(ρ) = br(ρ) = 1−ρ A(ρ), cl(ρ) = cr(ρ) = ρ A(ρ)Γ(ρ), Ml(t−ζ,ρ) = (t−ζ)ρ−1 and Mr(ζ −t,ρ) = (ζ −t)ρ−1.*

For more details on the mathematical characteristics of the fractional derivatives and integrations, we refer the interested reader to the studies by Podlubny [1], Losada and Nieto [26] and Abdeljawad and Baleanu [27].

3. Problem Formulation

This section is devoted to the FOCP formulation, in which the dynamic system includes the FDs in the form of Eq. (1). In the following, we first formulate the problem, and then we derive the necessary conditions for the optimality of this problem.

3.1. The Statement of the Problem

Here, we define the FOCP governed by minimizing the following cost functional

$$ J = \frac{1}{2} \int_0^T \left(y^T(t) Q y(t) + u^T(t) R u(t) \right) dt, \tag{5} $$

subject to the dynamic constraints

$$ {}_0D_t^\rho y(t) = Ay(t) + Bu(t) + f(y(t)), \; 0 < t < T, \tag{6} $$

and the initial condition

$$ y(t_0) = y_0, \tag{7} $$

where y(t) = (y1(t),...,yn(t)) is the state trajectory while u(t) = (u1(t),...,um(t)) denotes the control variable. The weighting coefficients Q and R within the cost functional (5) are positive semi–definite and positive definite matrices, respectively. The expression 0Dρ t y(t) represents the FD operator as defined by Eq. (1). The parameters A,B are constant matrices, f : Rn → Rn is a continuously differentiable function such that its gradient is Lipschitz continuous over the domain, and y0 ∈ Rn is a specified constant vector. In the problem formulation (5)–(7), the objective is to find the control function u∗(t) and the corresponding state trajectory y∗(t) satisfying Eqs. (6)–(7), which minimize the quadratic cost functional (5).

3.2. Necessary Optimality Conditions

To derive the necessary optimality conditions corresponding to the FOCP (5)–(7), one can follow the same procedure as in Biswas and Sen [10]. Note that, satisfying the fractional integration–by–parts formula is a key point for the necessary optimality conditions in fractional sense. The proof of this formula in terms of the Caputo and AB–Caputo has been given by Podlubny [1] and Abdeljawad and Baleanu [27], respectively. Following the same procedure as in Abdeljawad and Baleanu [27], we can show that the integration–by–parts formula is also satisfied in the CF sense. However, for the other types of derivative operators in the form of Eqs. (1) and (2), the correctness of this formula should be checked and verified. Following the same procedure used by Biswas and Sen [10], the optimal control law is obtained from

$$u^*(t) = -R^{-1}B^T p^*(t), 0 < t < T, \tag{8}$$

while p∗(t) is the solution of the following boundary value problem

$$\begin{cases} {}_0D_t^\rho y(t) = Ay(t) - Cp(t) + f(y(t)),\ 0 < t < T, \\ {}_tD_T^\rho p(t) = Qy(t) + A^T p(t) + g(y(t))p(t),\ 0 < t < T, \\ y(0) = y_0,\ p(T) = 0, \end{cases} \tag{9}$$

From the system of equations (9), it is obvious that the state equation in (9) involves the left–sided FD while the costate equation includes the right–sided one, simultaneously. This causes some difficulties to find the analytic solution of these equations effectively. To solve this problem, we will present an efficient approximation method in the next section.

4. Numerical Method

Applying the FI operator (3) to the both sides of the state equation in (9), the state equation is converted into a Volterra integral equation

$$y(t) = y_0 + b_l(\rho)\varphi(y(t), p(t)) + c_l(\rho)\int_0^t \varphi(y(\zeta), p(\zeta))M_l(t-\zeta,\rho)d\zeta, \tag{10}$$

where ϕ(y(t),p(t)) := Ay(t)−Cp(t)+f(y(t)). From Eq. (10) at t = tk+1 and in the i-th iteration of the proposed algorithm we obtain

$$\begin{aligned} y_{i,k+1} = y_0 &+ b_l(\rho)\varphi(y_{i,k+1}, p_{i-1,k+1}) \\ &+ c_l(\rho)\int_0^{t_{k+1}} \varphi(y_i(\zeta), p_{i-1}(\zeta))M_l(t_{k+1}-\zeta,\rho)d\zeta, \end{aligned} \tag{11}$$

where the values of the costate variable have been assumed to be known from previous iteration. Using the trapezoidal quadrature rule, we approximate the integration part in (11) as follows

$$\begin{aligned} I_{i,k+1} &= \int_0^{t_{k+1}} \varphi(y_i(\zeta), p_{i-1}(\zeta))M_l(t_{k+1}-\zeta,\rho)d\zeta \\ &\approx \int_0^{t_{k+1}} \hat{\varphi}_{i,k+1}(\zeta)M_l(t_{k+1}-\zeta,\rho)d\zeta, \end{aligned} \tag{12}$$

where ˆ ϕi,k+1(ζ) is a piecewise linear interpolation polynomial computed from

$$\begin{aligned} \hat{\varphi}_{i,k+1}(\zeta)\big|_{\zeta\in[t_j,t_{j+1}]} &\approx \frac{t_{j+1}-\zeta}{t_{j+1}-t_j}\varphi(y_{i,j}, p_{i-1,j}) \\ &+ \frac{\zeta-t_j}{t_{j+1}-t_j}\varphi(y_{i,j+1}, p_{i-1,j+1}),\ 0 \le j \le k. \end{aligned} \tag{13}$$

Using Eq. (13) in (12) we derive

$$\begin{aligned} I_{i,k+1} &\approx \sum_{j=0}^{k} \varphi(y_{i,j}, p_{i-1,j})I_{1,j} + \varphi(y_{i,j+1}, p_{i-1,j+1})I_{2,j} \\ &= \sum_{j=0}^{k+1} \alpha_{k+1,j}\varphi(y_{i,j}, p_{i-1,j}), \end{aligned} \tag{14}$$

5. Conclusion Remarks

In this section, we present an efficient iterative technique to solve the necessary optimality conditions stated in Eq. (9). For this purpose, first we divide the interval [0,T] into N equal subintervals with the mesh points tk = kh, 0 ≤ k ≤ N, is the time step size and N is an arbitrary positive integer. Let us denote yi(t), pi(t), ui(t) as the numerical approximations of y(t), p(t), u(t) in the i–th iteration of the proposed algorithm, respectively. Moreover, we consider the notations yi,k, pi,k, ui,k for the approximate values of yi(tk), pi(tk), ui(tk), respectively. Then, we continue with the derivation of predictor–corrector method for the state and costate equations in (9) forward and backward in time, respectively. Finally, we combine these two approaches for the state and costate equations by employing a forward–backward sweep iterative algorithm.

5.1. Funding Acknowledgement

This research was in part enlightened by the grant 106EFA0101550 of Ministry of Science and Technology, Taiwan.

5.2. Conflict of interest statement

The authors declare no conflict of interest in preparing this article.

References

[1] Podlubny, I., 1999. *Fractional differential equations, An introduction to fractional derivatives, frac- tional differential equations, to methods of their solution and some of their applications*. New York: Academic Press.

[2] Kilbas, A. A., Srivastava, H. H., and Trujillo, J. J., 2006. "Theory and applications of fractional differentialequations. New york " *Elsevier,*

[3] Baleanum, D., Diethelm, K., Scalas, E., and Trujillo, J. J., 2012. *Fractional calculus, Models and numerical methods*. Berlin: World Scientific.

[4] Jajarmi, A., M., H., and Baleanu, D., 2017. "New aspects of the adaptive synchronization and hyperchaos suppression of a financial model." *Chaos, Solitons & Fractals,* vol. 99, pp. 285–296.

[5] Moghaddam, B. P. and Machado, J. A. T., 2017. "A stable three-level explicit spline finite difference scheme for a class of nonlinear time variable order fractional partial differential equations. ." *Computers & Mathematics with Applications,* vol. 73, pp. 1262–1269.

[6] Jajarmi, A. and Baleanu, D., 2018. "A new fractional analysis on the interaction of hiv with cd4+ t-cells." *Chaos, Solitons & Fractals,* vol. 113, pp. 221–229.

[7] Dabiri, A., Moghaddam, B. P., and Machado, J. A. T., 2018. "Optimal variable–order fractional PID controllers for dynamical systems." *Journal of Computational and Applied Mathematics,* vol. 339, pp. 40–48.

[8] David, S. A., Fischer, C., and Machado, J. A. T., 2018. "Fractional electronic circuit simulation of a nonlinearmacroeconomic model." *AEU-International Journal of Electronics and Communications,* vol. 84, pp. 210–220.

[9] Agrawal, O. P., 2004. "A general formulation and solution scheme for fractional optimal control problems." *Nonlinear Dynamics,* vol. 38, pp. 323–337.

[10] Biswas, R. K. and Sen, S., 2010. "Fractional optimal control problems with specified final time." *Journal of Computational and Nonlinear Dynamics,* vol. 6,

[11] Frederico, G. S. F. and Torres, D. F. M., 2008. "Fractional conservation laws in optimal control theory." *Nonlinear Dynamics,* vol. 53, pp. 215–222.

[12] Almeida, R. and Torres, D. F. M., 2015. "A discrete method to solve fractional optimal control problems." *Nonlinear Dynamics,* vol. 80, pp. 1811–1816.

[13] Sweilam, N. H. and Al-Mekhlafi, S. M., 2016. "On the optimal control for fractional multi-strain TB model." *Optimal Control Applications and Methods,* vol. 37, pp. 1355–1374.

[14] Ejlali, N. and Hosseini, S. M., 2017. "A pseudospectral method for fractional optimal control problems." *Journal of Optimization Theory and Applications,* vol. 174, pp. 83–107.

[15] Jahanshahi, S. and Torres, D. F. M., 2017. "A simple accurate method for solving fractional variational and optimal control problems." *Journal of Optimization Theory and Applications,* vol. 174, pp. 156–175.

[16] Bhrawy, A. H., Ezz-Eldien, S. S., Doha, E. H., Abdelkawy, M. A., and Baleanu, D., 2017. "Solving fractional optimal control problems within a chebyshev–legendre operational technique." *International Journal of Control,* vol. 90, pp. 1230–1244.

[17] Rabiei, K., Ordokhani, Y., and Babolian, E., 2017. "The boubaker polynomials and their application to solve fractional optimal control problems." *Nonlinear Dynamics,* vol. 88, pp. 1013–1026.

[18] Lotfi, A., 2017. "A combination of variational and penalty methods for solving a class of fractional optimal control problems." *Journal of Optimization Theory and Applications,* vol. 174, pp. 65-82.

[19] Zaky, M. A., 2018. "A Legendre collocation method for distributed-order fractional optimal control problems." *Nonlinear Dynamics,* vol. 91, pp. 2667–2681.

[20] Mashayekhi, S. and Razzaghi, M., 2018. "An approximate method for solving fractional optimal control problems by hybrid functions." *Journal of Vibration and Control,* vol. 24, pp. 1621–1631.

[21] Sahu, P. K. and Ray, S. S., 2018. "Comparison on wavelets techniques for solving fractional optimal control problems." *Journal of Vibration and Control,* vol. 24, pp. 1185–1201.

[22] Jajarmi, A., Hajipour, M., Mohammadzadeh, E., and Baleanu, D., 2018. "A new approach for the nonlin- ear fractional optimal control problems with external persistent disturbances." *Journal of the Franklin Institute,* vol. 335, pp. 3938–3967.

[23] Luchko, Y. and Yamamoto, M., 2016. "General time–fractional diffusion equation, Some uniqueness and existence results for the initial–boundary–value problems." *Fractional Calculus and Applied Analysis,* vol. 19, pp. 676–695.

[24] Caputo, M. and Fabrizio, M., 2015. "A new defnition of fractional derivative without singular kernel." *Progress in Fractional Differentiation and Applications,* vol. 1, pp. 73–85.

[25] Atangana, A. and Baleanu, D., 2016. "New fractional derivatives with nonlocal and non-singular kernel, Theory and application to heat transfer model." *Thermal Science,* vol. 20, pp. 763–769.

[26] Losada, J. and Nieto, J. J., 2015. "Properties of a new fractional derivative without singular kernel." *Progress in Fractional Differentiation and Applications,* vol. 1, pp. 87–92.

[27] Abdeljawad, T. and Baleanu, D., 2017. "Integration by parts and its applications of a new nonlocal frac- tional derivative with mittag–leffler nonsingular kernel." *Journal of Nonlinear Sciences and Applications,* vol. 10, pp. 1098–1107.