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Lack of the Lower Bound for the Shannon Entropy

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Abstract

Shannon entropy is a basic characteristic of communications from the energetic point of view. Despite this fact, an expression for entropy as a function of the signal-to-noise ratio is still missing. In this paper, that shortage has been corrected first. Using that expression, lower bound for entropy has been investigated. We prove that such finite nonzero bound does not exist, therefore there is no theoretical limit for reduction of the effect of noise. The proof is valid for QAM modulation of arbitrary order.

Keywords: Shannon entropy; Probability of successful transmission; Bit error rate; Signal to noise ratio; Bound for noise reduction.

1. Introduction

In this year, we celebrate the 73rd anniversary of the Shannon theory [1]. An essential tool of the theory is the Shannon entropy [2]:

$$H = -\sum_i p_i \, ld \, p_i \quad (1)$$

Where p_i is the probability of the successful transmission of the i^th message, ld is the logarithm of base 2. For binary messages,

$$H = -p \, ld \, p - (1-p) ld(1-p) \quad (2)$$

In Section 2, H has been expressed as a function of signal to noise ratio. In Section 3, it has been proved that H has no extremum, as a function of the detection threshold. In Section 4, it is proved based on the previous Section, that there is no limit for reduction of the effect of noise.

2. An Expression for Shannon Entropy

From (2) we have the following ideA.1. i) The probability of a successful message p is in close connection with the bit error rate (BER). ii) BER can be expressed as a function of the signal to noise ratio (SNR) for any specified types of digital modulation [3]. iii) Then entropy can be expressed as a function of the detection threshold L, see please a few lines below. iv) Then extremum of entropy can be investigated.

This idea has been motivated by the problem if successful communication of bits is really limited by the condition when power density of the signal is identical to that of the noise. We suspect that the answer is negative, therefore we introduce the detection threshold $0 \leq L \leq 1$ and say, that the successful communication is limited by:

$$PDS = L.PDN \quad (3)$$

where PDS and PDN are the signal and noise power densities. That means if $PDS > L.PDN$ then the message is detected successfully, otherwise not. Our plan is to express $H(L)$ and try to determine the value of L corresponding to the minimum of H at a given SNR.

i) Denote g and b the number of good and bad detection of bits, respectively. Then

$$p = \frac{g}{g+b} \quad (4)$$

$$BER = \frac{b}{g+b} \quad (5)$$

By combining (4) and (5),

$$p = 1 - BER \quad (6)$$

From (2) and (6),

$$H = -BER \, ld \, BER - (1-BER) ld(1-BER) \quad (7)$$

ii) Expression for BER as a function of SNR is found in [3]:

$$BER = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}} \quad (8)$$

where E_b is bit energy, N_0 is noise power density and 4QAM modulation was assumed. Bit energy is the same as signal power density, so (8) can be written as follows:

$$BER = \frac{1}{2} erfc\sqrt{\frac{PDS}{PDN}} \quad (9)$$

where PDS and PDN are the signal and noise power densities, respectively. In this paper, we define the signal to noise ratio SNR as

$$SNR = \frac{PDS}{PDN} \quad (10)$$

This definition, instead of $SNR = \frac{S}{N}$, where S and N are the signal and noise power, respectively, is based on our modification of the Shannon formula [4].

Assume that averaging is applied for our signal that decreases PDN and leaves PDS intact:

$$SNR' = \frac{PDS}{L.PDN} = \frac{1}{L} SNR \quad (11)$$

Thus (9) is rewritten as

$$BER = \frac{1}{2} erfc\sqrt{L.SNR'} \quad (12)$$

iii) Applying (7) and (12), $H(L)$ has been obtained, for a given SNR'.

iv) Now it is to investigate if $H(L)$ has a finite nonzero minimum.

3. *H* has no Extremum

In Fig. 1, we plotted the negative derivative.

Fig-1. Horizontal axis: L. Vertical axis: $10\, log10 \frac{-dH(L)}{dL}$, at some SNR' values: .1 (cyan), .3 (blue), 1 (green), 3 (yellow), 10 (red). $H(L)$ has no minimum, it is strictly monotonically decreasing

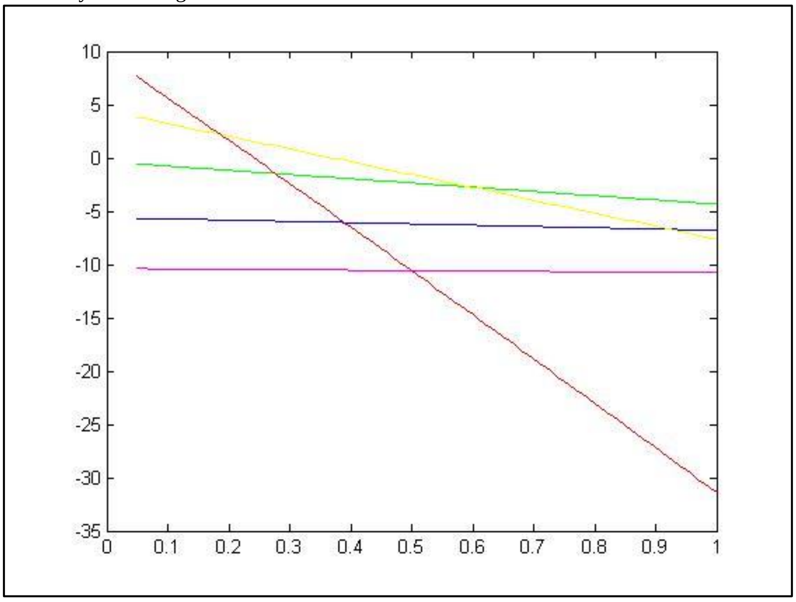

Fig. 1 has been provided using the following Matlab program:

```
SNRprime=1;
L=0.05:0.01:1;
dL=0.001;
BER=(.5*erfc(sqrt(L.*SNRprime)));
BER1=(.5*erfc(sqrt((L+dL).*SNRprime)));
H=-BER.*log2(BER)-(1-BER).*log2(1-BER);
H1=-BER1.*log2(BER1)-(1-BER1).*log2(1-BER1);
dH=H1-H;
plot(L,10*log10(-dH/dL));
```

This program has been repeated three times with different values of SNRprime.

4. Lack of Bound for Successful Transmission

Meaning of Fig. 1 is the following.

L can be arbitrarily small. That is, effect of noise can be decreased arbitrarily by averaging. That is done by repeating the message in noise and taking the arithmetic mean of the detected signals. This can be done without any bounds.

Analytical proofs have been found in the Appendix I and II.

5. Conclusions

We expected that by finding the extremum of $H(L)$, a specific L value can be found that is the bound of the averaging procedure. Surprisingly, we found that L can be arbitrarily small.

The result is significant. When trying to decrease the effect of noise by averaging, repetition can be done arbitrary times, without limit. Thus, equation (3) says that there is no theoretical limit for reduction of the effect of noise.

This result is in harmony with the similar result from the theory of coding. BER can be arbitrarily reduced by applying sufficient redundance.

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Appendix I

Analytical proof of the strict monotonicity of $H(L)$ for 4QAM

Strict negativity of $\frac{dH(L)}{dL}$ should be proved. We achieve it in the following way:

$$\frac{dH(L)}{dL} = \frac{dH(p)}{dp}\frac{dp}{dBER}\frac{dBER}{dSNR}\frac{dSNR}{dL} \qquad \text{(A.1.1)}$$

Let us start with (2) above:

$$H(p) = -p\, ld\, p - (1-p)ld(1-p) \qquad \text{(A.1.2)}$$

(A.1.2) yields:

$$\frac{dH(p)}{dp} = \frac{-1}{\ln 2\, dp}(p\, ln\, p + (1-p)ln(1-p)) = \frac{-1}{\ln 2}\left[\left(\ln p + p\frac{1}{p}\right) + \left(-\ln(1-p) + (1-p)\frac{-1}{1-p}\right)\right] = \frac{1}{\ln 2} ln\frac{1-p}{p} = ld\frac{1-p}{p} \qquad \text{(A.1.3)}$$

Continue with (6) above:

$$p = 1 - BER \qquad \text{(A.1.4))}$$

$$\frac{dp}{dBER} = -1 \qquad \text{(A.1.5))}$$

Continue with (9) above:

$$BER \approx \frac{1}{2} erfc\sqrt{SNR} \qquad \text{(A.1.6)}$$

$$\frac{dBER}{dSNR} = \frac{1}{2}\left(\frac{-2}{\sqrt{\pi}}\, e^{SNR}\right)\frac{1}{2\sqrt{SNR}} = \frac{-1}{2\sqrt{\pi\, SNR}} e^{SNR} \qquad \text{(A.1.7)}$$

Continue with (11) above:

$$SNR' = \frac{PDS}{L.PDN} = \frac{1}{L} SNR \qquad \text{(A.1.8)}$$

$$\frac{dSNR}{dL} = SNR' \qquad \text{(A.1.9)}$$

Combining the equations above:

$$\frac{dH(L)}{dL} = ld\frac{1-p}{p}\,(-1)\,\frac{-1}{2\sqrt{\pi\, SNR}} e^{SNR}\, SNR' \qquad \text{(A.1.10)}$$

$$ld\frac{1-p}{p}\,(-1) = ld\left(\frac{1}{BER} - 1\right) \qquad \text{(A.1.11)}$$

As in (A.1.6),

$$BER < \frac{1}{2} \qquad \text{(A.1.12)}$$

thus in (A.1.11),

$$ld\left(\frac{1}{BER} - 1\right) > 0 \qquad \text{(A.1.13)}$$

and in (A.1.10),

$$\frac{dH(L)}{dL} < 0 \qquad \text{(A.1.14)}$$

for any SNR' values. Therefore $H(L)$ has no extremum, it is strictly monotonically decreasing. That is, we wanted to prove.

Appendix II

Extension for QAM of arbitrary order

(A.1.6) is an approximation. That should be replaced with the exact formula that is valid for QAM of arbitrary order [5, 6].

Let us start with the exact formula for M-ary QAM symbol error rate [5, Eq. (15)]:

$$SER = 1 - \left(1 - \frac{2(\sqrt{M}-1)}{\sqrt{M}}\, Q\left(\sqrt{\frac{3\gamma_s}{M-1}}\right)\right)^2 \qquad \text{(A.2.1)}$$

where M is the order of QAM,

$$\gamma_s = \frac{E_s}{N_0} \qquad \text{(A.2.2)}$$

where E_s and N_0 are the average symbol energy and noise power density, respectively,

$$Q(x) = \frac{1}{2} erfc\ \frac{x}{\sqrt{2}} \qquad \text{(A.2.3)}$$

$$\gamma_b = \frac{E_b}{N_0} \qquad \text{(A.2.4)}$$

where E_b is the average bit energy, and

$$E_s = E_b\ ld\ M \qquad \text{(A.2.5)}$$

$$SER = BER\ ld\ M \qquad \text{(A.2.6)}$$

Combination of (A.2.1-6) yields:

$$BER = \frac{1}{ld\ M}\left[1 - \left(1 - \frac{(\sqrt{M}-1)}{\sqrt{M}}\, erfc\left(\sqrt{\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}\right)\right)^2\right] \qquad \text{(A.2.7)}$$

We use [7]:

$$\frac{d}{dz}\, erfc(z) = -\frac{2\, e^{-z^2}}{\sqrt{\pi}} \qquad \text{(A.2.8)}$$

Then we replace (A.1.7) with the following:

$$\frac{dBER}{dSNR} = \frac{-1}{ld\ M}\left[2\left(1 - \frac{(\sqrt{M}-1)}{\sqrt{M}}\, erfc\left(\sqrt{\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}\right)\right)\left(-\frac{(\sqrt{M}-1)}{\sqrt{M}}\right)\left(-\frac{2}{\sqrt{\pi}}\right) e^{-\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}\ \frac{1}{2\sqrt{\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}}\ \frac{3\ ld\ M}{(M-1)\sqrt{2}}\right] = \left[-2\left(1 - \frac{(\sqrt{M}-1)}{\sqrt{M}}\, erfc\left(\sqrt{\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}\right)\right)\left(\frac{(\sqrt{M}-1)}{\sqrt{M}}\right)\left(\frac{1}{\sqrt{\pi}}\right) e^{-\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}\ \frac{1}{\sqrt{\frac{3SNR\ ld\ M}{(M-1)\sqrt{2}}}}\ \frac{3}{(M-1)\sqrt{2}}\right] \qquad \text{(A.2.9)}$$

As $M \geq 2$, $\frac{dBER}{dSNR} < 0$. Substituting this into (A.1.1):

$$\frac{dH(L)}{dL} = \frac{dH(p)}{dp}\,\frac{dp}{dBER}\,\frac{dBER}{dSNR}\,\frac{dSNR}{dL} < 0 \qquad \text{(A.2.10)}$$

Because the first three terms are negative, and the last one is positive. Thus $H(L)$ is strictly monotonically decreasing for QAM of order M.